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Series ♯270. November 1994

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Gauge theory on a non-simply-connected domain and representations of canonical commutation relations

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A quantum system of a particle interacting with a (non-Abelian) gauge field on the non-simply-connected domain $M = \mathbf{R}^2 \setminus \{\mathbf{a}_n\}_{n=1}^N$ is considered, where $\mathbf{a}_n, n = 1, \cdots, N$, are fixed isolated points in $\mathbf{R}^2$. The gauge potential A of the gauge field is a $p \times p$ anti-Hermitian matrix-valued 1-form on M and may be strongly singular at the points $\mathbf{a}_n, n = 1, \cdots, N$. If A is flat, then the (non-canonical) momentum and the position operators $\{P_j, q_j\}_{j=1}^2$ of the particle satisfy the canonical commutation relations (CCR) with two degrees of freedom on a suitable dense domain of the Hilbert space $L^2(\mathbf{R}^2; \mathbf{C}^p)$. A necessary and sufficient condition for this representation to be the Schrödinger 2-system is given in terms of the Wilson loops of the rectangles not intersecting $\mathbf{a}_n, n = 1, \cdots, N$. This gives also a characterization for the representaion to be non-Schrödinger. It is proven that, for a class of gauge potentials, which is not necessarily flat, P_j is essentially self-adjoint. Moreover, an example, which gives a class of non-Schrödinger represenations of the CCR with two degrees of freedom, is discussed in some detail.

PACS numbers (1992):03.65.Db, 02.30.Tb

I. INTRODUCTION

In Ref.1, the author considered a quantum system of a charged particle with charge $q \in \mathbf{R} \setminus \{0\}$ moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field which may be singular at given fixed isolated points $\mathbf{a}_n = (a_{n1}, a_{n2}) \in \mathbf{R}^2, n = 1, \cdots, N$. In this quantum system, the position operators q_j and the (non-canonical) momentum operators $P_j = p_j - qA_j$ $(j = 1, 2)$ of the particle, where p_j is the canonical momentum operator with respect to q_j and (A_1, A_2) is a vector potential of the magnetic field, satisfy, on a dense domain of $L^2(\mathbf{R}^2)$, the canonical commutation relations (CCR) with two degrees of freedom if and only the magnetic field is concentrated on the set

$$S = \{\mathbf{a}_1, \cdots, \mathbf{a}_N\}. \tag{1.1}$$

A set $\{\widehat{P}_1, \cdots, \widehat{P}_n, \widehat{Q}_1, \cdots, \widehat{Q}_n\}$ of self-adjoint operators on a Hilbert space $\mathcal{H}$ is called a **Schrödinger n-system** if there exist mutually orthogonal closed subspaces $\mathcal{H}_\alpha$ of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ with the following properties: (i) each $\mathcal{H}_\alpha$ reduces all $\widehat{P}_j, \widehat{Q}_j$; (ii) the set $\{\widehat{P}_1, \cdots, \widehat{P}_n, \widehat{Q}_1, \cdots, \widehat{Q}_n\}$ is, in each $\mathcal{H}_\alpha$, irreducible and unitarily equivalent to the Schrödinger system $\{P_1^s, \cdots, P_n^s, Q_1^s, \cdots, Q_n^s\}$ where P_j^s and Q_j^s are self-adjoint operators on $L^2(\mathbf{R}^n)$ defined by $P_j^s = -i\partial_j$ (∂_j denotes the generalized partial differential operator in the j-th variable x_j of $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbf{R}^n$) and $Q_j^s = x_j$ (the multiplication operator by x_j)(Ref.2, p.81). We call a representation of the CCR with n degrees of freedom **non-Schrödinger** if it is not a Schrödinger n-system.

It was proven in Ref.1 that the above mentioned representation $\{\bar{P}_j, q_j\}_{j=1}^2$ of CCR, where $\bar{P}_j$ denotes the closure of P_j, is a Schrödinger 2-system if and only if the magnetic flux is locally quantized (i.e., the magnetic flux of every rectangle not intersecting $\mathbf{a}_n$ $(n = 1, \cdots, N)$ is an integer multiple of $2\pi/q$). This result, which generalizes the main result of Ref.3 concerning a special example in the case $N = 1$, shows that, if the magnetic field is concentrated on S and the magnetic flux is not locally quantized, then $\{\bar{P}_j, q_j\}_{j=1}^2$ is a non- Schrödinger representation of CCR. Thus a class of non-Schrödinger representations of CCR is constructed. An interesting point in these non-Schrödinger representations is that they correspond to the occurrence of the Aharonov-Bohm effect[4] in an idealized sense. From a mathematical point of view, the non-simply-connectedness of the base manifold

$$M = \mathbf{R}^2 \setminus S \tag{1.2}$$

is essential for the analysis just mentioned to be nontrivial. Further properties of the quantum system were investigated in terms of the Dirac-Weyl operator[5].

An operator-theoretical analysis related to Refs.1 and 3 has been made by Kurose and Nakazato[6], who has constructed a $*$-representation of the Weyl algebra with two degrees of freedom induced by a one-dimensional representation of the fundamental group of M and proved that the $*$-representation is unitarily equivalent to the $*$-algebra generated by $\{P_j, q_j\}_{j=1}^2$.

The quantum system discussed in Ref.1 is an example of Abelian gauge theory with gauge group $U(1)$, the one-dimensional unitary group. It is natural and interesting to extend the analysis made in Ref.1 to the case of non-Abelian gauge theory. With this motivation, we consider in this paper a quantum mechanical particle moving in M under the influence of a (non-Abelian) gauge field. Indeed, in this case too, the position and the (non-canonical) momentum operators of the particle give a representation of the CCR with two degrees of freedom if the gauge field strength is concentrated on S. We formulate a necessary and sufficient condition for the representation to be a Schrödinger 2-system. As in the case of Ref.1, this result gives a class of non-Schrödinger representations of CCR. These non-Schrödinger representations may physically correspond to the occurrence of the "Aharonov-Bohm effect" for a quantum mechanical particle moving under the influence of a (non-Abelian) gauge field. Our analysis is general in that it applies to any gauge theory on M with a finite dimensional unitary representation of a Lie group.

In Section I, under the assumption that the (non-canonical) momentum operators P_1, P_2 of the particle are essentially self-adjoint, we compute the commutation relations (in the strong sense) of the position operators q_1, q_2 of the particle and $\bar{P}_1, \bar{P}_2$. As a result, it is shown that $\{\bar{P}_j, q_j\}_{j=1}^2$ is a Schrödinger 2-system if and only if the Wilson loop of every rectangle not intersecting $\mathbf{a}_n$ $(n = 1, \cdots, N)$ is equal to the identity. In Section III, we derive, in terms of the gauge potential, a condition equivalent to the condition for the Wilson loops just mentioned. This can be done by employing the theory of product integral. Section IV is devoted to proof of essential self-adjointness of P_j for a wide class of gauge potentials. In the last section we discuss an example in some detail.

II. CCR IN (NON-ABELIAN) GAUGE THEORY

Let M be defined by (1.2) with S given by (1.1). For a natural number p, we denote by $M_p(\mathbf{C})$ the set of $p\times p$ matrices with comlex entries and by $M_p^{\rm as}(\mathbf{C})$ the set of $p\times p$ anti-Hermitian matrices. Let A_j $(j=1,2)$ be an $M_p^{\rm as}(\mathbf{C})$-valued continuously differentiable fucntions on M and set

$$A(\mathbf{r}) = A_1(\mathbf{r})dx + A_2(\mathbf{r})dy, \quad \mathbf{r}=(x,y)\in M,$$

an $M_p^{\rm as}(\mathbf{C})$-valued 1-form on M. This 1-form may be regarded as a gauge potential in a gauge theory on M with a p-dimensional unitary represenation of a Lie group.

We shall use the system of physical units where $\hbar$, the Planck constant divided by 2π, is equal to 1. Let ∂_1 and ∂_2 be the generalized partial differential operators in x and y, respectively, and set

$$p_j = -i\partial_j, \quad j=1,2.$$

Then the (non-canonical) momentum operator of a quantum mechanical particle interacting with the gauge potential A may be given by $\mathbf{P}=(P_1,P_2)$ with

$$P_j = p_j - iA_j, \quad j=1,2,$$

acting in the Hilbert space $L^2(\mathbf{R}^2;\mathbf{C}^p)$ $(\simeq L^2(M;\mathbf{C}^p))$ of $\mathbf{C}^p$-valued square integrable functions on $\mathbf{R}^2$. We denote by $C_0^m(M;\mathbf{C}^p)$ the set of $\mathbf{C}^p$-valued m times continuously differentiable fucntions on M with compact support. Each P_j is a symmetric operator with $C_0^2(M;\mathbf{C}^p)\subset D(P_1P_2)\cap D(P_2P_1)$ and

$$[P_1,P_2]\psi = -F_{12}\psi, \quad \psi\in C_0^2(M;\mathbf{C}^p), \tag{2.1}$$

where

$$F_{12} := \partial_1 A_2 - \partial_2 A_1 + [A_1,A_2]$$

is the component of the gauge field strength 2-form

$$F(A) := dA + A\wedge A = F_{12}dx\wedge dy.$$

We say that A is *flat on M* if $F(A)=0$ on M.

We denote by q_1 and q_2 the multiplication operators by x and y, respectively. Then we have the following fact.

Proposition 2.1: *Suppose that A is flat on M. Then $\{P_j,q_j\}_{j=1}^2$ satisfies the CCR with two degrees of freedom on $C_0^2(M;\mathbf{C}^p)$: for all $\psi\in C_0^2(M;\mathbf{C}^p)$,*

$$[P_j,P_k]\psi = 0, \tag{2.2}$$

$$[q_j,q_k]\psi = 0, \quad [q_j,P_k]\psi = i\delta_{jk}\psi, \quad j,k=1,2. \tag{2.3}$$

Proof: Equation (2.2) is a simple consequence of (2.1) together with the flatness of A. Equation (2.3) follows from a direct computation. ∎

Thus we have a representation of the CCR with two degrees of freedom. In what follows, we investigate under what conditions this representation of CCR is a Schrödinger 2-system. For this purpose, under the condition that P_j is essentially self-adjoint, we first compute the commutation relations of the operators $\exp(itq_j), \exp(it\bar{P}_j)$, $j = 1, 2$, where $\bar{P}_j$ denotes the closure of P_j.

For a continuous, piecewise continuously differentiable path C in M with parametrization $\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)), \tau \in [a, b]$ $(a < b, a, b \in \mathbf{R})$, we define the **Wilson loop** $W_A(C)$ by

$$W_A(C) = \prod_a^b e^{-\{A_1(\gamma(\tau))\dot{\gamma}_1(\tau)+A_2(\gamma(\tau))\dot{\gamma}_2(\tau)\}d\tau},$$

where $\dot{\gamma}_j(\tau) = d\gamma_j(\tau)/d\tau, j = 1, 2$, and the right hand side (RHS) is the product integral of the matrix-valued fucntion $-\{A_1(\gamma(\tau))\dot{\gamma}_1(\tau) + A_2(\gamma(\tau)\dot{\gamma}_2(\tau))\}$ on the interval $[a, b]$ (see Ref.7, §1.1). In the physicist notation,

$$W_A(C) = P\exp(-\int_C A).$$

It can be shown that $W_A(C)$ is independent of the choice of parametrizations of C (cf. Ref.7, §2.1). It follows from the anti-Hermiticity of A_j that $W_A(C)$ is in $U(p)$, the set of $p \times p$ unitary matrices.

For $x, y, s, t \in \mathbf{R}$, we define two hook-shaped paths $C^{\pm}_{x,y;s,t}$ from (x, y) to $(x+s, y+t)$ with parametrizations $\gamma^{\pm}_{x,y;s,t} : [0, 1] \to \mathbf{R}^2$ given by

$$\gamma^{-}_{x,y;s,t}(\tau) = \begin{cases} (x + 2\tau s, y) & ; 0 \le \tau \le \frac{1}{2} \\ (x + s, y + (2\tau - 1)t) & ; \frac{1}{2} \le \tau \le 1 \end{cases} \tag{2.4}$$

$$\gamma^{+}_{x,y;s,t}(\tau) = \begin{cases} (x, y + 2\tau t) & ; 0 \le \tau \le \frac{1}{2} \\ (x + (2\tau - 1)s, y + t) & ; \frac{1}{2} \le \tau \le 1 \end{cases} \tag{2.5}$$

and set

$$C_{x,y;s,t} = \{C^{+}_{x,y;s,t}\}^{-1} \circ C^{-}_{x,y;s,t},$$

which is the rectangle : $(x, y) \to (x + s, y) \to (x + s, y + t) \to (x, y + t) \to (x, y)$.

For each $s, t \in \mathbf{R}$, let

$$\mathbf{R}^{(1)}_s = \mathbf{R} \setminus \{a_{n1}, a_{n1} - s\}_{n=1}^N, \quad \mathbf{R}^{(2)}_t = \mathbf{R} \setminus \{a_{n2}, a_{n2} - t\}_{n=1}^N,$$

and

$$M_{s,t} = \mathbf{R}^{(1)}_s \times \mathbf{R}^{(2)}_t.$$

Note that, if $(x,y) \in M_{s,t}$, then $C^{\pm}_{x,y;s,t}$ do not intersect $\mathbf{a}_n, n = 1, \cdots, N$. Hence, for each $s,t \in \mathbf{R}$, we can define $U(p)$-valued functions $W^{A,\pm}_{s,t}, W^{A}_{s,t}$ on $M_{s,t}$ by

$$W^{A,\pm}_{s,t}(x,y) = W_A\left(C^{\pm}_{x,y;s,t}\right), \quad W^{A}_{s,t}(x,y) = W_A\left(C_{x,y;s,t}\right), \quad (x,y) \in M_{s,t}.$$

The two-dimensional Lebesgue measure of the set $\mathbf{R}^2 \setminus M_{s,t}$ is zero. Hence $W^{A,\pm}_{s,t}$ and $W^{A}_{s,t}$ can be regarded as almost everywhere (a.e.) finite $U(p)$-valued functions on $\mathbf{R}^2$, so that they define unitary operators on $L^2(\mathbf{R}^2;\mathbf{C}^p)$ as multiplication operators. We denote these unitary multiplication operators by the same symbols $W^{A,\pm}_{s,t}, W^{A}_{s,t}$, respectively.

Theorem 2.2: *Assume that P_j ($j = 1,2$) is essentially self-adjoint. Then, for all $s,t \in \mathbf{R}$,*

$$e^{itP_2}e^{isP_1} = W^{A}_{s,t}e^{isP_1}e^{itP_2}. \tag{2.6}$$

Proof: The method of proof is similar to the proof of Theorem 2.1 in Ref.1. By the present assumption, we can apply the Trotter product formula to obtain

$$\left\langle \phi, e^{itP_j}\psi \right\rangle = \lim_{m\to\infty} \left\langle \phi, \left(e^{itp_j/m}e^{tA_j/m}\right)^m \psi \right\rangle,$$

where $< \cdot, \cdot >$ denotes the inner product of $L^2(\mathbf{R}^2;\mathbf{C}^p)$. We denote by $L_{(x,y);(x',y')}$ the straight line from (x,y) to (x',y'). The srtaight line $L_{(x+s,y);(x,y)}$ is parametrized by

$$\gamma(\tau) = (x + (1-\tau)s, y), \quad \tau \in [0,1].$$

Then $\dot{\gamma}(\tau) = -s(1,0)$. In terms of $\gamma(\tau)$ and $\dot{\gamma}(\tau)$, we can write

$$\begin{aligned}
&\left(\left(e^{isp_1/m}e^{sA_1/m}\right)^m \psi\right)(x,y) \\
&= \exp\left(-A_1\left(\gamma\left(\frac{m-1}{m}\right)\right)\dot{\gamma}_1\left(\frac{m-1}{m}\right)\frac{1}{m}\right)\cdots\exp\left(-A_1\left(\gamma\left(\frac{1}{m}\right)\right)\dot{\gamma}_1\left(\frac{1}{m}\right)\frac{1}{m}\right) \\
&\quad\times \exp\left(-A_1\left(\gamma(0)\right)\dot{\gamma}_1(0)\frac{1}{m}\right)\psi(x+s,y),
\end{aligned}$$

which converges to $W_A(L_{(x+s,y);(x,y)})\psi(x+s,y)$ a.e.$(x,y) \in M_{s,t}$ as $m \to \infty$. Moreover, by the fact that $e^{-A_1(\gamma(\tau))\dot{\gamma}_1(\tau)/m} \in U(p)$, we have

$$||\phi(x,y)\left(e^{isp_1/m}e^{sA_1/m}\right)^m \psi(x,y)||_{\mathbf{C}^p} \le ||\phi(x,y)||_{\mathbf{C}^p}||\psi(x+s,y)||_{\mathbf{C}^p}$$

and

$$\int_{\mathbf{R}^2} ||\phi(x,y)||_{\mathbf{C}^p}||\psi(x+s,y)||_{\mathbf{C}^p} d\mathbf{r} \le \left(\int_{\mathbf{R}^2} ||\phi(\mathbf{r})||^2_{\mathbf{C}^p} d\mathbf{r}\right)^{1/2}\left(\int_{\mathbf{R}^2} ||\psi(\mathbf{r})||^2_{\mathbf{C}^p} d\mathbf{r}\right)^{1/2} < \infty.$$

Hence, by the dominated convergence theorem, we obtain for all $\phi, \psi \in L^2(\mathbf{R}^2; \mathbf{C}^p)$

$$\left\langle \phi, e^{is\bar{P}_1}\psi \right\rangle = \int_{\mathbf{R}^2} \langle \phi(x,y), W_A(L_{(x+s,y);(x,y)})\psi(x+s,y)\rangle_{\mathbf{C}^p}\, d\mathbf{r},$$

which implies that

$$\left(e^{is\bar{P}_1}\psi\right)(x,y) = W_A(L_{(x+s,y);(x,y)})\psi(x+s,y), \quad \text{a.e.}(x,y) \in \mathbf{R}^2.$$

Similarly we can show that

$$\left(e^{it\bar{P}_2}\psi\right)(x,y) = W_A(L_{(x,y+t);(x,y)})\psi(x,y+t), \quad \text{a.e.}(x,y) \in \mathbf{R}^2.$$

Combining these formulas, we obtain

$$e^{is\bar{P}_1}e^{it\bar{P}_2} = \left(W_{s,t}^{A,-}\right)^{-1} e^{isp_1}e^{itp_2}, \quad e^{it\bar{P}_2}e^{is\bar{P}_1} = \left(W_{s,t}^{A,+}\right)^{-1} e^{isp_1}e^{itp_2},$$

which, together with the fact $(W_{s,t}^{A,+})^{-1}W_{s,t}^{A,-} = W_{s,t}^{A}$, imply (2.6). ∎

As for the commutation relations of $\exp(isq_j)$ and $\exp(it\bar{P}_j)$ $(s,t \in \mathbf{R})$, we have the following.

Lemma 2.3: *Assume that P_j $(j = 1,2)$ is essentially self-adjoint. Then, for all $s,t \in \mathbf{R}$,*

$$e^{isq_j}e^{it\bar{P}_k} = e^{-ist\delta_{jk}}e^{it\bar{P}_k}e^{isq_j}, \quad j,k = 1,2.$$

Proof: Simlar to the proof of Theorem 2.1. Note that $\{e^{itq_j}, e^{itp_j} | t \in \mathbf{R}, j = 1,2\}$ satisfies the Weyl relations with two degrees of freedom (e.g, Ref.2, p.81). ∎

Theorem 2.2 and Lemma 2.3 imply the following:

Theorem 2.4: *Assume that P_j $(j = 1,2)$ is essentially self-adjoint. Then the set $\{e^{itq_j}, e^{it\bar{P}_j} | t \in \mathbf{R}, j = 1,2\}$ satisfies the Weyl relations with two degrees of freedom if and only if $W_{s,t}^A = I$ for all $s,t \in \mathbf{R}$, where I is the identity operator on $L^2(\mathbf{R}^2; \mathbf{C}^p)$.*

As a corollary, we obtain the following:

Corollary 2.5: *Assume that P_j $(j = 1,2)$ is essentially self-adjoint and A is flat on M. Then $\{\bar{P}_j, q_j\}_{j=1}^2$ is a Schrödinger 2-system if and only if $W_{s,t}^A = I$ for all $s,t \in \mathbf{R}$.*

Proof. We need only apply the well known fact that a representation of the CCR with n degrees of freedom satisfying the Weyl relations with the same degrees of freedom is a Schrödinger n-system (von Neumann's theorem[8], Ref.2, p.81, Theorem 4.11.1). ∎

Corollary 2.5 shows that the representation $\{\bar{P}_j, q_j\}_{j=1}^2$ of CCR appearing in the case where A is flat on M is characterized in terms of the Wilson loops for the rectangles $C_{x,y;s,t}$. In the next section we derive a condition equivalent to the condition that $W_{s,t}^A = I$ for all $s, t \in \mathbf{R}$.

III. THE WILSON LOOPS $W_{s,t}^A$

Let

$$\delta_0 = \min_{n \neq m\, ;\, n,m=1,\cdots,N} |\mathbf{a}_n - \mathbf{a}_m|. \tag{3.1}$$

For a positive constant $\varepsilon < \delta_0$ and $\mathbf{r} \in M$ with $|\mathbf{r} - \mathbf{a}_n| = \varepsilon$, we denote by $C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)$ the circle of radius ε with center $\mathbf{a}_n$ and initial point $\mathbf{r}$, where the direction of the circle is taken to be anticlockwise. In this section we prove the following theorem.

Theorem 3.1: *The equality $W_{s,t}^A = I$ holds for all $s, t \in \mathbf{R}$ if and only if A is flat on M and there exists a constant $\delta \in (0, \delta_0)$ such that for all $\varepsilon < \delta$ and some $\mathbf{r}_n \in M$ with $|\mathbf{r}_n - \mathbf{a}_n| = \varepsilon$,*

$$W_A(C_\varepsilon^{\mathbf{r}_n}(\mathbf{a}_n)) = I, \quad n = 1, \cdots, N. \tag{3.2}$$

Remark: One can easily show that, if $W_A(C_\varepsilon^{\mathbf{r}_n}(\mathbf{a}_n)) = I$ for some $\mathbf{r}_n$ with $|\mathbf{r}_n - \mathbf{a}_n| = \varepsilon$, then $W_A(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) = I$ for all $\mathbf{r}$ with $|\mathbf{r} - \mathbf{a}_n| = \varepsilon$.

We denote by $D_{x,y;s,t}$ the interior domain of the closed path $C_{x,y;s,t}$.

Lemma 3.2.: *For all $(x, y) \in M$,*

$$W_{t,t}^A(x,y) = I - \int_{D_{x,y;t,t}} F_{12}(\mathbf{r}')d\mathbf{r}' + O(|t|^3) \tag{3.3}$$

as $t \to 0$.

Proof: Informal proofs of this lemma can be found in the physics literature (e.g., Ref.9, pp.52–53). For the sake of completeness, we give a rigorous proof of it. Let $\gamma_{x,y;s,t} : [0,1] \to M$ be the parametrization of the path $C_{x,y;s,t}$ such that $\gamma_{x,y;s,t}(\tau) = \gamma_{x,y;s,t}^-(2\tau)$ for $0 \le \tau \le 1/2$ and $\gamma_{x,y;s,t}(\tau) = \gamma_{x,y;s,t}^+(1 - 2(\tau - 1/2))$ for $1/2 \le \tau \le 1$, where $\gamma_{x,y;s,t}^\pm(\tau)$ are given by (2.4) and (2.5). Let

$$B_{x,y;s,t}(\tau) = A_1(\gamma_{x,y;s,t}(\tau))(\dot{\gamma}_{x,y;s,t})_1(\tau) + A_2(\gamma_{x,y;s,t}(\tau))(\dot{\gamma}_{x,y;s,t})_2(\tau), \quad 0 \le \tau \le 1.$$

and, for $k \geq 1$, define

$$J_k(x,y;s,t) = \int_0^1 \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} B_{x,y;s,t}(\tau_1) B_{x,y;s,t}(\tau_2) \cdots B_{x,y;s,t}(\tau_k) d\tau_k \cdots d\tau_2 d\tau_1.$$

Then, applying Theorem 4.3 in §1.4 of Ref.7 (p.31), we have for all $s, t \in \mathbf{R}$

$$W^A_{s,t}(x,y) = I + \sum_{k=1}^{\infty} (-1)^k J_k(x,y;s,t), \quad (x,y) \in M_{s,t}. \tag{3.4}$$

There is a positive constant $t_0 < 1$ such that for all $|t| \in (0, t_0)$, $\mathbf{a}_n \notin D_{x,y;t,t}, n = 1, \cdots, N$. Let $0 < |t| < t_0$ and $a = \sup\{||A_j(\mathbf{r}')||; \mathbf{r}' \in D_{x,y;t_0,t_0} \cup D_{x,y;-t_0,-t_0}, j = 1, 2\}$. Then we have

$$||B_{x,y;t,t}(\tau)|| \leq 8a|t|,$$

which implies that

$$||J_k(x,y;t,t)|| \leq \frac{(8a|t|)^k}{k!}, \quad k \geq 1.$$

Hence we have

$$\sum_{k=3}^{\infty} ||J_k(x,y;t,t)|| \leq Ct^3, \tag{3.5}$$

where $C = \sum_{k=3}^{\infty} (8a)^k/k! < \infty$.

By Stokes' theorem, we have

$$J_1(x,y;t,t) = \int_{D_{x,y;t,t}} (\partial_1 A_2(\mathbf{r}') - \partial_2 A_1(\mathbf{r}')) d\mathbf{r}'. \tag{3.6}$$

We can write

$$J_2(x,y;t,t) = \frac{1}{2}(J_+ + J_-),$$

where

$$J_\pm = \int_0^1 \int_0^{\tau_1} \left(B_{x,y;t,t}(\tau_1) B_{x,y;t,t}(\tau_2) \pm B_{x,y;t,t}(\tau_2) B_{x,y;t,t}(\tau_1) \right) d\tau_2 \tau_1.$$

By symmetry, we have

$$J_+ = \frac{1}{2} J_1(x,y;t,t)^2.$$

By (3.6), we have

$$||J_1(x,y;t,t)|| \leq bt^2, \quad 0 < |t| < t_0,$$

where $b = \sup\{||\partial_1 A_2(\mathbf{r}') - \partial_2 A_1(\mathbf{r}')||; \mathbf{r}' \in D_{x,y;t_0,t_0} \cup D_{x,y;-t_0,-t_0}\}$. Hence

$$||J_+|| \leq \frac{b^2}{2} t^4. \tag{3.7}$$

To estimate J_-, we note that

$$B_{x,y;t,t}(\tau) = A_1(x,y)(\dot{\gamma}_{x,y;t,t})_1(\tau) + A_2(x,y)(\dot{\gamma}_{x,y;t,t})_2(\tau) + O(t^2)$$

as $t \to 0$ uniformly in $\tau \in [0,1]$. Hence

$$\begin{aligned}
&B_{x,y;t,t}(\tau_1)B_{x,y;t,t}(\tau_2) - B_{x,y;t,t}(\tau_2)B_{x,y;t,t}(\tau_1) \\
&= \sum_{\mu,\nu=1}^{2} [A_\mu(x,y), A_\nu(x,y)](\dot{\gamma}_{x,y;t,t})_\mu(\tau_1)(\dot{\gamma}_{x,y;t,t})_\nu(\tau_2) + O(|t|^3).
\end{aligned}$$

It follows that

$$\begin{aligned}
J_- &= -[A_1(x,y), A_2(x,y)] \int_{C_{x,y;t,t}} (x'dy' - y'dx') + O(|t|^3) \\
&= -2\int_{D_{x,y;t,t}} [A_1(\mathbf{r}'), A_2(\mathbf{r}')]d\mathbf{r}' + O(|t|^3).
\end{aligned} \tag{3.8}$$

Substituting (3.5)–(3.8) into (3.4), we obtain (3.3). ∎

The following lemma is well known.

Lemma 3.3: *Suppose that A is flat on M and each component A_j is m times continuously differentiable on M. Then, for every simply-connected domain D of M, there exists a $U(p)$-valued, $m+1$ times continuously differentiable function g on D such that $A_j = g^{-1}\partial_j g$ on D.*

Proof: See, e.g., Ref.10. ∎

Lemma 3.4: *Suppose that A is flat on M. Then the following (i)-(iii) hold:*

(i) *Let C be any continuous, piecewise continuously differentiable closed path in M which is contractible to a point within M. Then*

$$W_A(C) = I. \tag{3.9}$$

(ii) *Let $C_\varepsilon^{\mathbf{r}_1}(\mathbf{a}_n) \subset D_{x,y;s,t}$ with $|\mathbf{r}_1 - \mathbf{a}_n| = \varepsilon < \delta_0$ and $D_{x,y;s,t} \cap \{\mathbf{a}_1, \cdots, \mathbf{a}_N\} = \{\mathbf{a}_n\}$. Then there exists a unitary matrix U such that*

$$W_{s,t}^A(x,y) = UW_A(C_\varepsilon^{\mathbf{r}_1}(\mathbf{a}_n))U^{-1}. \tag{3.10}$$

(iii) *Let $0 < \varepsilon_1 < \varepsilon_2 < \delta_0$ and $\mathbf{r}_j \in \mathbf{R}^2, j = 1,2$, be such that $|\mathbf{r}_j - \mathbf{a}_n| = \varepsilon_j$. Then*

$$W_A(C_{\varepsilon_2}^{\mathbf{r}_2}(\mathbf{a}_n)) = W_A(L_{\mathbf{r}_2;\mathbf{r}_1})^{-1}W_A(C_{\varepsilon_1}^{\mathbf{r}_1}(\mathbf{a}_n))W_A(L_{\mathbf{r}_2;\mathbf{r}_1}), \tag{3.11}$$

where $L_{\mathbf{r}_2;\mathbf{r}_1}$ denotes the straight line from $\mathbf{r}_2$ to $\mathbf{r}_1$.

Proof: (i) The path C is included in a simply connected domain D of M. By Lemma 3.3, there exists a $U(p)$-valued twice differentiable function g on D such that $A_j = g^{-1}\partial_j g$ on D $(j = 1, 2)$. In terms of $h := g^{-1}$, we can write $A_j = -(\partial_j h)h^{-1}$. Let $\gamma : [0, 1] \to M$ be a parametrization of C $(\gamma(0) = \gamma(1))$. Then we have

$$-\{A_1(\gamma(\tau))\dot{\gamma}_1(\tau) + A_2(\gamma(\tau))\dot{\gamma}_2(\tau)\} = \frac{dh(\gamma(\tau))}{d\tau} h(\gamma(\tau))^{-1}.$$

Hence, applying Theorem 3.1 on p.20 in Ref.7, we have $W_A(C) = h(\gamma(1))h(\gamma(0))^{-1} = I$. Thus we obtain (3.9).

(ii) We decompose $D_{x,y;s,t}$ as is shown in Fig. 3.1.

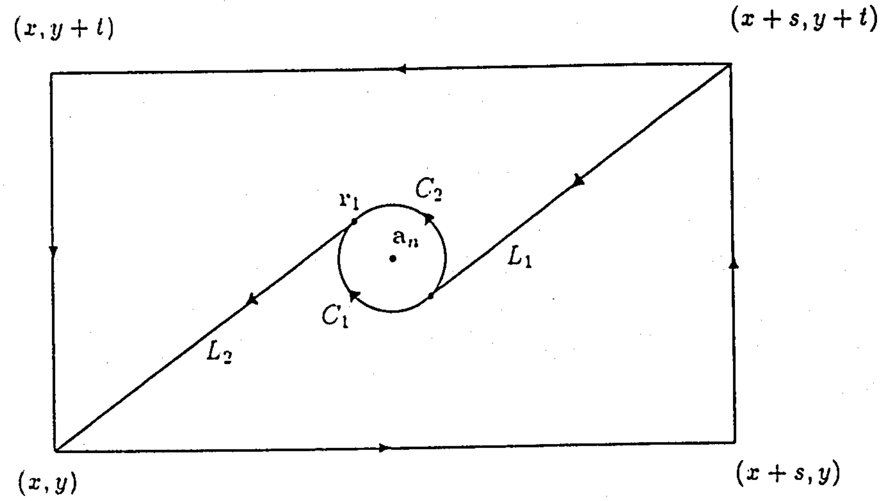

Fig. 3.1

Let

$$V_j = W_A(C_j),\ W_j = W_A(L_j),\ j = 1, 2.$$

Then, by part (i), we have

$$W_2 V_1 W_1 W_{s,t}^{A,-}(x, y) = I, \qquad W_1^{-1} V_2^{-1} W_2^{-1} (W_{s,t}^{A,+})^{-1} = I,$$

which imply $W_2 V_1 V_2^{-1} W_2^{-1} W_{s,t}^{A}(x, y) = I$. We have

$$V_2 V_1^{-1} = W_A(C_\varepsilon^{\mathrm{r}_1}(\mathrm{a}_n)).$$

Thus, taking $U = W_2$, we obtain (3.10).

(iii) We need only repeat the proof of part (ii) with ε and $D_{x,y;s,t}$ replaced by ε_1 and $C^{\mathrm{r}_2}_{\varepsilon_2}(\mathrm{a}_n)$, respectively. ∎

Proof of Theorem 3.1

Suppose that $W^A_{s,t} = I$ holds for all $s, t \in \mathbf{R}$. Then Lemma 3.2 implies that $F_{12}(\mathbf{r}) = 0$ for all $\mathbf{r} \in M$. Hence A is flat on M. Then, using Lemma 3.4 (i) and (ii), we obtain (3.2).

Conversely, suppose that A is flat on M and (3.2) holds. Then, by Lemma 3.4(i), we have $W^A_{s,t}(x,y) = I$ for all $C_{x,y;s,t}$ contractible to a point within M. Let $\mathrm{a}_n \in D_{x,y;s,t}$, but $\mathrm{a}_m \notin D_{x,y;s,t}$ for $m \neq n$. Then, by Lemma 3.4 (ii), we have

$$W^A_{s,t}(x,y) = UW_A(C^{\mathrm{r}_1}_{\varepsilon}(\mathrm{a}_n))U^{-1} = UIU^{-1} = I.$$

Hence, in this case, $W^A_{s,t}(x,y) = I$. Finally we consider the case where $D_{x,y;s,t}$ include $\{\mathrm{a}_{i_1}, \cdots, \mathrm{a}_{i_k}\}$, where $2 \leq k \leq N$ and $\{i_1, \cdots, i_k\}$ is a subset of $\{1, \cdots, N\}$. In this case we decompose $D_{x,y;s,t}$ as is shown in Fig.3.2, where we set $\mathrm{b}_j = \mathrm{a}_{i_j}, j = 1, \cdots, k$.

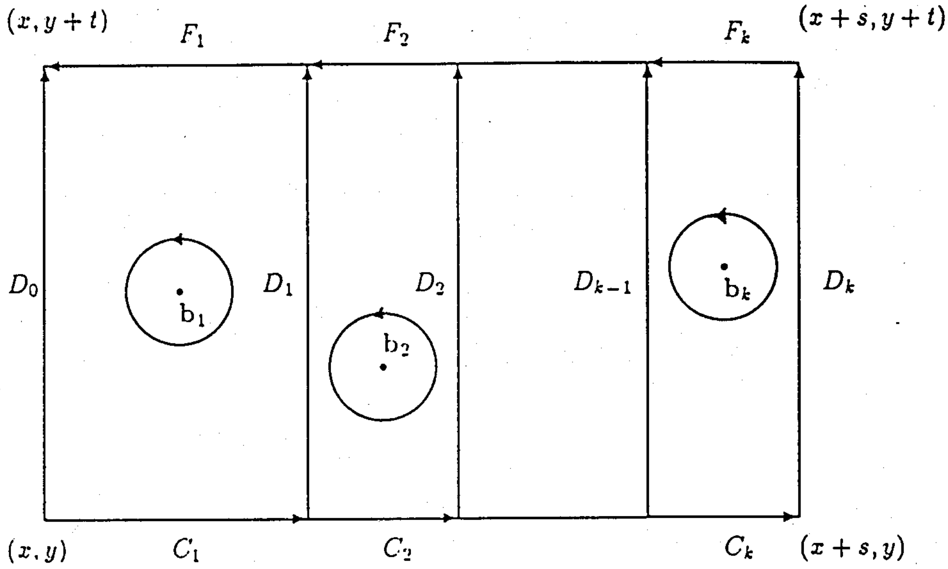

Fig. 3.2

Then, by Lemma 3.4(ii), there exist $U_j \in U(p), j = 1, , \cdots, k$ such that for $j = 1, \cdots, k$,

$$U_j W_A(C^{\mathrm{r}_j}_{\varepsilon}(\mathrm{b}_j))U_j^{-1} = W_A(D_{j-1}^{-1} \circ F_j \circ D_j \circ C_j)$$

By (3.2), each of the RHS's of these equalities turns out to be the identity. Then the resulting equalities give $W^A_{s,t}(x,y) = I$. ∎

IV. ESSENTIAL SELF-ADJOINTNESS OF P_j

For $j = 1, 2$, we set

$$S_j = \mathbf{R} \setminus \{a_{nj}\}_{n=1}^{N}.$$

Let

$$\mathcal{D}_1^m = C_0^m(\mathbf{R} \times S_2; \mathbf{C}^p), \quad \mathcal{D}_2^m = C_0^m(S_1 \times \mathbf{R}; \mathbf{C}^p).$$

For a subset V of $M_p(\mathbf{C})$ and an open subset D of M, we denote by $C^m(D; V)$ the set of V-valued, m times continuously differentiable functions on D. We introduce a class of gauge potentials.

Definition 4.1: We say that a 1-form A on M is in the set $\mathfrak{A}_m$ if there exist $g_1 \in C^{m+1}(\mathbf{R} \times S_2; U(p))$ and $g_2 \in C^{m+1}(S_1 \times \mathbf{R}; U(p))$ such that

$$A_1 = g_1^{-1}\partial_1 g_1 \quad \text{on } \mathbf{R} \times S_2, \quad A_2 = g_2^{-1}\partial_2 g_2, \quad \text{on } S_1 \times \mathbf{R}.$$

Theorem 4.2: *Assume that $A \in \mathfrak{A}_{m-1}$ ($m \geq 1$). Then P_j ($j = 1, 2$) is essentially self-adjoint on $\mathcal{D}_j^m$.*

Proof: Let g_j be as in Definition 4.1. Then g_j is a bijective mapping from $\mathcal{D}_j^m$ to itself. Note that

$$P_j\psi = g_j^{-1} p_j g_j \psi, \quad \psi \in \mathcal{D}_j^m.$$

Hence we need only to show that p_j is essentially self-adjoint on $\mathcal{D}_j^m$. But this is a well known fact (see the proof of Theorem 3.2 in Ref.1). ∎

Theorem 4.3: *Suppose that $A_j \in C^m(M; M_p^{\rm as}(\mathbf{C}))$($j = 1, 2$) ($m \geq 1$) and $A = A_1 dx + A_2 dy$ is flat on M. Then $A \in \mathfrak{A}_m$. In particular, P_j is essentially self-adjoint on $\mathcal{D}_j^{m+1}$.*

Proof: Let $b_1, \cdots, b_k$ ($k \leq N$) be the numbers different each other in the set $\{a_{n2}\}_{n=1}^{N}$ with $b_1 < b_2 < \cdots < b_k$. Let $M_\ell = \{(x, y) \in \mathbf{R}^2 | b_{\ell-1} < y < b_\ell\}, \ell = 1, \cdots, k+1$, with $b_0 = -\infty, b_{k+1} = +\infty$. Then each M_ℓ is simply-connected and $\mathbf{R} \times S_2 = \cup_{\ell=1}^{k+1} M_\ell$. By Lemma 3.3, there exists a function $h_\ell \in C^{m+1}(M_\ell; U(p))$ such that $A_j = h_\ell^{-1}\partial_j h_\ell$ ($j = 1, 2$) on M_ℓ. Defining $g_1 \in C^{m+1}(\mathbf{R} \times S_2; U(p))$ by $g_1(\mathbf{r}) = h_\ell(\mathbf{r})$ if $\mathbf{r} \in M_\ell$, we have $A_j = g_1^{-1}\partial_j g_1$ on $\mathbf{R} \times S_2$. In particular, $A_1 = g_1^{-1}\partial_1 g_1$ on $\mathbf{R} \times S_2$. Similarly we can show that there exists a function $g_2 \in C^{m+1}(S_1 \times \mathbf{R}; U(p))$ such that $A_2 = g_2^{-1}\partial_2 g_2$ on $S_1 \times \mathbf{R}$. ∎

V. NON-SCHRÖDINGER REPRESENTATIONS

Corollary 2.5, Theorems 3.1 and 4.3 yield the following result.

Theorem 5.1: *Suppose that $A_j \in C^m(M; M_p^{\rm as}(\mathbf{C}))$($j = 1, 2$) ($m \geq 1$) and A is flat on M. Then, the representation $\{\bar{P}_j, q_j\}_{j=1}^2$ of CCR is a Schrödinger 2-system if and* *only if there exists a constant $\delta \in (0, \delta_0)$ such that for all $\varepsilon < \delta$ and some $\mathbf{r}_n \in M$ with $|\mathbf{r}_n - \mathbf{a}_n| = \varepsilon$,*

$$W_A(C_\varepsilon^{\mathbf{r}_n}(\mathbf{a}_n)) = I, \quad n = 1, \cdots, N.$$

Theorem 5.1 can be rephrased as follows: *Under the same assumption as that of Theorem 5.1,, $\{\bar{P}_j, q_j\}_{j=1}^2$ is a non-Schrödinger representation of the CCR with two degrees of freedom if and only if*

(A) *there exists a sequence $\{\varepsilon_\ell\}_\ell$ of positive numbers converging to zero such that the following holds: for each ℓ, there exist a number $n_\ell \in \{1, \cdots, N\}$ and $\mathbf{r}_\ell$ with $|\mathbf{r}_\ell - \mathbf{a}_{n_\ell}| = |\varepsilon_\ell|$ such that $W_A(C_{\varepsilon_\ell}^{\mathbf{r}_\ell}(\mathbf{a}_{n_\ell})) \neq I$.*

Let $\mathcal{A}^m(M)$ be the set of $M_p^{\mathrm{as}}(\mathbf{C})$)-valued, flat 1-forms with $A_j \in C^m(M; M_p^{\mathrm{as}}(\mathbf{C})), j = 1, 2$, satisfying condition (A). Then, the above result shows that, for each $A \in \mathcal{A}^m(M)$, there is a non-Schrödinger representation of CCR.

VI. EXAMPLE

In this section we discuss a class of $M_p^{\mathrm{as}}(\mathbf{C})$-valued, flat 1-forms on M. Let S_n and T_n be $p \times p$ Hermitian constant matrices such that, for all $n \neq m$ $(n, m = 1, \cdots, N)$,

$$[S_n, S_m] = 0, \quad [T_n, T_m] = 0, \quad [S_n, T_m] = 0,$$

but S_n does not necessarily commute with T_n. Let ϕ_n be a real-valued, continuously differentiable function on $\mathbf{R}^2$ $(n = 1, \cdots, N)$ (so that ϕ_n and $\partial_j \phi_n$ have no singularity at $\mathbf{r} = \mathbf{a}_n, n = 1, \cdots, N$). Then, for each $\mathbf{r} \in M$, $e^{\pm i S_n \phi_n(\mathbf{r})}$ are in $U(p)$ and the matrix

$$K_n(\mathbf{r}) := e^{-iS_n\phi_n(\mathbf{r})} T_n e^{iS_n\phi_n(\mathbf{r})}$$

is Hermitian. Hence

$$A_1(\mathbf{r}) := -i \sum_{n=1}^N \left\{ \frac{(y - a_{n2})}{|\mathbf{r} - \mathbf{a}_n|^2} K_n(\mathbf{r}) - S_n \partial_1 \phi_n(\mathbf{r}) \right\}$$

and

$$A_2(\mathbf{r}) := i \sum_{n=1}^N \left\{ \frac{(x - a_{n1})}{|\mathbf{r} - \mathbf{a}_n|^2} K_n(\mathbf{r}) + S_n \partial_2 \phi_n(\mathbf{r}) \right\}$$

are in $M_p^{\mathrm{as}}(\mathbf{C})$. It is easy to see that the 1-form $A = A_1 dx + A_2 dy$ is flat on M. Note that, for $p \geq 2$, this example is a non-Abelian generalization of examples in Ref.1 and Ref.3.

We want to compute the Wilson loop $W_A(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n))$ in the present case. We first note that we can write A_j in the form

$$A_1(\mathbf{r}) = -\frac{i(y - a_{n2})}{|\mathbf{r} - \mathbf{a}_n|^2} K_n(\mathbf{a}_n) + F_n^{(1)}(\mathbf{r}),$$

$$A_2(\mathbf{r}) = \frac{i(x - a_{n1})}{|\mathbf{r} - \mathbf{a}_n|^2} K_n(\mathbf{a}_n) + F_n^{(2)}(\mathbf{r}),$$

where $F_n^{(j)}$ $(j = 1, 2)$ is a function continuous in the domain

$$D_n = \{\mathbf{r} \in \mathbf{R}^2 |\ 0 < |\mathbf{r} - \mathbf{a}_n| < \delta\}$$

$(\delta < \delta_0)$ with

$$\sup_{\mathbf{r} \in D_n} ||F_n^{(j)}(\mathbf{r})|| < \infty, \quad j = 1, 2.$$

Lemma 6.1: *Let $B = B_1 dx + B_2 dy$ be an $M_p(\mathbf{C})$-valued, continuously differentiable 1-form on M. Suppose that there exist a constant matrix $S \in M_p(\mathbf{C})$ and $M_p(\mathbf{C})$-valued contiuous functions G_1, G_2 on D_n such that*

$$C_j := \sup_{\mathbf{r} \in D_n} ||G_j(\mathbf{r})|| < \infty, \quad j = 1, 2,$$

and

$$B_1(\mathbf{r}) = -\frac{i(y - a_{n2})}{|\mathbf{r} - \mathbf{a}_n|^2} S + G_1(\mathbf{r}),$$
$$B_2(\mathbf{r}) = \frac{i(x - a_{n1})}{|\mathbf{r} - \mathbf{a}_n|^2} S + G_2(\mathbf{r}),$$

for all $\mathbf{r} \in D_n$. Then

$$\lim_{\varepsilon \downarrow 0} W_B(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) = e^{2\pi i S}$$

independently of the choice of the initial point $\mathbf{r}$ with $|\mathbf{r} - \mathbf{a}_n| = \varepsilon$.

Proof: We parametrize the circle $C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)$ by $\gamma(\theta) = \mathbf{a}_n + (\varepsilon \cos(\theta + \theta_0), \varepsilon \sin(\theta + \theta_0)), 0 \leq \theta \leq 2\pi$, where $\mathbf{r} = \mathbf{a}_n + (\varepsilon \cos\theta_0, \varepsilon \sin\theta_0)$ $(\varepsilon < \delta)$. Then we have

$$B_1(\gamma(\theta))\dot{\gamma}_1(\theta) + B_2(\gamma(\theta))\dot{\gamma}_2(\theta) = iS + \varepsilon F(\varepsilon, \theta),$$

where $F(\varepsilon, \theta) = G_2(\gamma(\theta)) \cos(\theta + \theta_0) - G_1(\gamma(\theta)) \sin(\theta + \theta_0)$. Hence we have

$$W_B(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) = \lim_{m \to \infty} e^{2\pi[iS + \varepsilon F(\varepsilon, \theta_m)]/m} e^{2\pi[iS + \varepsilon F(\varepsilon, \theta_{m-1})]/m} \cdots e^{2\pi[iS + \varepsilon F(\varepsilon, \theta_1)]/m},$$

where $\theta_j = 2\pi j/m,\ j = 1, \cdots, m$. By the condition for $G_j, j = 1, 2$, we have

$$||F(\varepsilon, \theta)|| \leq C$$

where $C = C_1 + C_2$. In general, we can show that, for all $M_j, N_j \in M_p(\mathbf{C}), j = 1, \cdots, k$ $(k = 1, 2, \cdots)$,

$$||e^{M_1 + N_1} \cdots e^{M_k + N_k} - e^{M_1} \cdots e^{M_k}||$$
$$\leq (\sum_{j=1}^{k} ||N_j||) e^{2 \sum_{j=1}^{k} ||M_j||} e^{\sum_{j=1}^{k} ||N_j||}.$$

Applying this estimate to $M_j = 2\pi iS/m, N_j = 2\pi\varepsilon F(\varepsilon,\theta_j)/m$, we obtain

$$||e^{2\pi[iS+\varepsilon F(\varepsilon,\theta_m)]/m}e^{2\pi[iS+\varepsilon F(\varepsilon,\theta_{m-1})]/m}\cdots e^{2\pi[iS+\varepsilon F(\varepsilon,\theta_1)]/m} - e^{2\pi iS}|| \le C'\varepsilon$$

with $C' = 2\pi C\exp(4\pi||S|| + 2\pi C\varepsilon)$. Hence

$$||W_B(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) - e^{2\pi iS}|| \le C'\varepsilon,$$

which implies the desired result. ∎

Applying Lemma 6.1 to the present example, we obtain the following.

Lemma 6.2: *For all* $n = 1,\cdots,N$,

$$\lim_{\varepsilon\downarrow 0} W_A(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) = e^{2\pi iK_n(\mathbf{a}_n)}$$

independently of the choice of the initial point $\mathbf{r}$ *with* $|\mathbf{r}-\mathbf{a}_n| = \varepsilon$.

Lemma 6.3: *Let* $0 < \varepsilon_1 < \varepsilon_2 < \delta_0$ *and* $\mathbf{r}_j \in \mathbf{R}^2, j = 1,2$, *be such that* $|\mathbf{r}_j - \mathbf{a}_n| = \varepsilon_j$ *and* $\mathbf{r}_1 - \mathbf{a}_n = \alpha(\mathbf{r}_2 - \mathbf{r}_1)$ *with a constant* $\alpha > 0$. *Then*

$$W_A(L_{\mathbf{r}_2;\mathbf{a}_n}) := \lim_{\varepsilon_1\downarrow 0} W_A(L_{\mathbf{r}_2;\mathbf{r}_1})$$

exists.

Proof: The straight line $L_{\mathbf{r}_2;\mathbf{a}_n}$ is parametrized by

$$\ell(\tau) = (1-\tau)\mathbf{r}_2 + \tau\mathbf{a}_n, \quad 0 \le \tau \le 1.$$

There exists a number $\tau_1 \in (0,1)$ such that $\mathbf{r}_1 = \ell(\tau_1)$. Then we have for $\tau \in [0,1)$

$$A_1(\ell(\tau))\dot{\ell}_1(\tau) + A_2(\ell(\tau))\dot{\ell}_2(\tau) = f_n(\tau)$$

where $f_n(\tau) = (F_n^{(1)}(\ell(\tau)), F_n^{(2)}(\ell(\tau)))\cdot(\mathbf{a}_n - \mathbf{r}_2)$. It is easy to see that

$$C_n := \lim_{\tau\uparrow 1} f_n(\tau)$$

exists. Hence f_n can be extended to a continuous function on $[0,1]$ with $f_n(1) = C_n$. We have

$$W_A(L_{\mathbf{r}_2;\mathbf{r}_1}) = \prod_0^{\tau_1} e^{f_n(\tau)d\tau}.$$

On the other hand, $\prod_0^t e^{f_n(\tau)d\tau}$ is continuous in $t \in [0,1]$. Thus the desired result follows. ∎

Lemma 6.4: *Let $0 < \delta < \delta_0$ be fixed. Then, for all $\varepsilon \in (0,\delta), n = 1, \cdots, N$, and all $\mathbf{r} \in M$ with $|\mathbf{r} - \mathbf{a}_n| = \varepsilon$,*

$$W_A(C_\varepsilon^{\mathbf{r}}(\mathbf{a}_n)) = e^{2\pi i K_n(\mathbf{a}_n)}.$$

Proof: By Lemmas 6.2 and 6.3, we can take the limit $\varepsilon_1 \downarrow 0$ of the RHS of (3.11) to obtain

$$W_A(C_{\varepsilon_2}^{\mathbf{r}_2}(\mathbf{a}_n)) = W_A(L_{\mathbf{r}_2;\mathbf{a}_n})^{-1} e^{2\pi i K_n(\mathbf{a}_n)} W_A(L_{\mathbf{r}_2;\mathbf{a}_n}).$$

Then, taking $\varepsilon_2 \downarrow 0$, we have

$$e^{2\pi i K_n(\mathbf{a}_n)} = W_A(L_{\mathbf{r}_2;\mathbf{a}_n})^{-1} e^{2\pi i K_n(\mathbf{a}_n)} W_A(L_{\mathbf{r}_2;\mathbf{a}_n}),$$

which implies that $e^{2\pi i K_n(\mathbf{a}_n)}$ commutes with $W_A(L_{\mathbf{r}_2;\mathbf{a}_n})$. Thus the desired result follows. ∎

By Lemma 6.4 and Theorem 5.1, we obtain the following theorem.

Theorem 6.5: *In the present example, the representation $\{\bar{P}_j, q_j\}_{j=1}^2$ of CCR is a Schrödinger 2-system if and only if, for all $n = 1, \cdots, N$, all the eigenvalues of T_n are integers.*

Proof: We need only consider the condition that $e^{2\pi i K_n(\mathbf{a}_n)} = I$ for all $n = 1, \cdots, N$, which is equivalent to the condition that $e^{2\pi i T_n} = I$ for all $n = 1, \cdots, N$ (note that $K_n(\mathbf{a}_n)$ is unitarily equivalent to T_n). Since T_n is Hermitian, $e^{2\pi i T_n} = I$ if and only if all the eigenvalues of T_n are integers. ∎

Theorem 6.5 implies the following: *Let*

$$\mathfrak{A} = \{A = A_1 dx + A_2 dy | \textit{ at least one } T_n \textit{ has a non-integer eigenvalue}\}.$$

Then, for each $A \in \mathfrak{A}$, $\{\bar{P}_j, q_j\}_{j=1}^2$ is a non-Schrödinger representation of the CCR with two degrees of freedom. Thus we obtain a class of non-Schrödinger representations of CCR associated with $M_p^{\mathrm{as}}(\mathbf{C})$-valued, flat 1-forms on M.

VII. CONCLUDING REMARKS

We have considered a represenation of the CCR with two degrees of freedom appearing in gauge theory on the non-simply-connected domain M. We have seen that it is possible for the represenation to be non-Schrödinger, which is due to the non-simply-connectedness of M, and a necessary and sufficient condition for that is characterized in terms of the

Wilson loops of the rectangles not intersecting $\mathbf{a}_n, n = 1, \cdots, N$. Moreover we have discussed in some detail a non-trivial example of non-Abelian gauge theory, which can give a non-Schrödinger representation of the CCR.

The following subjects may be interesting for further investigations in connection with the present work:

(S.1) Study of the Dirac-Weyl operator in the non-Abelian case (cf. Ref.5 for the Abelian case). It is expected that properties of this operator may depend on the dimension p of the representation of the gauge group.

(S.2) A complete characterization of the set of flat 1-forms on M; in other words, to find all the solutions (up to gauge transformations) to the distribution equation

$$\partial_1 A_2(\mathbf{r}) - \partial_2 A_1(\mathbf{r}) + [A_1(\mathbf{r}), A_2(\mathbf{r})] = \sum_{n=1}^{N} \sum_{\alpha,\beta \geq 0}^{\text{finite}} T_{\alpha\beta}^{(n)} \partial^\alpha \partial^\beta \delta(\mathbf{r} - \mathbf{a}_n)$$

on $\mathbf{R}^2$, where $T_{\alpha\beta}^{(n)} \in M_p^{\text{as}}(\mathbf{C})$ and $\delta(\mathbf{r})$ is the two-dimensional Dirac's delta distribution. The example in Section VI is a solution to an equation of this type.

(S.3) Extension of the theory given in this paper to a gauge theory on a more general non-simply-connected manifold (including the case of higher dimensions).

(S.4) Analysis from the view-point of Ref.6.

ACKNOWLEDGEMENT

This work is supported by the Grant-In-Aid 06640188 for science research from the Ministry of Education, Japan.

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