# GINZBURG LANDAU EQUATION AND STABLE STEADY STATE SOLUTIONS IN A NON-TRIVIAL DOMAIN

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# GINZBURG LANDAU EQUATION AND STABLE STEADY STATE SOLUTIONS IN A NON-TRIVIAL DOMAIN

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ABSTRACT. The Ginzburg Landau equation with a large parameter is studied in a bounded domain with the Neumann B.C. It is shown that if $n = 2$ or $n = 3$ and the domain is not simply connected, many kinds of non-constant stable equilibrium solutions exist.

## §1. Introduction

In this paper we study the existence of non-constant stable equilibrium solutions of the following Ginzburg Landau equation,

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial \Phi}{\partial t} = \Delta \Phi + \lambda(1 - |\Phi|^2)\Phi \quad \text{in} \quad (0,\infty) \times \Omega, \\ \dfrac{\partial \Phi}{\partial \nu} = 0 \quad \text{on} \quad (0,\infty) \times \partial\Omega \quad (\Phi : \mathbb{C} - \text{valued}), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^3$ boundary and $\lambda > 0$ is a parameter.

Ginzburg and Landau (1950) introduced a model of the superconductivity phenomena in the low temperature. Afterwards the important features of this phenomenon were analyzed through the GL equation. Moreover the successful idea of GL model has been used in many other fields of physics. The original GL equation is the system of equations of the wave function $\Phi$ of the electrons in a material and the vector potential $A$ of the magnetic field made by the current of the electrons. The equation (1.1) is one of the simplified models and it is obtained by putting $A \equiv 0$ in the original equation and it is sometimes used in the case that the magnetic effect is expected to be very small and negligible inside the superconductor $\Omega$. We are concerned with the occurrence of a permanent current of electrons without outer (magnetic) driving force. Mathematically speaking, it is the problem of existence of non-constant stable equilibrium solution of (1.1). In [15] it was proved that if $\Omega$ is convex, there is no non-constant stable solution to (1.1) for any $\lambda > 0$. This shows that the domain $\Omega$ should be more or less complicated so that (1.1) has non-constant stable steady state solutions. In [15], [16] it was proved that non-constant stable equilibria exist in a ring shaped domain (solid torus) for large $\lambda > 0$. In this paper we deal with more general domains than that in [15], [16]. Precisely speaking, we will obtain the same result for the domains, which are not simply connected in the case $n = 2$ or $n = 3$ (cf. Fig. 1). From this result we see that the situation of solutions of the GL equation is very different from that of the (real valued) single reaction diffusion equation and the competition diffusion system (2 species) with Neumann B.C., because these equations do not have non-constant stable equilibria in an annulus (see also [7], [17], [18], [19], [20]).

Let us fix the meaning of the stability of an equilibrium solution. (1.1) defines a dynamical system in the function space $X = C^0(\overline{\Omega}; \mathbb{C})$. We consider the stability in the sense of Lyapunov, which is defined as follows.

**Definition 1.1.** An equilibrium solution $\Psi$ of (1.1) is *stable* iff for any neighborhood $W$ ($\ni \Psi$) in $X$ there exists a neighborhood of $W'$ ($\ni \Psi$) such that any solution $\Phi(t,\cdot)$ of (1.1) with $\Phi(0,\cdot) \in W'$ satisfies $\Phi(t,\cdot) \in W$ ($\forall t \geqq 0$).

For other important problems (vortices, motion of votices, magnetic quantization) of Ginzburg Landau equation with or without magnetic effect, see [2], [3], [4], [8], [9], [14], [25] and the references therein.

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## §2. Main results

In this section we present the main results. Let $\Omega \subset \mathbb{R}^n$ ($n \geqq 2$) be a bounded domain with $C^3$ boundary and consider the following equation,

$$(2.1) \qquad \begin{cases} \Delta\Phi + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial \Phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \quad (\Phi : \mathbb{C}-\text{valued}), \end{cases}$$

where $\nu$ is the outward normal vector on $\partial\Omega$. We deal with the domains with the following condition.

(A) There exists a continuous map $\theta_0 : \overline{\Omega} \longrightarrow S^1$ which is not homotopy equivalent to a constant value map.

Now we present the main theorem of this paper.

**Theorem 2.1.** Assume the condition (A). Then there exists a non-constant stable equilibrium solution $\Phi_\lambda$ to (2.1) for large $\lambda > 0$. Moreover $\Phi_\lambda(x) \neq 0$ in $\Omega$ and the map

$$(2.2) \qquad \overline{\Omega} \ni x \longrightarrow \Phi_\lambda(x)/|\Phi_\lambda(x)| \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

is homotopic to $\theta_0$ for large $\lambda > 0$.

In the case of low dimensions, the condition (A) reduces to a more comprehensive property of $\Omega$. Precisely speaking, (A) holds if $n = 2$ or 3 and $\Omega$ is not simply connected (cf. Proposition A.1 in Appendix). Hence the next result follows immediately from the above theorem.

**Corollary 2.2.** Assume that $n = 2$ or $n = 3$. If $\Omega$ is not simply connected, the same result as Theorem 2.1 holds.

**Remark 2.3.** In the case that $\Omega$ is a solid torus in $\mathbb{R}^3$, the same result was obtained in [16].

The proofs of the above results will be given in the remaining sections of this paper.

## §3. Preliminaries and $S^1$–valued (harmonic) functions

In this section we prepare a formulation which we will use in the proof of the construction of non-constant solutions to (2.1). Let $\widehat{\Omega}$ be the universal covering space of $\Omega$ which is endowed with the canonical metric such that the projection $\iota_1 : \widehat{\Omega} \longrightarrow \Omega$ is locally isometric. It is well known that $\mathbb{R}$ is the universal cover of $S^1$. Let $\iota_2 : \mathbb{R} \longrightarrow S^1$ be the covering map. Remark also that the fundamental group $\pi_1(\Omega)$ acts as an isometric diffeomorphism on $\widehat{\Omega}$ (the covering transformation group). We denote its action as follows

$$\widehat{\Omega} \times \pi_1(\Omega) : (z, \gamma) \longmapsto z \cdot \gamma \in \widehat{\Omega}.$$

We denote the generators of $\pi_1(\Omega)$ by $\beta_1$, ..., $\beta_m$ with the relations

$$\begin{gathered}\beta_{m(1,i)}{}^{p(1,i)} \cdot \beta_{m(2,i)}{}^{p(2,i)} \cdots \beta_{m(n(i),i)}{}^{p(n(i),i)} = e \\ (1 \leqq m(j,i) \leqq m,\ p(j,i) = 1 \text{ or } -1\ , 1 \leqq j \leqq n(i),\ n(i) \in \mathbb{N},\ 1 \leqq i \leqq k)\end{gathered}$$

where $e \in \pi_1(\Omega)$ is the unit. For any continuous map $\theta : \Omega \to S^1$, there exists a continuous map $F : \widehat{\Omega} \to \widehat{S^1} = \mathbb{R}$ such that the following diagram is commutative (cf. Proposition 5.3 in [13]).

$$\begin{array}{ccc} \widehat{\Omega} & \xrightarrow{F} & \widehat{S^1} = \mathbb{R} \\ {\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\ \Omega & \xrightarrow{\theta} & S^1 \end{array}$$

$$\theta(\iota_1(z)) = \iota_2(F(z)) \quad \text{for} \quad \forall z \in \widehat{\Omega}. \tag{3.1}$$

$F$ is unique up to an additive constant (integer $\times\ 2\pi$). Such $F$ is called a lift of $\theta$. On the other hand, what is the totality of $\mathbb{R}$–valued continuous functions in $\widehat{\Omega}$, which are obtained by lifting all the continuous maps on $\Omega$ into $S^1$ ? A standard argument in the theory of the covering spaces gives the following result.

**Lemma 3.1.** For a $\mathbb{R}-$valued continuous function $F$ in $\widehat{\Omega}$, there exists a continuous map $\theta$ on $\Omega$ into $S^1$ satifying (3.1) if and only if there exists a unique set of integers $(\ell_1, ..., \ell_m) \in \mathbb{Z}^m$ with

$$p(1,i)\ell_{m(1,i)} + p(2,i)\ell_{m(2,i)} \cdots + p(n(i),i)\ell_{m(n(i),i)} = 0 \qquad (1 \leqq i \leqq k)$$

such that

$$F(z \cdot \beta_i) = F(z) + 2\pi\ell_i \quad (\forall z \in \widehat{\Omega}, \quad 1 \leqq i \leqq m). \tag{3.2}$$

Here $(\ell_1, ..., \ell_m) \in \mathbb{Z}^m$ depends only on the homotopy class of $\theta$.

**Remark 3.2.** $F$ satisfies the condition (3.2) for $(\ell_1, ..., \ell_m) = (0, ..., 0)$ if and only if there exists a $\mathbb{R}-$valued continuous function $\rho$ in $\Omega$ such that $\rho(\iota_1(z)) = F(z)$ for $z \in \widehat{\Omega}$. This correspondence is one-to-one. Hence, $F$ can be identified with a function on $\Omega$. By the same consideration, for a $C^1$ class $\mathbb{R}-$valued function $F$ on $\widehat{\Omega}$ which satisfies (3.2) for $(\ell_1, ..., \ell_m)$, $\nabla_z F$ can be identified with a $\mathbb{R}^n-$valued function on $\Omega$ because $\nabla_z F(z \cdot \beta_j) = \nabla_z F(z)$ for $z \in \widehat{\Omega}$ and $1 \leqq j \leqq m$.

$$\begin{array}{ccc} \widehat{\Omega} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle \iota_1}\big\downarrow & & \\ \Omega & & \end{array}$$

Using the above formulation, we will prove that there exists a $S^1-$valued harmonic function in $\Omega$ (with the Neumann B. C.) in each homotopy class of continuous maps on $\overline{\Omega}$ into $S^1$. This fact is known in more general framework in the theory of harmonic maps on manifolds (cf. [12]). However we give the proof below because the following argument is important in the construction of the solutions of (2.1) in §4. The equation is written as follows,

$$\operatorname{div}(\nabla\theta) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\theta}{\partial\nu} \equiv \langle\nabla\theta \cdot \nu\rangle = 0 \quad \text{on} \quad \partial\Omega. \tag{3.3}$$

Here $\nabla\theta$ is well-defined as a $\mathbb{R}^n-$valued function in $\Omega$, while $\theta$ itself is a $S^1-$valued function. We suppose the solutions of (3.3) to be regular up to the boundary because of the elliptic regularity theory and hence we naturally regard solution $\theta$ as a function on $\overline{\Omega}$.

**Lemma 3.3.** There exists a solution to (3.3) in each homotopy class of continuous mappings on $\overline{\Omega}$ into $S^1$. Moreover, the solution is unique up to uniform rotation of values in $S^1$.

(Proof of Lemma 3.3) Given any continuous map $\theta_0$ on $\overline{\Omega}$ into $S^1$, we will make a solution of (3.3) with the same homotopy as $\theta_0$. We can assume, without loss of generality, that $\theta_0$ is $C^3$ up to $\overline{\Omega}$, because we can mollify $\theta_0$ (if necessary) in each local patch of $\overline{\Omega}$ without changing the homotopy. We "lift" the equation (3.3) to one on the universal covering space $\widehat{\Omega}$ so that we can argue about solutions which are $\mathbb{R}$−valued functions. Using Lemma 3.1, $\widehat{\theta}_0$ satisfies (3.2) for a $(\ell_1, ..., \ell_m) \in \mathbb{Z}^m$. The equation for the lift $\widehat{\theta} : \widehat{\Omega} \longrightarrow \mathbb{R}$, is

$$(3.4) \qquad \begin{cases} \mathrm{div}_z(\nabla_z \widehat{\theta}) = 0 \quad \text{in} \quad \widehat{\Omega}, \quad \dfrac{\partial \widehat{\theta}}{\partial \nu_z} = 0 \quad \text{on} \quad \partial\widehat{\Omega}, \\ \text{with} \quad \xi(z \cdot \beta_i) = \xi(z) + 2\pi\ell_i \quad \text{for} \quad \forall z \in \widehat{\Omega} \quad \text{and} \quad 1 \leqq i \leqq m. \end{cases}$$

Thus the problem turns to seek for $\widehat{\theta}$ satisfying (3.4) and the condition (3.2) for the same $(\ell_1, ..., \ell_m)$ as $\widehat{\theta}_0$. By transforming the unknown variable $\widehat{\theta}$ to $\xi$ by $\xi = \widehat{\theta} - \widehat{\theta}_0$, we get the equation

$$(3.5) \qquad \begin{cases} \mathrm{div}_z(\nabla_z \xi) = -\mathrm{div}(\nabla \widehat{\theta}_0) \quad \text{in} \quad \widehat{\Omega}, \quad \dfrac{\partial \xi}{\partial \nu_z} = -\dfrac{\partial \widehat{\theta}_0}{\partial \nu_z} \quad \text{on} \quad \partial\widehat{\Omega}, \\ \text{with} \quad \xi(z \cdot \beta_i) = \xi(z) \quad \text{for} \quad \forall z \in \widehat{\Omega} \quad \text{and} \quad 1 \leqq i \leqq m. \end{cases}$$

From Remark 3.2, $\xi$ in (3.5) can be identified with a function in $\mathbb{R}$−valued function in $\Omega$ (also denoted by $\xi$) and hence the equation (3.5) reduces to the equation on $\Omega$,

$$(3.6) \qquad \mathrm{div}(\nabla \xi) = -\mathrm{div}(\nabla \theta_0) \quad \text{in} \quad \Omega, \quad \frac{\partial \xi}{\partial \nu} = -\frac{\partial \theta_0}{\partial \nu} \quad \text{on} \quad \partial\Omega.$$

It is well-known that (3.6) has a unique equation up to an arbitrary additive real constant and thus we obtain a required solution with the same homotopy class as $\theta_0$ to the original equation (3.3). □

## §4. Construction of equilibrium solutions

Before the construction of solutions, we prepare an auxiliary inequality (cf. (4.2) ) due to S. Campanato [6], which gives an estimate of norm of the resolvent of the 2nd order elliptic operator in the Hölder space. We rely on this inequality in the proof of the existence of the solution and its asymptotic behavior. We state this inequality in the simplest version for our later use. Consider

$$(4.1) \qquad -\Delta U + \lambda U = f \quad \text{in} \quad \Omega, \quad \frac{\partial U}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where $\lambda > 0$ is a positive parameter. Let $\alpha \in (0, 1)$ be an arbitrarily fixed constant. It is well known that for any $f \in C^\alpha(\overline{\Omega})$ and $\lambda > 0$, there exists a unique solution $U_\lambda \in C^{2+\alpha}(\overline{\Omega})$. Moreover the following estimate holds for this solution $U_\lambda$.

**Proposition 4.1 (Campanato [6]).** There exists a constant $K > 0$ (independent of $\lambda > 0$) such that

$$(4.2) \qquad \|U_\lambda\|_{C^\alpha(\overline{\Omega})} \leqq \frac{K}{\lambda} \|f\|_{C^\alpha(\overline{\Omega})} \quad (\lambda > 0).$$

We will construct equilibrium solutions to the Ginzburg Landau equation in the following form

$$\Phi(x) = w(x)e^{i\theta(x)} \tag{4.3}$$

where

$$w : \Omega \longrightarrow (0,\infty), \quad \theta : \Omega \longrightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

From the equation (2.1) we easily derive the following equations for $w$ and $\theta$.

$$\begin{cases} \Delta w + \left(\lambda(1-w^2) - |\nabla\theta|^2\right) w = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.4}$$

$$\operatorname{div}(w^2\nabla\theta) = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial\theta}{\partial\nu} \equiv \langle\nabla\theta\cdot\nu\rangle = 0 \quad \text{on} \quad \partial\Omega. \tag{4.5}$$

We remark again that $\nabla\theta$ is well-defined as a $\mathbb{R}^n$−valued function in $\Omega$ for a $C^1$ map $\Omega \longrightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The purpose of this section is to construct a nontrivial solution $(w,\theta)$ to (4.4)-(4.5). We will prove the following result.

**Proposition 4.2.** There exists a $\lambda_* > 0$ such that the system of equations (4.4) and (4.5) has a pair of solutions $(w_\lambda, \theta_\lambda)$ such that $\theta_\lambda$ is homotopic to $\theta_0$ (given in the assumption in Theorem 2.1) with

$$1 - \frac{c}{\lambda} \leqq w_\lambda \leqq 1 \qquad \text{in} \quad \Omega, \tag{4.6}$$

$$\limsup_{\lambda\to\infty} \|\nabla\theta_\lambda\|_{L^\infty(\Omega)} < \infty, \quad \lim_{\lambda\to\infty} \|\Delta\theta_\lambda\|_{L^\infty(\Omega)} = 0, \tag{4.7}$$

$$\lim_{\lambda\to\infty} \|\lambda\,(1-w_\lambda^2) - |\nabla\theta_\lambda|^2\|_{L^\infty(\Omega)} = 0. \tag{4.8}$$

The sketch of the proof is as follows. A solution is obtained by seeking for the fixed point of the map $\Psi : \theta \longmapsto w \longmapsto \widetilde{\theta}$ where $\theta$ is homotopic to $\theta_0$, $w$ is a positive solution of (4.4) for this $\theta$ and $\widetilde{\theta}$ is a solution of (4.5) (for this $w$) which is homotopic to $\theta_0$. Then using several kinds of estimates, we set up a situation where the Schauder fixed point theorem works.

(Proof of Propostion 4.2)

(Step 1:) Without loss of generality, we can assume that $\theta_0 : \overline{\Omega} \longrightarrow S^1$ is $C^3$. We fix arbitrary points $p \in \Omega$ and $q \in S^1$. From Proposition 3.1, there exists a unique harmonic map $\theta_* : \Omega \longrightarrow S^1$ with Neumann B.C. such that $\theta_*(p) = q$. Fix arbitrary points $\widehat{p} \in \widehat{\Omega}$ and $\widehat{q} \in \widehat{S^1} = \mathbb{R}$ such that $\iota_1(\widehat{p}) = p$ and $\iota_2(\widehat{q}) = q$. Remark that any continuous map $\theta : \Omega \longrightarrow S^1$ with $\theta(p) = q$ is uniquely lifted to a map $\widehat{\theta} : \widehat{\Omega} \longrightarrow \mathbb{R}$ such that $\widehat{\theta}(\widehat{p}) = \widehat{q}$ (cf. [13]). We introduce a certain class of continuous maps, in which we will find a solution of (4.5)

$$E = \{\theta \in C^{1+\alpha}(\overline{\Omega}; S^1) \mid \theta(p) = q,\ \theta \,\text{is homotopic to}\ \theta_*,\ d(\theta,\theta_*) \leqq 1\},$$

where

$$d(\theta_1, \theta_2) = \|\widehat{\theta}_1 - \widehat{\theta}_2\|_{C^{1+\alpha}(\overline{\Omega})},$$

for $\theta_1, \theta_2 \in C^{1+\alpha}(\overline{\Omega}; S^1)$ such that $\theta_i$ is homotopic to $\theta_*$ and $\theta_i(p) = q$ $(i = 1, 2)$. Remark that $\widehat{\theta}_1 - \widehat{\theta}_2$ is well-defined as a $\mathbb{R}$–valued function in $\Omega$ (see Remark 3.2). Under this metric $d$, $E$ is a complete metric space and it is isometrically diffeomorphic to the following set

$$\{\rho \in C^{1+\alpha}(\overline{\Omega}) \mid \rho(p) = 0, \ \|\rho\|_{C^{1+\alpha}(\overline{\Omega})} \leqq 1\}.$$

(Step 2:) Applying the same arguments in [15] (making upper-lower solution pair), we have a $\lambda_0 > 0$ such that (4.4) has a unique positive solution $w = w(\lambda, \theta)$ for any $\lambda \geqq \lambda_0$ and $\theta \in E$ and an estimate

$$1 - c/\lambda \leqq w(\lambda, \theta; x) \leqq 1 \quad (x \in \Omega), \tag{4.9}$$

for a constant $c$ which is independent of $\lambda$ and $\theta \in E$ and this inequality yields

$$\sup_{\lambda \geqq \lambda_0} \sup_{\theta \in E} \sup_{x \in \Omega} |\lambda(1 - w(\lambda, \theta; x)^2) w(\lambda, \theta; x) - |\nabla\theta|^2 w(\lambda, \theta; x)| < +\infty.$$

Applying the Schauder estimate to the equation (4.4), we obtain $\{w(\lambda, \theta)\}_{\lambda \geqq \lambda_0, \theta \in E}$ is bounded in $C^{1+\alpha}(\overline{\Omega})$. To see a more detailed asymptotics of $w(\lambda, \theta)$. Define a function $g$ by the relation $w(\lambda, \theta; x) = 1 - g(\lambda, \theta; x)/\lambda$ and derive the equation satisfied by $g$.

$$\begin{cases} (2 - \dfrac{1}{\lambda}\Delta) g + (w + 2)(w - 1) g - |\nabla\theta|^2 w = 0 & \text{in} \quad \Omega, \\ \partial g / \partial \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.10}$$

Applying Propostion 3.1 to (4.5), we have

$$\begin{aligned} \|g\|_{C^\alpha(\overline{\Omega})} &\leqq c_1 \|(w + 2)(w - 1) g - |\nabla\theta|^2 w\|_{C^\alpha(\overline{\Omega})} \\ &\leqq c_1 (\epsilon \|g\|_{C^\alpha(\overline{\Omega})} + c_2 \|g\|_{C^0(\overline{\Omega})}) + c_3, \end{aligned}$$

for $\lambda \geqq \lambda_1(\epsilon)$. $\epsilon > 0$ can be taken arbitrarily small. By taking $\epsilon > 0$ such that $0 < c_1 \epsilon \leqq 1/2$, we obtain that there exists $c_4 > 0$ such that

$$\|g(\lambda, \theta)\|_{C^\alpha(\overline{\Omega})} = \lambda \|w(\lambda, \theta) - 1\|_{C^\alpha(\overline{\Omega})} \leqq c_4 \tag{4.11}$$

for $\lambda \geqq \lambda_1$ and $\theta \in E$ and consequently there exists also $c_5 > 0$ such that

$$\|\lambda w(\lambda, \theta)(1 - w(\lambda, \theta)^2) - |\nabla\theta|^2 w(\lambda, \theta)\|_{C^\alpha(\overline{\Omega})} \leqq c_5 \quad (\lambda \geqq \lambda_1, \ \theta \in E).$$

Applying again the Schauder estimate in (4.4), we get that $\|w(\lambda, \theta)\|_{C^{2+\alpha}(\overline{\Omega})}$ is bounded uniformly in $\theta \in E$ and $\lambda \geqq \lambda_1$. From the compactness of the inclusion map $C^{2+\alpha}(\overline{\Omega}) \hookrightarrow C^2(\overline{\Omega})$, we obtain

$$\lim_{\lambda \to \infty} \sup_{\theta \in E} \|w(\lambda, \theta) - 1\|_{C^2(\overline{\Omega})} = 0. \tag{4.12}$$

(Step 3:) Next we consider (4.5) for $w = w(\lambda, \theta)$ obtained above. To seek for a solution $\theta$ of (4.5) which is homotopic to $\theta_0$ and $\theta(p) = q$, we do as in §3. We consider the following lifted equation of (4.5) in $\widehat{\Omega}$, which is satisfied by $\widehat{\theta}$ (lift of $\theta$).

$$\text{(4.13)} \qquad \operatorname{div}_z(w(\iota_1(z))^2 \nabla_z \widehat{\theta}) = 0 \quad \text{in} \quad \widehat{\Omega}, \quad \frac{\partial \widehat{\theta}}{\partial \nu_z} = 0. \quad \text{on} \quad \partial\widehat{\Omega},$$

where the unknown function $\widehat{\theta}$ is a $\mathbb{R}$−valued function in $\widehat{\Omega}$. By changing the unknown function from $\widehat{\theta}$ to $\eta'$ by $\eta' = \widehat{\theta} - \widehat{\theta}_0$, we obtain the equation

$$\operatorname{div}_z(w(\iota_1(z))^2 \nabla_z \widehat{\eta}) = -\operatorname{div}_z(w(\iota_1(z))^2 \nabla_z \widehat{\theta}_0) \quad \text{in} \quad \widehat{\Omega}, \quad \frac{\partial \widehat{\eta}}{\partial \nu_z} = -\frac{\partial \widehat{\theta}_0}{\partial \nu_z} \quad \text{on} \quad \partial\widehat{\Omega}.$$

Now remark that we are seeking for $\theta$ which is homotopic to $\theta_0$ and hence, from Lemma 3.1 and Remark 3.2, $\eta' = \widehat{\theta} - \widehat{\theta}_0$ can be regarded as a $\mathbb{R}$−valued function in $\Omega$. That is, a unique $\eta : \Omega \longrightarrow \mathbb{R}$ corresponds to $\eta'$ in such a way that $\eta'(z) = \eta(\iota_1(z))$ in $\widehat{\Omega}$. The equation for $\eta$ is

$$\text{(4.14)} \qquad \operatorname{div}(w^2 \nabla \eta) = -\operatorname{div}(w^2 \nabla \theta_0) \quad \text{in} \quad \Omega, \quad \frac{\partial \eta}{\partial \nu} = -\frac{\partial \theta_0}{\partial \nu} \quad \text{on} \quad \partial\Omega.$$

It is easy to see that (4.14) has the unique solution $\eta(x)$ such that $\eta(p) = \widehat{q} - \widehat{\theta}_0(\widehat{p})$. Thus we obtain the solution $\widehat{\theta}(z) = \widehat{\theta}_0 + \eta(\iota_1(z))$ of (4.13). Hence $\widetilde{\theta}(x) = \iota_2(\widehat{\theta}(z))$ with $\iota_1(z) = x \in \Omega$ is a solution of (4.5) for $w(x) = w(\lambda, \theta)$ such that $\widetilde{\theta}(p) = q$. Denote this solution $\widetilde{\theta}$ by $\Psi_\lambda(\theta)$.

(Step 4:) We will check the continuity of the map $\Psi_\lambda : E \longrightarrow E$. Fisrt, the continuity of the map

$$E \ni \theta \longrightarrow w(\lambda, \theta) \in C^{2+\alpha}(\overline{\Omega})$$

is verified by estimating the difference $w(\lambda, \theta_1) - w(\lambda, \theta_2)$ through

$$\text{(4.15)} \qquad \begin{cases} (\Delta - 2\lambda - |\nabla\theta_2|^2)(w_2 - w_1) + \lambda(w_2 - w_1)(3 - (w_1^2 + w_1 w_2 + w_2^2)) \\ \qquad\qquad = w_1(|\nabla\theta_2|^2 - |\nabla\theta_1|^2) \quad \text{in} \quad \Omega, \\ (\partial/\partial\nu)(w_2 - w_1) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $w_i(x) = w(\lambda, \theta_i; x)$ for $i = 1, 2$. By the aid of the Schauder estimate and (4.6), we see that $\|w_2 - w_1\|_{C^{2+\alpha}(\overline{\Omega})}$ is small if $d(\theta_1, \theta_2)$ is small for $\theta_1, \theta_2 \in E$. Here we used that the coefficient of the second term is bounded for large $\lambda$ and hence it is absorbed in "$\lambda$" in the first term. Next we consider the difference of $\widetilde{\theta}_1 \equiv \Psi_\lambda(\theta_1)$ and $\widetilde{\theta}_2 \equiv \Psi_\lambda(\theta_2)$. Similarly as in the above argument, the difference $\xi = \widetilde{\theta}_1 - \widetilde{\theta}_2$ can be regarded as a $\mathbb{R}$−valued function in $\Omega$, which satisfies

$$\text{(4.16)} \qquad \begin{cases} \operatorname{div}(w_1^2 \nabla\xi) = \operatorname{div}((w_2^2 - w_1^2)\nabla\theta_2) \quad \text{in} \quad \Omega, \\ \partial\xi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega \quad \text{with} \quad \xi(p) = 0. \end{cases}$$

Applying again the Schauder estimate to (4.16) and obtain that $\|\xi\|_{C^{2+\alpha}(\overline{\Omega})}$ is small if $\|w_2 - w_1\|_{C^{1+\alpha}(\overline{\Omega})}$ is small. Consequently we complete the proof of the continuity of $\Psi_\lambda$ in $E$. In the above argument it is easy to see that $\Psi_\lambda(E)$ is relatively compact in $E$ for large $\lambda > 0$. Applying the Schauder fixed point theorem, we get a fixed point $\theta_\lambda$ in $E$ and complete the proof of the existence of soution $(w_\lambda, \theta_\lambda)$ by putting $w_\lambda = w(\lambda, \theta_\lambda)$. The estimates (4.6), (4.7), (4.8) follow from (4.9), (4.12) and (4.5). □

**§5. Stability of $\Phi_\lambda = w_\lambda e^{i\theta_\lambda}$**

In this section, we will discuss the stability of the equilibrium solution $\Phi_\lambda$ of (1.1), which has been constructed in the previous section. For the later discussion, we express $\Phi_\lambda$ in terms of real functions. Let $u_\lambda$ and $v_\lambda$ be the real and imaginary part of $\Phi_\lambda$, i.e., $\Phi_\lambda(x) = u_\lambda(x) + v_\lambda(x)i$. Clearly $(u_\lambda, v_\lambda)$ is a solution of the following system of equations,

$$(5.1) \qquad \begin{cases} \Delta \begin{pmatrix} u \\ v \end{pmatrix} + \lambda(1 - u^2 - v^2)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \partial\Omega. \end{cases}$$

Let us consider the linearized eigenvalue problem of (5.1) at $\Phi_\lambda = (u_\lambda, v_\lambda)$.

$$(5.2) \qquad \begin{cases} \Delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \lambda M(u_\lambda, v_\lambda)\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } \partial\Omega, \end{cases}$$

where

$$(5.3) \qquad M(u, v) = \begin{pmatrix} 1 - 3u^2 - v^2 & -2uv \\ -2uv & 1 - u^2 - 3v^2 \end{pmatrix}.$$

It is easy to see that (5.2) is a selfadjoint eigenvalue problem with the real eigenvalues of finite multiplicity.

**Definition 5.1.** Let $\{\mu_k(\lambda)\}_{k=1}^{\infty}$ be the set of the eigenvalues of (5.2) which is arranged in increasing order (with counting multiplicity). It is known that these eigenvalues are real values with finite multiplicity and the corresponding eigenfunctions can be chosen as real valued.

For the solution $(u_\lambda, v_\lambda)$, ${}^t(-v_\lambda, u_\lambda)$ is an eigenfunction of (5.2) for an eigenvalue $\mu = 0$. This implies $\{\mu_k(\lambda)\}_{k=1}^{\infty} \ni 0$. Remark also that $\{e^{ic}\Phi_\lambda \mid c \in \mathbb{R}\}$ forms a one dimensional family of solutions of (2.1). Hence, to prove the stability (cf. Definition 1.1) of $\Phi_\lambda$, it suffices to show that 0 is a simple eigenvalue and all other eigenvalues are positive. Such argument to prove the stability of an equilibrium point in a continuum of equilibria (of finite dimension) in a dynamical system, is standard and can be found in several literature (cf. Chapter 5 in [11]). We can prove the following results.

**Proposition 5.2.**

$$\lim_{\lambda\to\infty}\mu_k(\lambda)=\mu_k \quad (k\geqq 1), \tag{5.4}$$

where $\{\mu_k\}_{k=1}^{\infty}$ are the set of the eigenvalues arranged in increasing order (counting multiplicity) of the Laplacian with Neumann boundary condition, i.e.,

$$\begin{cases} \Delta\psi+\mu\psi=0 & \text{in} \quad \Omega, \\ \dfrac{\partial\psi}{\partial\nu}=0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.5}$$

Moreover, let $\{(\phi_{k,\lambda},\psi_{k,\lambda})\}_{k=1}^{\infty}$ be an system of the corresponding eigenfunctions of (5.2) which is orthonormalized in $L^2(\Omega)\times L^2(\Omega)$. Then

$$\begin{aligned} \lim_{\lambda\to\infty}(\|\nabla(\phi_{k,\lambda}\cos\theta_\lambda+\psi_{k,\lambda}\sin\theta_\lambda)\|^2_{L^2(\Omega)} \\ +\lambda\|\phi_{k,\lambda}\cos\theta_\lambda+\psi_{k,\lambda}\sin\theta_\lambda)\|^2_{L^2(\Omega)})=0 \quad (k\geqq 1). \end{aligned} \tag{5.6}$$

This yields the following result which implies the stability of $\Phi_\lambda$ for large $\lambda>0$.

**Corollary 5.3.** There exists $\lambda_*>0$ and $d>0$ ($d$ is independent of $\lambda$ ) such that

$$\mu_1(\lambda)\equiv 0, \quad \mu_2(\lambda)\geqq d \quad \text{for} \quad \lambda\geqq\lambda_*. \tag{5.7}$$

(Proof of Corollary 5.3.) As was stated before, $\{\mu_k(\lambda)\}_{k=1}^{\infty}\ni 0$, $\forall\lambda>0$ and hence from $\mu_1=0$ and $\mu_2>0$, Propostion 5.2 yields that $\mu_1(\lambda)\equiv 0$. The other part follows directly. □

**Proof of Proposition 5.2.**
To see more detailed situation of the eigenvalues of (5.2), transform the unknown variable $(\phi,\psi)$ into $(\widehat{\phi},\widehat{\psi})$ by

$$\begin{pmatrix}\phi(x)\\ \psi(x)\end{pmatrix}=\begin{pmatrix}\cos\theta_\lambda(x) & -\sin\theta_\lambda(x)\\ \sin\theta_\lambda(x) & \cos\theta_\lambda(x)\end{pmatrix}\begin{pmatrix}\widehat{\phi}(x)\\ \widehat{\psi}(x)\end{pmatrix} \tag{5.8}$$

and get a new eigenvalue problem (which is still selfadjoint with the same eigenvalues),

$$\begin{cases} \Delta\begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix}+\begin{pmatrix}-\nabla\theta_\lambda\nabla\widehat{\psi}\\ \nabla\theta_\lambda\nabla\widehat{\phi}\end{pmatrix}+(\lambda(1-w_\lambda^2)-|\nabla\theta_\lambda|^2)\begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix} \\ \quad +\Delta\theta_\lambda\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix}-2\lambda w_\lambda^2\begin{pmatrix}\widehat{\phi}\\ 0\end{pmatrix}+\mu\begin{pmatrix}\widehat{\phi}\\ \widehat{\psi}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{in} \quad \Omega, \\ \begin{pmatrix}\partial\widehat{\phi}/\partial\nu\\ \partial\widehat{\psi}/\partial\nu\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.9}$$

For the simplicity of notation we use $(\phi, \psi)$ in place of $(\widehat{\phi}, \widehat{\psi})$ in the following arguments. To compare the eigenvalue problems (5.5) and (5.9), we use the variational characterization of the eigenvalues of a selfadjoint operator (cf. [23]). Let us define $\mathcal{E}_\lambda(\phi, \psi)$ and $\mathcal{E}_\infty(\psi)$ as follows,

$$
\begin{aligned}
(5.10) \qquad \mathcal{E}_\lambda(\phi, \psi) = \int_\Omega \{ & |\nabla\phi|^2 + |\nabla\psi|^2 + (\nabla\theta_\lambda \cdot \nabla\psi)\phi - (\nabla\theta_\lambda \cdot \nabla\phi)\psi \\
& - (\lambda(1 - w_\lambda^2) - |\nabla\theta_\lambda|^2)(\phi^2 + \psi^2) + 2\lambda w_\lambda^2 \phi^2\} dx,
\end{aligned}
$$

$$
(5.11) \qquad \mathcal{E}_\infty(\psi) = \int_\Omega |\nabla\psi|^2 dx.
$$

Then the eigenvalues $\{\mu_k(\lambda)\}_{k=1}^\infty$ of (5.9) and $\{\mu_k\}_{k=1}^\infty$ of (5.5) and the corresponding systems of the eigenfunctions $\{(\phi_{k,\lambda}, \psi_{k,\lambda})\}_{k=1}^\infty$ and $\{\psi_k\}_{k=1}^\infty$, are characterized inductively by the following series of minimizing problems (5.12) and (5.13), respectively. Put $\phi_{0,\lambda} = \psi_{0,\lambda} = \psi_0 = 0$ in $\Omega$ for a convenience of the definition.

$$
(5.12) \qquad \mu_k(\lambda) = \min\{\mathcal{E}_\lambda(\phi, \psi); (\phi, \psi) \in E_k(\lambda), \|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 = 1\}
$$

where

$$
E_k(\lambda) = \{(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega); \int_\Omega (\phi\phi_{\ell,\lambda} + \psi\psi_{\ell,\lambda}) dx = 0,\ 0 \leqq \ell \leqq k-1\}.
$$

Let $(\phi_{k,\lambda}, \psi_{k,\lambda})$ be a minimizer (which exists) of (5.12).

$$
(5.13) \qquad \mu_k = \min\{\mathcal{E}_\infty(\psi); \psi \in E_k(\infty), \|\psi\|_{L^2} = 1\},
$$

where

$$
E_k(\infty) = \{\psi \in H^1(\Omega); \int_\Omega \psi\psi_\ell dx = 0,\ 0 \leqq \ell \leqq k-1\}.
$$

Let $\psi_k$ be a minimizer (which exists) of (5.13).
We first prove that each eigenvalue $\mu_k(\lambda)$ is bounded from below when $\lambda \to \infty$. That is,

$$
(5.14) \qquad \liminf_{\lambda \to \infty} \mu_k(\lambda) > -\infty \quad \text{for} \quad k \geqq 1.
$$

From Proposition 4.2-(4.7), $\|\nabla\theta_\lambda\|_{L^\infty(\Omega)}$ is bounded in $\lambda$, there exists a $c > 0$ (independent of $\lambda > 0$) such that

$$
\begin{aligned}
& |\nabla\theta_\lambda \cdot \nabla\psi_{k,\lambda}\phi_{k,\lambda} - \nabla\theta_\lambda \cdot \nabla\phi_{k,\lambda}\psi_{k,\lambda}| \\
& \qquad \leqq \frac{1}{2}(|\nabla\phi_{k,\lambda}|^2 + |\nabla\psi_{k,\lambda}|^2) + c(\phi_{k,\lambda}^2 + \psi_{k,\lambda}^2).
\end{aligned}
$$

and consequently,

$$\mu_k(\lambda) = \mathcal{E}_\lambda(\phi_{k,\lambda},\psi_{k,\lambda}) \geqq \frac{1}{2}\int_\Omega \left(|\nabla\phi_{k,\lambda}|^2 + |\nabla\psi_{k,\lambda}|^2\right) dx \\ - c'\int_\Omega \left(\phi_{k,\lambda}^2 + \psi_{k,\lambda}^2\right) dx + 2\lambda\int_\Omega \phi_{k,\lambda}^2 dx \quad (k \geqq 1). \tag{5.15}$$

Hence, using $\|\phi_{k,\lambda}\|^2_{L^2(\Omega)} + \|\psi_{k,\lambda}\|^2_{L^2(\Omega)} = 1$, we obtain (5.14).

We can prove that $\lim_{\lambda\to\infty}\mu_k(\lambda) = \mu_k$ for $k = 1,2,3,...$ inductively. We are going to carry out this procedure only for $k = 1,2$ because even these cases ($k = 1,2$) are enough to deduce Corollary 5.3, which is the main purpose of this section (stability of $\Phi_\lambda$) and also because it is easy to see how the further arguments for $k = 3,4,....$ proceed, from these cases $k = 1,2$. Similar arguments which will appear hereafter, are also found in [16].

**CASE :** $k = 1$

**Claim 1.**

$$\limsup_{\lambda\to\infty}\mu_1(\lambda) \leqq \mu_1. \tag{5.16}$$

(Proof of Claim 1)
From (5.12)

$$\mu_1(\lambda) \leqq \mathcal{E}_\lambda(0,\psi_1) = \int_\Omega \left(|\nabla\psi_1|^2 + (\lambda(1-w_\lambda^2) - |\nabla\theta_\lambda|^2)\psi_1^2\right) dx.$$

Using Proposition 4.2-(4.8), we get (5.16).

From Claim 1 and the inequality (5.15) for $k = 1$, we have,

**Claim 2.**

$$\limsup_{\lambda\to\infty}\int_\Omega (|\nabla\phi_{1,\lambda}|^2 + |\nabla\psi_{1,\lambda}|^2)dx < \infty, \quad \limsup_{\lambda\to\infty}\lambda\int_\Omega \phi_{1,\lambda}^2 dx < \infty. \tag{5.17}$$

The next claim completes the case $k = 1$.

**Claim 3.**

$$\lim_{\lambda\to\infty}\mu_1(\lambda) = \mu_1, \quad \lim_{\lambda\to\infty}\int_\Omega (|\nabla\phi_{1,\lambda}|^2 + \lambda\phi_{1,\lambda}^2)dx = 0. \tag{5.18}$$

(Proof of claim 3)
Take an arbitrary positive sequence $\{\lambda_m\}_{m=1}^\infty$ which tends to $\infty$ as $m\to\infty$. From (5.14) and Claim 1, $|\mu_1(\lambda)|$ is bounded when $\lambda\to\infty$. By the aid of Claim 2, there exist a subsequence $\{\eta_m\}$ of $\{\lambda_m\}$ and $\psi_1' \in H^1(\Omega)$ such that

$$\begin{cases} \lim_{m\to\infty}\mu_1(\eta_m) \equiv \exists\,\mu \quad (\leqq \mu_1), \\ \lim_{m\to\infty}\psi_{1,\eta_m} = \psi_1' \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and strongly in} \quad L^2(\Omega). \end{cases} \tag{5.19}$$

From (5.10),

$$(5.20)\quad \mu_1(\lambda) = \mathcal{E}_\lambda(\phi_{1,\lambda}, \psi_{1,\lambda}) =$$
$$\int_\Omega \{|\nabla\phi_{1,\lambda}|^2 + |\nabla\psi_{1,\lambda}|^2 + (\nabla\theta_\lambda \cdot \nabla\psi_{1,\lambda})\phi_{1,\lambda} - (\nabla\theta_\lambda \cdot \nabla\phi_{1,\lambda})\psi_{1,\lambda}$$
$$- (\lambda(1 - w_\lambda^2) - |\nabla\theta_\lambda|^2)(\phi_{1,\lambda}^2 + \psi_{1,\lambda}^2) + 2\lambda w_\lambda^2 \phi_{1,\lambda}^2\}dx$$

From Propostion 4.2, (5.19) and the lower semicontinuity of the norm under the weak convergence in Hilbert space,

$$(5.21)\quad (\mu_1 \geqq)\ \ \mu = \lim_{m\to\infty} \mu_1(\eta_m) = \lim_{m\to\infty} \mathcal{E}_{\eta_m}(\phi_{1,\eta_m}, \psi_{1,\eta_m})$$
$$= \limsup_{m\to\infty} \int_\Omega \{|\nabla\phi_{1,\eta_m}|^2 + |\nabla\psi_{1,\eta_m}|^2 + 2\eta_m w_{\eta_m}^2 \phi_{1,\eta_m}^2\}dx$$
$$= \liminf_{m\to\infty} \int_\Omega \{|\nabla\phi_{1,\eta_m}|^2 + |\nabla\psi_{1,\eta_m}|^2 + 2\eta_m w_{\eta_m}^2 \phi_{1,\eta_m}^2\}dx$$
$$\geqq \limsup_{m\to\infty} \int_\Omega |\nabla\psi_{1,\eta_m}|^2 dx \geqq \liminf_{m\to\infty} \int_\Omega |\nabla\psi_{1,\eta_m}|^2 dx$$
$$\geqq \int_\Omega |\nabla\psi_1'|^2 dx = \mathcal{E}_\infty(\psi_1').$$

From $\|\psi_1'\|_{L^2(\Omega)} = 1$ and (5.13), $\mathcal{E}_\infty(\psi) \geqq \mu_1$. Therefore, from (5.21),

$$\mathcal{E}_\infty(\psi_1') = \mu = \mu_1 = \lim_{m\to\infty} \mu_1(\eta_m)$$

follows and $\psi_1$ is an eigenfunction of (5.5) corresponding to $\mu_1$. Accordingly, we get

$$\lim_{m\to\infty} \eta_m \int_\Omega \phi_{1,\eta_m}^2 dx = 0, \quad \lim_{m\to\infty} \int_\Omega |\nabla\phi_{1,\eta_m}|^2 dx = 0.$$

The whole sequence $\{\lambda_m\}$ was arbitrary and hence these facts (just proved) concludes Claim 3 and consequently, (5.3) and (5.4) for $k = 1$.

**CASE :** $\quad k = 2$

**Claim 1.**

$$(5.22)\qquad \limsup_{\lambda\to\infty} \mu_2(\lambda) \leqq \mu_2.$$

(Proof of Claim 1)
Take an arbitrary positive sequence $\{\lambda_m\}$ such that $\lim_{m\to\infty} \lambda_m = \infty$. By the argument in the case $k = 1$, there exists a subsequence $\{\eta_m\}$ and an eigenfunction $\psi_1'$ of (5.5) corresponding to $\mu_1$ such that

$$(5.23)\qquad \lim_{m\to\infty} \psi_{1,\eta_m} = \psi_1' \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and strongly in} \quad L^2(\Omega).$$

We can take $\psi \in L.h.[\psi_1, \psi_2] \equiv \{a\psi_1 + b\psi_2 \mid a, b \in \mathbb{R}\}$ such that

$$(5.24) \qquad \|\psi\|_{L^2(\Omega)} = 1, \quad (\psi, \psi_1')_{L^2(\Omega)} = 0.$$

Put

$$\begin{pmatrix} \phi_\lambda' \\ \psi_\lambda' \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix} - (\psi \cdot \psi_{1,\lambda})_{L^2} \begin{pmatrix} \phi_{1,\lambda} \\ \psi_{1,\lambda} \end{pmatrix}.$$

Remark that $\lim_{m\to\infty}(\psi, \psi_{1,\eta_m})_{L^2(\Omega)} = 0$ and $\mathcal{E}_\infty(\psi) \leqq \mu_2$. From (5.12),

$$\mu_2(\eta_m) \leqq \mathcal{E}_{\eta_m}(\phi_{\eta_m}', \psi_{\eta_m}')/(\|\phi_{\eta_m}'\|_{L^2}^2 + \|\psi_{\eta_m}'\|_{L^2}^2).$$

Using (5.23),(5.24) with (5.18), we have,

$$\mathcal{E}_{\eta_m}(\phi_{\eta_m}', \psi_{\eta_m}') \leqq \int_\Omega |\nabla\psi|^2 dx + o(1) \leqq \mu_2 + o(1) \quad \text{as} \quad m \to \infty$$

$$\|\phi_{\eta_m}'\|_{L^2}^2 + \|\psi_{\eta_m}'\|_{L^2}^2 = \|\psi_1'\|_{L^2} = 1 + o(1) \quad \text{as} \quad m \to \infty.$$

Thus $\limsup_{m\to\infty} \mu_2(\eta_m) \leqq \mu_2$ follows. Since the sequence $\{\lambda_m\}$ $(\uparrow \infty)$ was arbitrary, (5.22) is proved.

From (5.15), Claim 1 and Proposition 4.2, we have,

**Claim 2.**

$$(5.25) \qquad \limsup_{\lambda\to\infty} \int_\Omega (|\nabla\phi_{2,\lambda}|^2 + |\nabla\psi_{2,\lambda}|^2) dx < \infty, \quad \limsup_{\lambda\to\infty} \lambda \int_\Omega \phi_{2,\lambda}^2 dx < \infty.$$

The next claim completes the case $k = 2$.

**Claim 3.**

$$(5.26) \qquad \lim_{\lambda\to\infty} \mu_2(\lambda) = \mu_2, \quad \lim_{\lambda\to\infty} \int_\Omega (|\nabla\phi_{2,\lambda}|^2 + \lambda\phi_{2,\lambda}^2) dx = 0.$$

(Proof of Claim 3)
Take an arbitrary positive sequence $\{\lambda_m\}$ such that $\lim_{m\to\infty} \lambda_m = \infty$. From Claim 1, 2 and the case $k = 1$, there exist a subsequence $\{\eta_m\} \subset \{\lambda_m\}$ and an eigenfunction $\psi_1'$ of (5.5) corresponding to $\mu_1$ and $\psi_2' \in H^1(\Omega)$ such that

$$(5.27) \qquad \begin{cases} \lim_{m\to\infty} \mu_2(\eta_m) \equiv \exists\, \mu \quad (\leqq \mu_2), \\ \lim_{m\to\infty} \psi_{1,\eta_m} = \psi_1' \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and strongly in} \quad L^2(\Omega), \\ \lim_{m\to\infty} \phi_{2,\eta_m} = 0 \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and strongly in} \quad L^2(\Omega), \\ \lim_{m\to\infty} \psi_{2,\eta_m} = \psi_2' \quad \text{weakly in} \quad H^1(\Omega) \quad \text{and strongly in} \quad L^2(\Omega). \end{cases}$$

Using Proposition 4.2 and (5.27), we get

$$(5.28)\quad (\mu_2 \geqq)\ \ \mu = \lim_{m\to\infty} \mu_2(\eta_m) = \lim_{m\to\infty} \mathcal{E}_{\eta_m}(\phi_{2,\eta_m}, \psi_{2,\eta_m})$$
$$= \limsup_{m\to\infty} \int_\Omega \{|\nabla\phi_{2,\eta_m}|^2 + |\nabla\psi_{2,\eta_m}|^2 + 2\eta_m w_{\eta_m}^2 \phi_{1,\eta_m}^2\}dx$$
$$= \liminf_{m\to\infty} \int_\Omega \{|\nabla\phi_{2,\eta_m}|^2 + |\nabla\psi_{2,\eta_m}|^2 + 2\eta_m w_{\eta_m}^2 \phi_{1,\eta_m}^2\}dx$$
$$\geqq \limsup_{m\to\infty} \int_\Omega |\nabla\psi_{2,\eta_m}|^2 dxdx \geqq \liminf_{m\to\infty} \int_\Omega |\nabla\psi_{2,\eta_m}|^2 dxdx$$
$$\geqq \int_\Omega |\nabla\psi_2'|^2 dx = \mathcal{E}_\infty(\psi_2').$$

On the other hand, from

$$(\phi_{2,\eta_m}, \phi_{1,\eta_m})_{L^2} + (\psi_{2,\eta_m}, \psi_{1,\eta_m})_{L^2} = 0, \quad \|\phi_{2,\eta_m}\|_{L^2} + \|\psi_{2,\eta_m}\|_{L^2} = 1,$$

we obtain $(\psi_2', \psi_1')_{L^2(\Omega)} = 0$ and $\|\psi_2'\|_{L^2} = 1$ respectively. Hence we have $\mathcal{E}_\infty(\psi_2') \geqq \mu_2$. (5.28) and this inequality prove that

$$\mathcal{E}_\infty(\psi_2') = \mu = \mu_2 = \lim_{m\to\infty} \mu_2(\eta_m)$$

and $\psi_2'$ is an eigenfunction of (5.5) corresonding to $\mu_2$.

$$(5.30)\qquad \lim_{m\to\infty} \int_\Omega (|\nabla\phi_{2,\eta_m}|^2 + \eta_m \phi_{2,\eta_m}^2)dx = 0,$$

follows simultaneously. Since the whole sequence $\{\lambda_m\}_{m=1}^\infty$ was arbitrary, (5.26) is proved. Consequenctly the case $k = 2$ is completed.

The cases for $k \geqq 3$ proceed similarly.

## Appendix.

We will prove the topological result which we used in §2 to deduce Corollary 2.2 from Theorem 2.1. Concerning the following arguments, we are indebted to Professor M. Morimoto in Okayama university. We are very grateful to him. Let $\Omega$ be a domain in §2, that is, $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^3$ boundary. We have the following proposition.

**Proposition A.1 ([21]).** If $n = 2$ and $n = 3$, the condition (A) (in §2) holds if and only if $\Omega$ is not simply connected.

(Proof of Proposition A.1)
We omit the case $n = 2$ because it is easy. We prove the case $n = 3$. Put $X = \overline{\Omega}$. Let $\Omega$ be simply connected. Let $f : X \longrightarrow S^1$ be an arbitrary continous map. Because $X$ is simply connected, there exists a continuous map $\widehat{f} : X \to \mathbb{R}$ such that $f(x) = \iota(\widehat{f}(x))$ $(x \in X)$ where $\iota : \mathbb{R} \to S^1$ is a covering map (cf. Proposition

5.3 in [13]). Define $f_t(x) = \iota(t\widehat{f}(x))$ $(0 \leqq t \leqq 1)$. From this, it is easy to see that $f$ is homotopic to a constant map. Conversely, assume that any continuous map $X \longrightarrow S^1$ is homotopic to a constant map. Then $H^1(X) = 0$ (1-dim cohomology vanishes) (cf. Theorem 7.1 in [13]). The coefficient ring of the homology groups and cohomology groups in this proof is $\mathbb{Z}$. Remark that $X$ is a 3-dimensional orientable compact smooth manifold with $C^3$ boundary $\partial X$. Using the Poincaré duality theorem, we have $H_2(X, \partial X) \cong H^1(X) = 0$. From the universal coefficient theorem, $H^1(X) \cong Hom(H_1(X), \mathbb{Z}) = 0$ and hence $H_1(X)$ is a finite abelian group. On the other hand, $\partial X$ is a finite disjoint union of orientable 2 dimensional compact manifolds and so $H_1(\partial X)$ is tortion free, i.e. $H_1(\partial X) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Hence, from the homology exact sequence,

$$\cdots \longrightarrow H_2(X, \partial X) \longrightarrow H_1(\partial X) \longrightarrow H_1(X) \longrightarrow \cdots$$

we have that $H_1(\partial X) = 0$. This implies that the euler number of each component of $\partial X$ is 2 and hence it is diffeomorphic to $S^2$. By deforming the components of $\partial X$ (the spheres) which are located inside of $X$ to points, we see that $X$ is simply connected. □

**Remark A.2.** For $n \geqq 4$, it is known that there exists $\Omega \subset \mathbb{R}^n$ which does not satisfy the condition (A) while it is not simply connected.

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