ON THE CRITICAL DECAY AND POWER FOR SEMILINEAR WAVE EQUATIONS IN ODD SPACE DIMENSIONS

Hideo Kubo

Series ♯274. December 1994

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 248 H. Kikuchi, Sheaf cohomology theory for measurable spaces, 12 pages. 1994.

♯ 249 A. Inoue, Tauberian theorems for Fourier cosine transforms, 9 pages. 1994.

♯ 250 S. Izumiya, G. T. Kossioris, Singularities for viscosity solutions of Hamilton-Jacobi equations, 23 pages. 1994.

♯ 251 H. Kubo, K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\Box u = |u|^p$ for super critical values p in odd space dimensions, 51 pages. 1994.

♯ 252 T. Mikami, Large Deviations and Central Limit Theorems for Eyraud-Farlie-Gumbel-Morgenstern Processes, 9 pages. 1994.

♯ 253 T. Nishimori, Some remarks in a qualitative theory of similarity pseudogroups, 19 pages. 1994.

♯ 254 T. Suwa, Residues of complex analytic foliations relative to singular invariant subvarieties, 15 pages. 1994.

♯ 255 T. Tsujishita, On Triple Mutual Information, 7 pages. 1994.

♯ 256 T. Tsujishita, Construction of Universal Modal World based on Hyperset Theory, 15 pages. 1994.

♯ 257 A. Arai, Trace Formulas, a Golden-Thompson Inequality and Classical Limit in Boson Fock Space, 35 pages. 1994.

♯ 258 Y-G. Chen, Y. Giga, T. Hitaka and M. Honma, A Stable Difference Scheme for Computing Motion of Level Surfaces by the Mean Curvature, 18 pages. 1994.

♯ 259 K. Iwata, J. Schäfer, Markov property and cokernels of local operators, 7 pages. 1994.

♯ 260 T. Mikami, Copula fields and its applications, 14 pages. 1994.

♯ 261 A. Inoue, An Abel-Tauber theorem for Fourier sine transforms, 6 pages. 1994.

♯ 262 N. Kawazumi, Homology of hyperelliptic mapping class groups for surfaces, 13 pages. 1994.

♯ 263 Y. Giga, M. E. Gurtin, A comparison theorem for crystalline evolution in the plane, 14 pages. 1994.

♯ 264 J. Wierzbicki, On Commutativity of Diagrams of Type II_1 Factors, 26 pages. 1994.

♯ 265 N. Hayashi, T. Ozawa, Schrödinger Equations with nonlinearity of integral type, 12 pages. 1994.

♯ 266 T. Ozawa, On the resonance equations of long and short waves, 8 pages. 1994.

♯ 267 T. Mikami, A sufficient condition for the uniqueness of solutions to a class of integro-differential equations, 9 pages. 1994.

♯ 268 Y. Giga, Evolving curves with boundary conditions, 10 pages. 1994.

♯ 269 A. Arai, Operator-theoretical analysis of representation of a supersymmetry algebra in Hilbert space, 12 pages. 1994.

♯ 270 A. Arai, Gauge theory on a non-simply-connected domain and representations of canonical commutation relations, 18 pages. 1994.

♯ 271 S. Jimbo, Y. Morita and J. Zhai, Ginzburg landau equation and stable steady state solutions in a non-trivial domain, 17 pages. 1994.

♯ 272 S. Izumiya, A. Takiyama, A time-like surface in Minkowski 3-space which contains light-like lines, 7 pages. 1994.

♯ 273 K. Tsutaya, Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations, 11 pages. 1994.

ON THE CRITICAL DECAY AND POWER FOR SEMILINEAR WAVE EQUATIONS IN ODD SPACE DIMENSIONS

HIDEO KUBO

Department of Mathematics Hokkaido University, Sapporo 060, Japan

1. Introduction and Statements of Results. In this paper we consider the initial value problem to semilinear wave equations:

$$\begin{aligned} &u_{tt} - \Delta_x u = F(u) \quad \text{in} \quad \mathbb{R}^n_x \times [0,\infty), \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \quad \text{for} \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where u is a real valued function and ϕ, ψ have appropriate regularity. To begin with, we mention a typical example of the nonlinear term, that is, $F(u) = |u|^p$ with $p > 1$. It is known that if the initial data are large in a certain sense, the solution to the initial value problem (1.1) blows up in finite time. (See e.g. [5]). Therefore it is necessary to exist a global solution that the initial data are sufficiently small. However under this smallness assumption on the initial data, F. John [8] obtained such a remarkable result that if $1 < p < p_0(n)$, any nontrivial solution blows up in finite time and that if $p > p_0(n)$, then a unique global solution exists, when $n = 3$ and the initial data are compactly supported. Here $p_0(n)$ is the positive root of the quadratic equation:

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{1.2}$$

When $2 \le n \le 4$, we know the criticality of the value $p_0(n)$ in the above sense. (See for instance [6], [7], [15] and [20]). Moreover when $n = 2, 3$, J. Schaeffer [14] proved blow-up of solutions if $p = p_0(n)$. Furthermore if $1 < p < p_0(n)$, T.C. Sideris [15] showed the blow-up result in general space dimensions. On the other hand, when $n \ge 5$, only for the following cases we know the global existence results; p is large enough or $F(u)$ is sufficiently smooth with $F(0) = F'(0) = 0$. (See [3], [4], [12] and [13]).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX

We now tern our attention to the decay rate of the initial data, because it plays an important role to determine a global behavior of the solutions as well as the power p. Let the initial data satisfy either

$$\sum_{|\alpha|\le[n/2]+2} |\partial_x^\alpha \phi(x)| + \sum_{|\alpha|\le[n/2]+1} |\partial_x^\alpha \psi(x)| \le \varepsilon(1+|x|)^{-(\kappa+1)} \tag{1.3}$$

or

$$\phi \equiv 0, \quad \psi(x) \ge \varepsilon(1+|x|)^{-(\kappa+1)}, \tag{1.4}$$

where κ and ε are positive parameters. When $n = 3$, F. Asakura [2] proved that if $\kappa > \kappa_0 = 2/(p-1)$ and $p > p_0(n)$, a unique global solution exists by assuming (1.3) with ε sufficiently small. Moreover it is proved that if $0 < \kappa < \kappa_0$ and (1.4) holds, any classical solution blows up in finite time.

This result extended to two space dimensional case. And for the critical case where $\kappa = \kappa_0$, the global existence results obtained when $n = 2, 3$. (See [1], [11] and [17-19]). Moreover when $n \ge 4$, H. Takamura [16] proved the blow-up result if $0 < \kappa < \kappa_0$. Furthermore when $0 < \kappa < \kappa_0$ and $n \ge 2$, we know the following upper bound of the lifespan:

$$T_\varepsilon \le C\varepsilon^{-(p-1)/(2-(p-1)\kappa)}, \quad C = C(n,p,\kappa) > 0. \tag{1.5}$$

(See [1], [16], [17, 19]).

On the other hand, in [9] the author has proved the global existence result and obtained the lower bound of the lifespan, provided the initial data are radially symmetric, the space dimension is odd and either $p > \max(p_0(n), (n-3)/2)$ or the nonlinear term is sufficiently smooth. Therefore when $n = 5$, we can show the existence of a global solution to the problem (1.6) below for arbitrary $p > p_0(5)$.

The aim of this paper is to consider the case where $n \ge 7$ and to remove the lower restriction $p > (n-3)/2$. Then our problem is written as

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) \quad \text{in} \quad \Omega_T = (0,\infty)\times[0,T), \tag{1.6a}$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for} \quad r > 0, \tag{1.6b}$$

where $n = 2m+3$ with m a positive integer and $T > 0$. We assume that $f \in C^2([0,\infty))$,

$g \in C^1([0,\infty))$ and that

$$\sum_{j=0}^{2} |f^{(j)}(r)| \langle r \rangle^{\kappa+j} + \sum_{j=0}^{1} |g^{(j)}(r)| \langle r \rangle^{1+\kappa+j} \le \varepsilon, \tag{1.7}$$

where κ, ε are positive parameters and we have set $\langle r \rangle = \sqrt{1+r^2}$.

As to $F(u)$ we suppose the following hypothesis (H):

$F(\lambda) \in C^1(\mathbb{R}),\ F(0) = F'(0) = 0$ and there are constants $p > 1,\ A > 0$ such that

$$p > p_0(n) \quad \text{and} \quad p < \frac{n+1}{n-3} = \frac{m+2}{m} \tag{H}$$

and

$$|F'(u) - F'(v)| \le Ap|u-v|^{p-1} \quad \text{for} \quad u,\ v \in \mathbb{R}.$$

We now define the lifespan T_ε by the maximal time interval where a solution $u \in C^2(\Omega_T)$ to the problem (1.6) exists uniquely. Then our main result is

Main Theorem. *Suppose that (1.7) and (H) hold. We also assume $m \ge 2$. Then there is a positive number $\varepsilon_0 = \varepsilon_0(n, F, \kappa)$ such that for $0 < \varepsilon \le \varepsilon_0$*

$$T = T_\varepsilon = \infty \qquad \text{if} \quad \kappa \ge \kappa_0 = 2/(p-1), \tag{1.8a}$$

$$T = T_\varepsilon \ge C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} \qquad \text{if} \quad 0 < \kappa < \kappa_0, \tag{1.8b}$$

where $C = C(n, F, \kappa) > 0$.

Remarks. 1) In [9], we have dealt with the case where $m = 1$. Indeed the conclusions are still holds, without assuming the upper bound of p in the condition (H). However when $m \ge 2$, we need the assumption in order to control the singularity of the solution at $r = 0$.

2) The lower bound in (1.8b) is optimal with respect to the order of ε by (1.5).

This paper is organized as follows. First we shall study a solution to a linear wave equation in Section 2. Section 3 is devoted to obtain preliminary estimates. In the final section we shall get basic a priori estimates to prove Main Theorem. Throughout this paper, we shall denote various constants depending only on n, F and κ by C, C_0, C_1 and so on.

At the end of this section, we mention that for the even space dimensional case, similar results obtained recently. The detail will be published elsewhere.

2. Linear Wave Equations. The aim of this section is to obtain decay estimates for a solution to a linear wave equation:

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = 0 \quad \text{in} \quad \Omega = (0,\infty) \times [0,\infty). \tag{2.1}$$

We consider a function defined by

$$u^0(r,t) = \int_{|t-r|}^{t+r} g(\lambda)K(\lambda,r,t)d\lambda + \partial_t \int_{|t-r|}^{t+r} f(\lambda)K(\lambda,r,t)d\lambda, \quad r \neq 0, \tag{2.2}$$

where we have set

$$K(\lambda,r,t) = \frac{(-1)^m}{2m!}(\frac{\lambda}{r})^{2m+1}(\frac{\partial}{\partial\lambda}\frac{1}{2\lambda})^m \phi^m(\lambda,r,t),$$
$$\phi(\lambda,r,t) = (\lambda - t + r)(t + r - \lambda).$$

Here we summarize some known results obtained in [10], Lemmas 2.2 and 2.3, and Section 3.

Lemma 2.1. *We suppose that $(r,t) \in \Omega$ and $|r-t| \leq \lambda \leq r+t$. Then we have*

$$|\partial^\alpha \phi^m(\lambda,r,t)| \leq Cr^{2m-|\alpha|} \quad \textit{for} \quad |\alpha| \leq 2m, \tag{2.3}$$

$$|K(\lambda,r,t)| \leq Cr^{-m-1}\lambda^{m+1}. \tag{2.4a}$$

Moreover if we further assume $0 < t \leq 2r$, we get

$$|\partial^\alpha K(\lambda,r,t)| \leq Cr^{-m-1}\lambda^{m+1-|\alpha|} \quad \textit{for} \quad 1 \leq |\alpha| \leq m+1, \tag{2.4b}$$

where $\partial^\alpha = \partial_\lambda^{\alpha_1}\partial_r^{\alpha_2}\partial_t^{\alpha_3}$ with $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ a multi-index.

Proposition 2.2. *We suppose $f \in C^2([0,\infty))$ and $g \in C^1([0,\infty))$. Then u^0 belongs to $C^2(\bar{\Omega})$ and satisfies (2.1) and (1.6b).*

We now state the main result of this section.

Proposition 2.3. *We suppose (1.7). Then we have for $(r,t) \in \Omega$*

$$|u^0(r,t)| \le C_0 \varepsilon r^{1-m} \langle r \rangle^{-1} \Psi_\kappa(r,t). \tag{2.5a}$$

$$|\partial_{r,t}^\alpha u^0(r,t)| \le C_0 \varepsilon r^{-m} \Psi_\kappa(r,t), \quad |\alpha| = 1, \tag{2.5b}$$

where we have set

$$\Psi_\kappa(r,t) = \begin{cases} \langle t+r \rangle^{-\kappa+m} & \text{if } \ 0 < \kappa < m+1, \\ \langle t+r \rangle^{-1}\big(1 + \log \dfrac{\langle t+r \rangle}{\langle t-r \rangle}\big) & \text{if } \ \kappa = m+1, \\ \langle t+r \rangle^{-1} \langle t-r \rangle^{-\kappa+m+1} & \text{if } \ \kappa > m+1. \end{cases}$$

Before we prove the above proposition, we prepare the following two lemmas.

Lemma 2.4. *Let $(r,t) \in \Omega$ satisfy $t \ge 2r > 0$. Then we have for $|\alpha| \le 1$*

$$|\partial_{r,t}^\alpha u^0(r,t)| \le C r^{-m-|\alpha|} \int_{t-r}^{t+r} \{ \sum_{j=0}^{m-1} \lambda^{m-j-1} |G_k(\lambda)| + |G_m(\lambda)| \} d\lambda, \tag{2.6}$$

where we have set

$$G_j(\lambda) = \lambda^{j+1} g(\lambda) + (\lambda^{j+1} f(\lambda))', \quad 0 \le j \le m-1,$$
$$G_m(\lambda) = -(\lambda^{m+1} g(\lambda))' - (\lambda^{m+1} f(\lambda))''.$$

Proof: Since $K(\lambda, r, t) = r^{-2m-1} \sum_{j=0}^m C_j \lambda^{j+1} \partial_\lambda^j \phi^m(\lambda, r, t)$, one can write $u^0(r,t)$ as

$$u^0(r,t) = r^{-2m-1} \sum_{j=0}^m C_j u_j(r,t), \tag{2.7}$$

where we have set

$$\begin{aligned} u_j(r,t) = & \int_{t-r}^{t+r} g(\lambda) \lambda^{j+1} \partial_\lambda^j \phi^m(\lambda, r, t) d\lambda \\ & + \partial_t \int_{t-r}^{t+r} f(\lambda) \lambda^{j+1} \partial_\lambda^j \phi^m(\lambda, r, t) d\lambda \\ \equiv & \ U_g + \partial_t U_f. \end{aligned}$$

First we assume $|\alpha| = 0$. When $0 \le j \le m-1$, we have

$$\partial_\lambda^j \phi^m(\lambda, r, t) \Big|_{\lambda = t \pm r} = 0,$$

hence we get

$$\partial_t U_f = -\int_{t-r}^{t+r} f(\lambda)\lambda^{j+1}\partial_\lambda^{j+1}\phi^m(\lambda,r,t)d\lambda$$
$$= \int_{t-r}^{t+r} (f(\lambda)\lambda^{j+1})'\partial_\lambda^{j}\phi^m(\lambda,r,t)d\lambda.$$

Therefore we have

$$u_j(r,t) = \int_{t-r}^{t+r} G_j(\lambda)\partial_\lambda^j\phi^m(\lambda,r,t)d\lambda \quad \text{for} \quad 0 \le j \le m-1, \tag{2.8a}$$

where $G_j(\lambda)$ is given in (2.6). When $j = m$, we have

$$u_m(r,t) = -\int_{t-r}^{t+r} (g(\lambda)\lambda^{m+1})'\partial_\lambda^{m-1}\phi^m(\lambda,r,t)d\lambda$$
$$- \partial_t \int_{t-r}^{t+r} (f(\lambda)\lambda^{m+1})'\partial_\lambda^{m-1}\phi^m(\lambda,r,t)d\lambda.$$

As to the second term, proceeding as before, we have

$$u_m(r,t) = \int_{t-r}^{t+r} G_m(\lambda)\partial_\lambda^{m-1}\phi^m(\lambda,r,t)d\lambda, \tag{2.8b}$$

where $G_m(\lambda)$ is given in (2.6). By (2.3) we have

$$|\partial_\lambda^j\phi^m(\lambda,r,t)| \le Cr^{2m-j} \le Cr^{m+1}\lambda^{m-j-1}, \quad 0 \le j \le m-1$$
$$|\partial_\lambda^{m-1}\phi^m(\lambda,r,t)| \le Cr^{m+1}.$$

Substitutuing these estimates into (2.8), we obtain (2.6) with $|\alpha| = 0$.

Next we assume $|\alpha| = 1$. It follows from (2.8) that

$$\begin{aligned} \partial_{r,t}^\alpha u_j(r,t) &= \int_{t-r}^{t+r} G_j(\lambda)\partial_{r,t}^\alpha\partial_\lambda^j\phi^m(\lambda,r,t)d\lambda, \quad 0 \le j \le m-1, \\ \partial_{r,t}^\alpha u_m(r,t) &= \int_{t-r}^{t+r} G_m(\lambda)\partial_{r,t}^\alpha\partial_\lambda^{m-1}\phi^m(\lambda,r,t)d\lambda. \end{aligned} \tag{2.9}$$

By (2.3) we have

$$|\partial_{r,t}^\alpha\partial_\lambda^j\phi^m(\lambda,r,t)| \le Cr^{2m-j-1} \le Cr^m\lambda^{m-j-1}, \quad 0 \le j \le m-1,$$
$$|\partial_{r,t}^\alpha\partial_\lambda^{m-1}\phi^m(\lambda,r,t)| \le Cr^m.$$

Substituting these estimates into (2.9), we obtain (2.6). The proof is complete. □

Lemma 2.5. *Let $(r,t)\in\Omega$ satisfy $0<t\le 2r$. Then we have*

$$\begin{aligned}|u^0(r,t)|\le\ & Cr^{-m-1}\int_{|t-r|}^{t+r}\{\lambda^{m+1}|g(\lambda)|+\lambda^m|f(\lambda)|\}d\lambda\\ &+Cr^{-m-1}|t\pm r|^{m+1}|f(|t\pm r|)|,\end{aligned}\tag{2.10a}$$

$$\begin{aligned}|\partial^\alpha_{r,t}u^0(r,t)|\le\ & Cr^{-m-1}\int_{|t-r|}^{t+r}\{\lambda^m|g(\lambda)|+\lambda^{m-1}|f(\lambda)|\}d\lambda\\ &+Cr^{-m-1}|t\pm r|^{m+1}\{|g(|t\pm r|)|+|f'(|t\pm r|)|\}\\ &+Cr^{-m-1}|t\pm r|^m|f(|t\pm r|)|\quad \textit{for}\quad |\alpha|=1.\end{aligned}\tag{2.10b}$$

Proof: First we prove (2.10a). From (2.2) we have

$$\begin{aligned}u^0(r,t)=&\int_{|t-r|}^{t+r}g(\lambda)K(\lambda,r,t)d\lambda+\int_{|t-r|}^{t+r}f(\lambda)\partial_tK(\lambda,r,t)d\lambda\\ &+f(r+t)K(r+t,r,t)-\partial_t|r-t|\cdot f(|r-t|)K(|r-t|,r,t).\end{aligned}\tag{2.11}$$

By (2.4) we have for $|r-t|\le\lambda\le r+t$

$$|K(\lambda,r,t)|\le Cr^{-m-1}\lambda^{m+1}\quad\text{and}\quad|\partial^\alpha_{r,t}K(\lambda,r,t)|\le Cr^{-m-1}\lambda^m,\quad |\alpha|=1,\tag{2.12}$$

hence (2.10a) follows.

Next we prove (2.10b). From (2.11) we have for $|\alpha|=1$

$$\begin{aligned}\partial^\alpha_{r,t}u^0(r,t)=&\int_{|t-r|}^{t+r}g(\lambda)\partial^\alpha_{r,t}K(\lambda,r,t)d\lambda+\int_{|t-r|}^{t+r}f(\lambda)\partial^\alpha_{r,t}\partial_tK(\lambda,r,t)d\lambda\\ &+g(r+t)K(r+t,r,t)-\partial^\alpha_{r,t}|r-t|\cdot g(|r-t|)K(|r-t|,r,t)\\ &+f(r+t)\partial_tK(r+t,r,t)-\partial^\alpha_{r,t}|r-t|\cdot f(|r-t|)\partial_tK(|r-t|,r,t)\\ &+\partial^\alpha_{r,t}\Big(f(r+t)K(r+t,r,t)\Big)-\partial_t|r-t|\cdot\partial^\alpha_{r,t}\Big(f(|r-t|)K(|r-t|,r,t)\Big).\end{aligned}$$

Employing (2.12), we get (2.10b). This completes the proof. □

End of Proof of Proposition 2.3: First we consider the case $t\ge 2r>0$. By (1.7) we have $|G_j(\lambda)|\le C\varepsilon\langle\lambda\rangle^{j-\kappa}$, $0\le j\le m-1$ and $|G_m(\lambda)|\le C\varepsilon\langle\lambda\rangle^{m-\kappa-1}$, where we have set G_j, $0\le j\le m$ in (2.6). Substituting these estimates into (2.6) we have for $|\alpha|\le 1$

$$\begin{aligned}|\partial^\alpha_{r,t}u^0(r,t)|\le& C\varepsilon r^{-m-|\alpha|}\int_{t-r}^{t+r}\langle\lambda\rangle^{m-\kappa-1}d\lambda\\ \le& C\varepsilon r^{1-m-|\alpha|}\langle t+r\rangle^{m-\kappa-1},\end{aligned}$$

which implies (2.5).

Next we consider the case where $0 < t \le 2r$. By (1.7) and (2.10a) we have

$$|u^0(r,t)| \le C\varepsilon r^{-m-1}\int_{|t-r|}^{t+r}\lambda^m\langle\lambda\rangle^{-\kappa}d\lambda + C\varepsilon r^{-m-1}|t\pm r|^{m+1}\langle t\pm r\rangle^{-\kappa}. \tag{2.13a}$$

When $0 < r \le 1$, it is enough to show $|u^0(r,t)| \le C\varepsilon r^{-m+1}$. Since $\lambda^m\langle\lambda\rangle^{-\kappa} \le 3r\langle\lambda\rangle^{m-1-\kappa}$ for $|t-r| \le \lambda \le t+r$, we get the above inequality.

When $r \ge 1$, we have $r^{-1} \le C\langle t+r\rangle^{-1}$. Therefore we have from (2.13a)

$$|u^0(r,t)| \le C\varepsilon r^{-m}\langle t+r\rangle^{-1}\left(\int_{|t-r|}^{t+r}\langle\lambda\rangle^{m-\kappa}d\lambda + \langle t\pm r\rangle^{m+1-\kappa}\right), \tag{2.14}$$

which implies (2.5a).

Finally we prove (2.5b). It follows from (1.7) and (2.10b) that for $|\alpha| = 1$

$$|\partial_{r,t}^\alpha u^0(r,t)| \le C\varepsilon r^{-m-1}\int_{|t-r|}^{t+r}\langle\lambda\rangle^{m-\kappa-1}d\lambda + C\varepsilon r^{-m-1}|t\pm r|^m\langle t\pm r\rangle^{-\kappa}. \tag{2.13b}$$

It is easy to see that $|\partial_{r,t}^\alpha u^0(r,t)| \le \varepsilon C r^{-m}$ for $0 < r \le 1$. When $r \ge 1$, noting that (2.13b) implies (2.14), we obtain (2.5b). The proof is complete. □

3. Preliminaries. In this section we derive some preliminery estimates. Since $p > p_0(n)$ by (H), we have $\kappa_0 = 2/(p-1) < (m+1)p-1$. Therefore without loss of generality we may assume

$$\kappa \le (m+1)p-1, \tag{3.1}$$

because if (1.7) holds for some $\tilde{\kappa}$, then (1.7) necessarily holds for any $\kappa \le \tilde{\kappa}$. If we notice that

$$m+1-mp = \begin{cases} p-1+1-(p-1)\kappa & \text{if} \quad \kappa = m+1, \\ p-1+m+2-(m+1)p & \text{if} \quad \kappa > m+1 \end{cases}$$

and employ Lemma 3.4 with $q = p$ in [9], we obtain

Lemma 3.1. *We suppose (3.1) and $p > p_0(n)$ hold. Then we have*

$$\int_{-\xi}^{\xi}\langle\frac{\xi+\eta}{2}\rangle^{m+1-mp}(2+\log\frac{1+\xi}{1+|\eta|})^p d\eta \le C\langle\xi\rangle^{p-1}\Phi_\kappa(\xi), \tag{3.2}$$

$$\int_{-\xi}^{\xi}\langle\frac{\xi+\eta}{2}\rangle^{m+1-mp}\langle\eta\rangle^{p(m+1-\kappa)}d\eta \le C\langle\xi\rangle^{p+m-\kappa}\Phi_\kappa(\xi) \quad \text{if} \quad \kappa > m+1, \tag{3.3}$$

where we have set $\Phi_\kappa(s) = \max(1, \langle s\rangle^{2-(p-1)\kappa})$ *for* $s \in \mathbb{R}$.

Next we prove the following lemma which will be used in the proof of Propositions 4.3 and 4.5 below.

Lemma 3.2. *We suppose (3.1),* $p > p_0(n)$ *and*

$$m + 2 - mp > 0 \tag{3.4}$$

hold. Then we have for $(r,t) \in \Omega = (0,\infty) \times [0,\infty)$

$$I \equiv \int_0^t d\tau \int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{m+1-mp} \Psi_\kappa^p(\lambda,\tau) d\lambda \le Cr\Psi_\kappa(r,t)\Phi_\kappa(t+r), \tag{3.5a}$$

$$J \equiv \int_0^t d\tau \int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p} \langle\lambda\rangle^{1-p} \Psi_\kappa^p(\lambda,\tau) d\lambda \le Cr\Psi_\kappa(r,t)\Phi_\kappa(t+r), \tag{3.5b}$$

where $\lambda_\pm = t - \tau \pm r$. *Moreover for* $t \ge 2r$ *and* $0 < r \le 1$ *we have*

$$J' \equiv \int_0^{t-2r} d\tau \int_{\lambda_-}^{\lambda_+} \lambda^{m+1-mp} \Psi_\kappa^p(\lambda,\tau) d\lambda \le Cr\Psi_\kappa(r,t)\Phi_\kappa(t+r). \tag{3.5c}$$

Furthermore for $r \ge 1$ *we have*

$$P_\pm \equiv \int_0^t \langle\lambda_\pm\rangle^{m+1-mp} \Psi_\kappa^p(|\lambda_\pm|,\tau) d\tau \le Cr\Psi_\kappa(r,t)\Phi_\kappa(t+r). \tag{3.6}$$

Here Ψ_κ *is given in (2.5).*

Proof: First we prove (3.5a). Let $0 < \kappa < m+1$. Introducing the characteristic coordinates $\xi = \lambda + \tau$, $\eta = \lambda - \tau$, we have

$$\begin{aligned} I =& \frac{1}{2} \int_{|t-r|}^{t+r} \langle\xi\rangle^{-p(\kappa-m)} d\xi \int_{r-t}^{\xi} \langle \frac{\xi+\eta}{2} \rangle^{m+1-mp} d\eta \\ \le& C \int_{|t-r|}^{t+r} \langle\xi\rangle^{m+2-p\kappa} d\xi, \end{aligned}$$

because the η-integral is dominated by $C\langle\xi\rangle^{m+2-mp}$ by (3.4). Therefore we have

$$I \le C\Phi_\kappa(t+r) I' \quad \text{with} \quad I' = \int_{|t-r|}^{t+r} \langle\xi\rangle^{-(\kappa-m)} d\xi, \tag{3.7}$$

because

$$m + 2 - p\kappa = -(\kappa - m) + 2 - (p-1)\kappa. \tag{3.8}$$

Note that (3.7) is also holds for the case where $\kappa \geq m+1$, if we use (3.2) (resp. (3.3)) when $\kappa = m+1$ (resp. $\kappa > m+1$). Therefore our aim becomes

$$I' \leq Cr\Psi_\kappa(r,t). \tag{3.9}$$

When either $t \geq 2r > 0$ or $0 < r \leq 1$ holds, since $\langle\xi\rangle$ is equivelent to $\langle t+r\rangle$, we have $I' \leq Cr\langle t+r\rangle^{-(\kappa-m)}$, which implies (3.9). When $0 < t \leq 2r$ and $r \geq 1$, we have $C \leq r\langle t+r\rangle^{-1}$. Therefore we get (3.9), hence obtain (3.5a).

Next we prove (3.5b). When $|\lambda_-| \geq 1$, we have

$$\lambda - \text{integral} \leq C\int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{m+1-mp}\Psi_\kappa^p(\lambda,\tau)d\lambda.$$

When $|\lambda_-| \leq 1$, we have

$$\begin{aligned}\lambda - \text{integral} \leq C\Big(&\int_{|\lambda_-|}^{\min(1,\lambda_+)} \lambda^{m-(m-1)p}\langle\tau\rangle^{-p(\kappa-m)}d\lambda \\ &+ \int_1^{\max(1,\lambda_+)} \langle\lambda\rangle^{m+1-mp}\Psi_\kappa^p(\lambda,\tau)d\lambda\Big),\end{aligned}$$

because $\lambda^{m-(m-1)p}\langle\lambda\rangle^{1-p} \leq C\langle\lambda\rangle^{m+1-mp}$ for $\lambda \geq 1$ and $\Psi_\kappa(\lambda,\tau) \leq C\langle\tau\rangle^{-p(\kappa-m)}$ for $0 \leq \lambda \leq 1$. Combining these estimates, we arrive at

$$J \leq C(\tilde{J} + I),$$

where I is given in (3.5a) and we have set

$$\begin{aligned}&\tilde{J} = \int_\Sigma \langle\tau\rangle^{-p(\kappa-m)}d\tau \int_{|\lambda_-|}^{\min(1,\lambda_+)} \lambda^{m-(m-1)p}d\lambda, \\ &\Sigma = \{\tau \in [0,t] :\ |\lambda_-| \leq 1\}.\end{aligned}$$

When $t \geq 2r > 0$ or $0 < r \leq 1$ or $0 < \kappa \leq m$, we have

$$\begin{aligned}\tilde{J} &\leq C\langle t-r\rangle^{-p(\kappa-m)} \int_{t-r-1}^{t-r+1} (1+|\lambda_-|)^{m-(m-1)p}d\tau \int_{\lambda_-}^{\lambda_+} d\lambda \\ &\leq Cr\langle t+r\rangle^{-p(\kappa-m)}.\end{aligned}$$

Here the last inequality follows from the following inequality deduced by (3.4):

$$m+1-(m-1)p > 0. \tag{3.10}$$

Since

$$-p(\kappa - m) = -(\kappa - m) + 2 - (p-1)\kappa - (m+2-mp), \tag{3.11}$$

noting (3.4), we get

$$\tilde{J} \le Cr\langle t+r\rangle^{-(\kappa-m)}\Phi_\kappa(t+r) \le Cr\Psi_\kappa(r,t)\Phi_\kappa(t+r).$$

When $0 < t \le 2r$, $r \ge 1$ and $\kappa > m$, we have

$$\begin{aligned}\tilde{J} \le& C\langle t-r\rangle^{-p(\kappa-m)}\int_{t-r-1}^{t-r+1} d\tau \int_0^1 \lambda^{m-(m-1)p} d\lambda \\ \le& C\langle t-r\rangle^{-p(\kappa-m)} \le C\langle t-r\rangle^{-(\kappa-m)}\Phi_\kappa(t-r),\end{aligned}$$

by (3.10), (3.11) and (3.4). If $m < \kappa \le m+1$, we have

$$\tilde{J} \le C\Phi_\kappa(t+r) \le Cr\langle t+r\rangle^{-(\kappa-m)}\Phi_\kappa(t+r),$$

because for any $\mu \in (0,1]$, $r\langle t+r\rangle^{-\mu}$ is bounded below for this case. If $\kappa > m+1$, we get

$$\tilde{J} \le Cr\langle t+r\rangle^{-1}\langle t-r\rangle^{-(\kappa-m)}\Phi_\kappa(t+r).$$

Therefore we obtain the desired estimate for $\tilde{J}$, hence (3.5b) holds.

Next we prove (3.5c). Analogously to the treatment of J, we have

$$J' \le CI + C\langle t\rangle^{-p(\kappa-m)}\int_{(t-r-1)_+}^{t-2r} (1+\lambda_-^{m+1-mp})d\tau \int_{\lambda_-}^{\lambda_+} d\lambda,$$

where I is given in (3.5a) and $(t-r-1)_+ = \max(t-r-1, 0)$. By (3.4) we have

$$\int_{(t-r-1)_+}^{t-2r} \lambda_-^{m+1-mp} d\tau \le C(t-r)^{m+2-mp} \le C\langle t\rangle^{m+2-mp}.$$

Therefore by (3.5a) and (3.8) we obtain (3.5c),

Finally we prove (3.6). We consider only P_-, because P_+ can be handled analogously. Let $t \ge 2r \ge 2$. When $0 < \kappa < m+1$, we have

$$P_- \le C\langle t+r\rangle^{-p(\kappa-m)}\int_0^t \langle\lambda_-\rangle^{m+1-mp} d\tau,$$

because $|\lambda_-| + \tau \geq C(t-r) \geq C(t+r)$. Changing the variable by $\lambda = \lambda_-$, we get

$$\begin{aligned} P_- \leq & C\langle t+r\rangle^{-p(\kappa-m)} \int_{-r}^{t-r} \langle\lambda\rangle^{m+1-mp} d\lambda \\ \leq & 2C\langle t+r\rangle^{-p(\kappa-m)} \int_0^{t-r} \langle\lambda\rangle^{m+1-mp} d\lambda \\ \leq & C\langle t+r\rangle^{m+2-p\kappa}, \end{aligned}$$

by (3.4). Using (3.8) we obtain (3.6), because $r \geq 1$.

When $\kappa = m+1$, we have

$$P_- \leq C\langle t+r\rangle^{-p}(Q_- + Q_+),$$

where we have set

$$\begin{aligned} Q_- &= \int_0^{t-r} \langle\lambda_-\rangle^{m+1-mp}\big(1 + \log\frac{\langle t-r\rangle}{\langle 2\tau - t + r\rangle}\big)^p d\tau, \\ Q_+ &= \int_{t-r}^{t} \langle -\lambda_-\rangle^{m+1-mp}\big(1 + \log\frac{\langle 2\tau - t + r\rangle}{\langle t-r\rangle}\big)^p d\tau. \end{aligned}$$

It is easy to see that

$$Q_+ \leq \big(1 + \log\frac{\langle t+r\rangle}{\langle t-r\rangle}\big)^p \int_{t-r}^{t} \langle -\lambda_-\rangle^{m+1-mp} d\tau \leq C\langle r\rangle^{m+2-mp},$$

by (3.4). Changing the variable by $\eta = t - r - 2\tau$, we get

$$\begin{aligned} Q_- \leq & \frac{1}{2}\int_{-(t-r)}^{t-r} \langle\frac{t-r+\eta}{2}\rangle^{m+1-mp}\big(1 + \log\frac{\langle t-r\rangle}{\langle\eta\rangle}\big)^p d\eta \\ \leq & C\langle t-r\rangle^{p-1}\Phi_\kappa(t-r), \end{aligned}$$

by (3.2) with $\xi = t-r$. Similarly we can deal with case where $\kappa > m+1$, if we use (3.3). We omit further detail.

Next we let both $0 < t \leq 2r$ and $r \geq 1$ hold. When $0 < \kappa \leq m$, changing the variable by $\lambda = \lambda_-$, we get

$$\begin{aligned} P_- \leq & C\langle t+r\rangle^{-p(\kappa-m)} \int_{-r}^{t-r} \langle\lambda\rangle^{m+1-mp} d\lambda \\ \leq & 2C\langle t+r\rangle^{-p(\kappa-m)} \int_0^{r} \langle\lambda\rangle^{m+1-mp} d\lambda \\ \leq & C\langle t+r\rangle^{m+2-p\kappa}, \end{aligned}$$

which implies (3.6) by (3.8).

When $m < \kappa \leq m+1$, we have $\Psi_\kappa(|\lambda_-|, \tau) \leq \langle |\lambda_-| + \tau \rangle^{-(\kappa-m)+\delta}$, where $\delta = 0$ if $m < \kappa < m+1$ and if $\kappa = m+1$, we took δ so small that $0 < p\delta < 1$ holds. Therefore we have

$$P_- \leq C \int_0^t \langle \lambda_- \rangle^{m+1-p\kappa+p\delta} d\tau \leq C \Phi_\kappa(t+r) \int_0^t \langle \lambda_- \rangle^{m-\kappa-1+p\delta} d\tau,$$

by (3.8). Since the integral is bounded, we get

$$P_- \leq C\Phi_\kappa(t+r) \leq Cr\langle t+r \rangle^{-(\kappa-m)} \Phi_\kappa(t+r),$$

because $m < \kappa \leq m+1$.

When $\kappa > m+1$, we have

$$P_- = C(Q'_- + Q'_+),$$

where we have set

$$\begin{aligned} Q'_- &= \int_0^{t-r} \langle \lambda_- \rangle^{m+1-mp} \langle t-r \rangle^{-p} \langle \tau - \lambda_- \rangle^{p(m+1-\kappa)} d\tau, \\ Q'_+ &= \int_{t-r}^t \langle -\lambda_- \rangle^{m+1-mp} \langle \tau - \lambda_- \rangle^{-p} \langle t-r \rangle^{p(m+1-\kappa)} d\tau. \end{aligned}$$

As to Q'_+, we have

$$Q'_+ \leq \langle t-r \rangle^{m+1-\kappa} \int_{t-r}^t \langle -\lambda_- \rangle^{m+1-(m+1)p} d\tau \leq C\langle t-r \rangle^{m+1-\kappa},$$

because $m+2-(m+1)p < 0$ if $p > p_0(n)$. Changing the variable by $\eta = t-r-2\tau$, we get

$$\begin{aligned} Q'_- \leq& \frac{1}{2} \langle t-r \rangle^{-p} \int_{-(t-r)}^{t-r} \langle \frac{t-r+\eta}{2} \rangle^{m+1-mp} \langle \eta \rangle^{p(m+1-\kappa)} d\eta \\ \leq& C\langle t-r \rangle^{m-\kappa} \Phi_\kappa(t-r), \end{aligned}$$

by (3.3) with $\xi = t-r$. Combining these estimates, we arrive at (3.6), because there is a constant $C > 0$ such that $C \leq r\langle t+r \rangle^{-1}$ in this case. The proof is complete. □

4. Basic A Priori Estimates. We consider the following integral equation associated with the initial value problem (1.6):

$$u(r,t) = u^0(r,t) + L(u)(r,t) \quad \text{in} \quad \Omega_T, \tag{4.1}$$

where u^0 is given by (2.2) and we have set

$$L(u)(r,t) = \int_0^t w(r,t,\tau)d\tau,$$
$$w(r,t,\tau) = \int_{|\lambda_-|}^{\lambda_+} G(\lambda,\tau)K(\lambda,r,t-\tau)d\lambda$$

with $\lambda_\pm = t-\tau\pm r$ and $G(\lambda,\tau) = F(u)(\lambda,\tau)$.

We now introduce a Banach space X on which we will construct a solution of (4.1):

$$X = \{u(r,t) \in C^{1,0}(\Omega_T) : \ \|u\| < \infty\},$$

where the norm $\|u\|$ is defined by

$$\|u\| = \sum_{i=0}^{1} \sup_{(r,t)\in\Omega_T} |r^{m+i-1}\partial_r^i u(r,t)|\langle r\rangle^{1-i}\Psi_\kappa^{-1}(r,t).$$

Here Ψ_κ is given in (2.5). Since (H) implies

$$|F'(u)| \le Ap|u|^{p-1}, \quad |F(u)| \le A|u|^p \quad \text{for} \quad u \in R, \tag{4.2}$$

we obtain

Lemma 4.1. *Suppose (H) holds. Setting $G(\lambda,\tau) = F(u)(\lambda,\tau)$ for $u \in X$, we have*

$$|G(\lambda,\tau)| \le C\|u\|^p\lambda^{-(m-1)p}\langle\lambda\rangle^{-p}\Psi_\kappa^p(\lambda,\tau), \quad (\lambda,\tau) \in \Omega_T, \tag{4.3}$$

$$|\partial_\lambda G(\lambda,\tau)| \le C\|u\|^p\lambda^{-(m-1)p-1}\langle\lambda\rangle^{1-p}\Psi_\kappa^p(\lambda,\tau), \quad (\lambda,\tau) \in \Omega_T. \tag{4.4}$$

For $u \in X$ the integral operator $L(u)$ satisfies the following inhomogeneous wave equation. (For the proof, see [10], Section 4).

Proposition 4.2. *We suppose that (3.1) and (H) hold and that u belongs to X. Then we have $L(u)(r,t) \in C^2(\Omega_T)$. Moreover $L(u)$ satisfies the zero initial data and*

$$(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r)L(u) = F(u) \quad \text{in} \quad \Omega_T.$$

In what follows, we examine the quantitative properties of $L(u)$.

Proposition 4.3. *Let the hypotheses of the preceding proposition be fulfilled. Then we have for* $(r,t) \in \Omega_T$

$$|\partial_{r,t}^{\alpha} L(u)(r,t)| \leq C_1 \|u\|^p r^{1-m-|\alpha|} \langle r \rangle^{|\alpha|-1} \Psi_\kappa(r,t) \Phi_\kappa(t+r), \quad |\alpha| \leq 1. \tag{4.5}$$

Proof: We prove (4.5) by dividing the argument into the cases.

Case 1: $0 < r \leq 1$.

It follows that for $|\alpha| \leq 1$

$$\partial_{r,t}^{\alpha} L(r,t) = \int_0^{(t-2r)_+} \partial_{r,t}^{\alpha} w(r,t,\tau) d\tau + \int_{(t-2r)_+}^{t} \partial_{r,t}^{\alpha} w(r,t,\tau) d\tau \equiv A_{|\alpha|} + B_{|\alpha|},$$

because $w(r,t,t) = 0$. Here we have set $(t-2r)_+ = \max(t-2r, 0)$. To begin with, we consider $A_{|\alpha|}$. Since $t - \tau \geq 2r$ for $\tau \leq t - 2r$, we get similarly to (2.6)

$$|\partial_{r,t}^{\alpha} w(r,t,\tau)| \leq C r^{-m-|\alpha|} \int_{|\lambda_-|}^{\lambda_+} \{\lambda^m |G(\lambda,\tau)| + |G_m(\lambda,\tau)|\} d\lambda,$$

where we have set $G_m(\lambda,\tau) = -\partial_\lambda(\lambda^{m+1} G(\lambda,\tau))$. From (4.3) and (4.4) we have

$$|G_m(\lambda,\tau)| \leq C \|u\|^p \lambda^{m-(m-1)p} \langle \lambda \rangle^{1-p} \Psi_\kappa^p(\lambda,\tau).$$

Therefore we have from this and (4.3)

$$|\partial_{r,t}^{\alpha} w(r,t,\tau)| \leq C \|u\|^p r^{-m-|\alpha|} \int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p} \langle \lambda \rangle^{1-p} \Psi_\kappa^p(\lambda,\tau) d\lambda,$$

hence we get

$$|A_{|\alpha|}| \leq C \|u\|^p r^{-m-|\alpha|} J \leq C \|u\|^p r^{1-m-i} \Psi_\kappa(r,t) \Phi_\kappa(t+r), \tag{4.6}$$

by (3.5b).

We now tern our attention to $B_{|\alpha|}$. By (2.4a) we have

$$|w(r,t,\tau)| \leq C r^{-m-1} \int_{|\lambda_-|}^{\lambda_+} \lambda^{m+1} |G(\lambda,\tau)| d\lambda. \tag{4.7a}$$

Moreover for $0 < t - \tau \leq 2r$, we get similarly to (2.10b)

$$|\partial_{r,t} w(r,t,\tau)| \leq C r^{-m-1} \int_{|\lambda_-|}^{\lambda_+} \lambda^m |G(\lambda,\tau)| d\lambda + C r^{-m-1} |\lambda_\pm|^{m+1} |G(|\lambda_\pm|,\tau)|. \tag{4.7b}$$

Since $0 < r \le 1$, we have

$$|\lambda_-| \le \lambda \le \lambda_+ \le 3r \le 3,$$

hence by (4.3) we have

$$|G(\lambda,\tau)| \le C\|u\|^p \lambda^{-(m-1)p} \langle \tau \rangle^{-p(\kappa-m)}. \tag{4.8}$$

Substituting this into (4.7a), we get

$$\begin{aligned} |w(r,t,\tau)| &\le C\|u\|^p r^{-m-1} \langle \tau \rangle^{-p(\kappa-m)} \int_0^{3r} \lambda^{m+1-(m-1)p} d\lambda \\ &\le C\|u\|^p r^{-m} \langle \tau \rangle^{-p(\kappa-m)}, \end{aligned}$$

because of (3.10). We also get from (4.7b), (4.8) and (3.10)

$$\begin{aligned} |\partial_{r,t} w(r,t,\tau)| \le\ & C\|u\|^p r^{-m-1} \langle \tau \rangle^{-p(\kappa-m)} \int_0^1 \lambda^{m-(m-1)p} d\lambda \\ & + C\|u\|^p r^{-m-1} \langle \tau \rangle^{-p(\kappa-m)} \\ \le\ & C\|u\|^p r^{-m-1} \langle \tau \rangle^{-p(\kappa-m)}. \end{aligned}$$

Combining these estimates, we obtain for $|\alpha| \le 1$

$$|B_{|\alpha|}| \le C\|u\|^p r^{-m-|\alpha|} \int_{(t-2r)_+}^t \langle \tau \rangle^{-p(\kappa-m)} d\tau \le C\|u\|^p r^{1-m-i} \langle t \rangle^{-p(\kappa-m)},$$

which implies (4.5) by virtue of (3.11).

Case 2: $r \ge 1$.

First we prove (4.5) with $\alpha = 0$. By (4.7a), (4.3) and (3.10) we have

$$|w(r,t,\tau)| \le C\|u\|^p r^{-m-1} \int_{|\lambda_-|}^{\lambda_+} \langle \lambda \rangle^{m+1-mp} \Psi_\kappa^p(\lambda,\tau) d\lambda,$$

hence we get

$$|L(r,t)| \le C\|u\|^p r^{-m-1} I \le C\|u\|^p r^{-m} \Psi_\kappa(r,t) \Phi_\kappa(t+r)$$

by (3.5a).

Next we prove (4.5) with $|\alpha| = 1$. We divide $\partial_{r,t} L$ into A_1 and B_1 as before. Note that (4.6) is still valid for $r \ge 1$, because we did not use the assumption $0 < r \le 1$ to derive it. Moreover (4.7b) also holds for $r \geq 1$ if $0 < t - \tau \leq 2r$. Therefore by (4.3) and (3.10) we have

$$|\partial_{r,t} w(r,t,\tau)| \leq C\|u\|^p r^{-m-1} \int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p} \langle\lambda\rangle^{-p} \Psi_\kappa^p(\lambda,\tau) d\lambda + C\|u\|^p r^{-m-1} \langle\lambda_\pm\rangle^{m+1-mp} \Psi_\kappa^p(|\lambda_\pm|,\tau).$$

Therefore by (3.5b) and (3.6), we get

$$|B_1| \leq C\|u\|^p r^{-m-1}(J + P_\pm) \leq C\|u\|^p r^{-m} \Psi_\kappa(r,t) \Phi_\kappa(t+r).$$

We thus obtain (4.5). The proof is complete. □

To proceed further, we now introduce an auxiliary norm $|||u|||$ for $u \in X$ by

$$|||u||| = \sup_{(r,t)\in\Omega} |r^m u(r,t)| \Psi_\kappa^{-1}(r,t).$$

Lemma 4.4. *Suppose that (H) holds. Set $\tilde{G}(\lambda,\tau) = F(u)(\lambda,\tau) - F(v)(\lambda,\tau)$ for u, $v \in X$. Then we have for $(\lambda,\tau) \in \Omega_T$*

$$|\tilde{G}(\lambda,\tau)| \leq CM\lambda^{-(m-1)p-1} \langle\lambda\rangle^{1-p} \Psi_\kappa^p(\lambda,\tau), \tag{4.9}$$

where we have set $M = |||u - v|||(\|u\|^{p-1} + \|v\|^{p-1})$. Moreover we have

$$|\tilde{G}(\lambda,\tau)| \leq CN_1 \lambda^{-(m-1)p} \langle\lambda\rangle^{-p} \Psi_\kappa^p(\lambda,\tau), \tag{4.10}$$

$$|\partial_\lambda \tilde{G}(\lambda,\tau)| \leq CN_1 \lambda^{-(m-1)p-1} \langle\lambda\rangle^{1-p} \Psi_\kappa^p(\lambda,\tau) + CN_2 \lambda^{-mp} \Psi_\kappa^p(\lambda,\tau), \tag{4.11}$$

where we have set $N_1 = \|u - v\|(\|u\|^{p-1} + \|v\|^{p-1})$ and $N_2 = |||u - v|||^{p-1}(\|u\|^m + \|v\|^m)$.

Proof: Noting that

$$\tilde{G}(\lambda,\tau) = (u - v) \int_0^1 F'(\theta u + (1-\theta)v) d\theta,$$

hence we have by (4.2)

$$|\tilde{G}(\lambda,\tau)| \leq Ap2^{p-1}|u - v|(|u|^{p-1} + |v|^{p-1}).$$

One can easily obtain (4.9) and (4.10) from the above.

By (4.2) and (H) we have

$$\begin{aligned}|\partial_\lambda \tilde{G}(\lambda,\tau)| \leq& |F'(u)||\partial_\lambda u - \partial_\lambda v| + |\partial_\lambda v||F'(u) - F'(v)| \\ \leq& Ap|u|^p|\partial_\lambda u - \partial_\lambda v| + Ap|\partial_\lambda v||u - v|^{p-1}.\end{aligned}$$

Using $|||\cdot|||$ for the second term, we get (4.11). □

Proposition 4.5. *Suppose (H) and (3.1) hold. let u, v belong to X. Then we have for $(r,t) \in \Omega_T$*

$$|L(u)(r,t) - L(v)(r,t)| \leq C_2 M r^{-m} \Psi_\kappa(r,t)\Phi_\kappa(t+r), \tag{4.12}$$

and for $i = 0,1$

$$|\partial_r^i(L(u) - L(v))(r,t)| \leq (C_3 N_1 + C_4 N_2) r^{1-m-i} \langle r\rangle^{i-1} \Phi_\kappa(t+r). \tag{4.13}$$

Moreover if we further assume $0 < \kappa < \kappa_0$, we then obtain (4.12) and (4.13) with $\Phi_\kappa(t+r)$ replaced by $\langle t\rangle^{2-(p-1)\kappa}$.

Proof: We have from (4.1)

$$\tilde{L}(r,t) \equiv L(u)(r,t) - L(v)(r,t) = \int_0^t \tilde{w}(r,t,\tau)d\tau, \tag{4.14}$$

where $\tilde{w}$ is equal to w defined in (4.1) with G replaced by $\tilde{G}$. First we prove (4.12). Since we have (4.7a) with w and G replaced by $\tilde{w}$ and $\tilde{G}$ respectively, by (4.9) we get

$$|\tilde{L}(r,t)| \leq CMr^{-m-1}J \tag{4.15}$$

where J is given in (3.5b). Therefore it is easy to see that (4.12) holds if we use (3.5b).

Next we prove (4.13). When $r \geq 1$ and $i = 0$, proceeding as in the proof of Proposition 4.3, we obtain (4.13), because by (4.10) $\tilde{G}$ has the same estimate as G with $\|u\|^p$ replaced by N_1. Therefore it is enough to consider the case where either $0 < r \leq 1$ or $r \geq 1$ and $i = 1$. Setting for $i = 0,1$

$$\tilde{A}_i = \int_0^{(t-2r)_+} \partial_r^i \tilde{w}(r,t,\tau)d\tau, \quad \tilde{B}_i = \int_{(t-2r)_+}^t \partial_r^i \tilde{w}(r,t,\tau)d\tau,$$

we then have $\partial_r^i \tilde{L}(r,t) = \tilde{A}_i + \tilde{B}_i$. Since by (4.10) $\tilde{B}_i$ can be handled similarly to B_i, we concentrate on estimating $\tilde{A}_i$. Therefore we may assum $0 \leq \tau \leq t - 2r$. Similarly to (2.6) we have

$$|\partial_r^i \tilde{w}(r,t,\tau)| \leq Cr^{-m-i} \int_{\lambda_-}^{\lambda_+} \{\lambda^m |\tilde{G}(\lambda,\tau)| + |\tilde{G}_m(\lambda,\tau)|\} d\lambda,$$

where we have set $\tilde{G}_m(\lambda,\tau) = -\partial_\lambda(\lambda^{m+1}\tilde{G}(\lambda,\tau))$. Employing (4.10) and (4.11) we have

$$|\partial_r^i \tilde{w}(r,t,\tau)| \leq Cr^{-m-i}\Big(N_1 \int_{\lambda_-}^{\lambda_+} \lambda^{m-(m-1)p}\langle\lambda\rangle^{1-p}\Psi_\kappa^p(\lambda,\tau)d\lambda + N_2 \int_{\lambda_-}^{\lambda_+} \lambda^{m+1-mp}\Psi_\kappa^p(\lambda,\tau)d\lambda\Big),$$

hence we get

$$|\tilde{A}_i| \leq Cr^{-m-i}(N_1 J + N_2 J'),$$

where J and J' is given in (3.5b) and (3.5c). We thus obtain (4.13).

Finally we consider the case where $0 < \kappa < \kappa_0$. When $t \geq r/2 > 0$ or $0 < r \leq 2$, the conclusion directly follows from (4.12) and (4.13). Therefore we assume $0 < t \leq r/2$ and $r \geq 2$ in what follows. Let $0 \leq \tau \leq t$ and $|\lambda_-| \leq \lambda \leq \lambda_+$. Since $r - t \geq r/2 \geq 1$ for the case, λ, $\lambda+\tau$ and $\lambda-\tau$ are equivelent to $\langle r\rangle$. In particular, for an arbitrary $\kappa > 0$ we get $\Psi_\kappa(\lambda,\tau) \leq C\langle r\rangle^{-(\kappa-m)}$. Here Ψ_κ is given in (2.5).

To begin with, we claim that

$$I \leq Cr\Psi_\kappa(r,t)\langle t\rangle^{2-(p-1)\kappa}, \tag{4.16a}$$

$$J \leq Cr\Psi_\kappa(r,t)\langle t\rangle^{2-(p-1)\kappa}, \tag{4.16b}$$

where I and J is given by (3.5a) and (3.5b). Indeed, from the above remarks we have

$$\begin{aligned} J &\leq CI \\ &\leq C\langle r\rangle^{-p(\kappa-m)} \int_0^t d\tau \int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{m+1-mp} d\lambda \\ &\leq C\langle r\rangle^{m+1-p\kappa} \int_0^t d\tau \int_{-\lambda_-}^{\lambda_+} d\lambda \\ &\leq C\langle r\rangle\Psi_\kappa(r,t)\langle t\rangle^{2-(p-1)\kappa}, \end{aligned}$$

because $m - p\kappa = -(\kappa-m) - (p-1)\kappa$. We thus get (4.16), because $r \geq 1$. Employing (4.15) and (4.16b), we find that the assertion concerning (4.12) holds.

Next we prove the assertion concerning (4.13). As to the case where $i = 0$, similarly to (4.15) we have

$$|\tilde{L}(r,t)| \leq CN_1 r^{-m-1} I,$$

if we use (4.10) instead of (4.9) and notice (3.10). By (4.16a) we get the desired result.

Next we consider the case where $i = 1$. Since $0 \le t - \tau \le 2r$ for $t \le r/2$ and $0 \le \tau \le t$, similarly to (2.10b) we have

$$|\partial_r \tilde{w}(r,t)| \le Cr^{-m-1} \int_{|\lambda_-|}^{\lambda_+} \lambda^m |\tilde{G}(\lambda,\tau)| d\lambda + |P'|,$$

where we have set

$$P' = \tilde{G}(\lambda_+,\tau)K(\lambda_+,r,t-\tau) - \tilde{G}(|\lambda_-|,\tau)K(|\lambda_-|,r,t-\tau).$$

Therefore we have by (4.10)

$$|\partial_r \tilde{L}(r,t)| \le CN_1 r^{-m-1} J + \int_0^t |P'| d\tau.$$

By (4.16b) we have only to consider P'. Setting $H(\lambda,r,t,\tau) = \tilde{G}(\lambda,\tau)K(\lambda,r,t-\tau)$, we obtain

$$P' = \int_0^1 \frac{d}{ds} H(\lambda(s),r,t,\tau) ds = (\lambda_+ - |\lambda_-|) \int_0^1 (\partial_\lambda H)(\lambda(s),r,t,\tau) ds,$$

where we put $\lambda(s) = s\lambda_+ + (1-s)|\lambda_-|$. By (2.4), (4.10) and (4.11), for $|\lambda_-| \le \lambda \le \lambda_+$ we get the following similarly to the proof of (4.16):

$$\begin{aligned}
|(\partial_\lambda H)(\lambda,r,t,\tau)| \le\ & CN_1 r^{-m-1} \lambda^{m-(m-1)p} \langle\lambda\rangle^{1-p} \Psi_\kappa^p(\lambda,\tau) \\
& + CN_2 r^{-m-1} \lambda^{m+1-mp} \Psi_\kappa^p(\lambda,\tau) \\
\le\ & C(N_1+N_2) r^{-m-1} \langle r\rangle^{m+1-p\kappa} \\
\le\ & C(N_1+N_2) r^{-m} \Psi_\kappa(r,t) \langle t\rangle^{-(p-1)\kappa}.
\end{aligned}$$

Since $\lambda_+ - |\lambda_-| = 2t$, we therefore get

$$\int_0^t |P'| d\tau \le C(N_1+N_2) r^{-m} \Psi_\kappa(r,t) \langle t\rangle^{2-(p-1)\kappa}.$$

We thus proved all the assertions of Proposition 4.5 $\square$

End of the Proof of Main Theorem: To begin with, we shall show that a solution to the integral equation (4.1) exists uniquely. We define a sequence of functions $\{u_k\}_{k=0}^\infty$ by

$$u_{k+1} = u_0 + L(u_k) \quad \text{for} \quad k \ge 0, \quad u_0 = u^0,$$

where u^0 is given by (2.2). By (2.5) we have $\|u_0\| \le C_0\varepsilon$. It follows from (4.5), (4.12) and (4.13) that for $u, v \in X$

$$\|L(u)\| \le C_1\|u\|^p\Phi_\kappa(T), \tag{4.17}$$

$$|||L(u)-L(v)||| \le C_2|||u-v|||(\|u\|^{p-1}+\|v\|^{p-1})\Phi_\kappa(T), \tag{4.18a}$$

$$\begin{aligned}\|L(u)-L(v)\| \le\ & C_3\|u-v\|(\|u\|^{p-1}+\|v\|^{p-1})\Phi_\kappa(T) \\ & + C_4|||u-v|||^{p-1}(\|u\|^m+\|v\|^m)\Phi_\kappa(T),\end{aligned} \tag{4.18b}$$

First we consider the case where $\kappa \ge \kappa_0$. Then we have $\Phi_\kappa(T) \le 1$. Therefore by (4.17), $u_k \in X$ for any k. Let ε_2 be so small that

$$2C_0\varepsilon_2 \le 1 \quad \text{and} \quad 2^{p+2}C_5(C_0\varepsilon_2)^{p-1} \le 1 \tag{4.19}$$

with $C_5 = \sum_{i=1}^4 C_i$. Then if we use (4.18), we find a solution $u \in X$ to the integral equation (3.1) for $0 < \varepsilon \le \varepsilon_2$ and arbitraly $T > 0$ by following [8], p.257-p.259. (See also [9], Section 5).

Next we treat the case where $0 < \kappa < \kappa_0$. We take a positive number ε_1 satisfying

$$\varepsilon_1 \le \varepsilon_2 \quad \text{and} \quad 2^{p+2}C_5(\sqrt{2})^{2-(p-1)\kappa}(C_0\varepsilon_1)^{p-1} < 1. \tag{4.20}$$

Since $2-(p-1)\kappa > 0$ by $0 < \kappa < \kappa_0$, there is a number $t_\varepsilon > 1$ uniqely such that

$$2^{p+2}C_5(\sqrt{2}t_\varepsilon)^{2-(p-1)\kappa}(C_0\varepsilon)^{p-1} = 1 \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_1]. \tag{4.21}$$

Then considering (4.19) and (4.20), for $0 < \varepsilon \le \varepsilon_1$ and $0 < T \le t_\varepsilon$ we have

$$2C_0\varepsilon \le 1 \quad \text{and} \quad 2^{p+2}C_5\langle T\rangle^{2-(p-1)\kappa}(C_0\varepsilon)^{p-1} \le 1,$$

because $\langle T\rangle \le \sqrt{2}\max(T,1)$. Hence for $0 < \varepsilon \le \varepsilon_1$ we get a solution on Ω_T by iteration.

Finally we claim that the unique solution $u \in X$ to (4.1) is also an unique solution to the initial value problem (1.6). It is easy to see that u satisfies (1.6) by Propositions 2.2 and 4.2. Moreover u becomes the unique solution to the problem. Indeed, if we set

$$v(r,t) = u(r,t) - (u^0(r,t) + L(u)(r,t)), \quad (r,t) \in \Omega_T.$$

Then $v(r,t)$ satisfies (2.1) in Ω_T with the zero initial data. Furthermore by (2.5b) and (4.5) with $|\alpha| = 1$, $\partial_{r,t} v(r,t) = O(r^{-m})$ as $r \to 0$. Then the uniqueness follows from the argument similar to [10], Lemma 3.2. Therefore we thus obtaine the unique solution to the problem (1.6). Moreover (1.8b) follows from (4.21), because $T_\varepsilon \geq t_\varepsilon$. The proof is complete. □

Acknowledgment

The author would like to acknowledge Professor Kôji Kubota for his valuable advices.

REFERENCES

1. R. Agemi and H. Takamura, *The lifespan of classical solutions to nonlinear wave equations in two space dimensions*, Hokkaido Math. J. **21** (1992), 517-542.
2. F. Asakura, *Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions*, Comm. in PDE **11(13)** (1986), 1459-1487.
3. Y. Choquet-Bruhat, *Global existence for solutions of* $\Box u = A|u|^p$ *small initial data*, J. Differential Equations **82** (1989), 98-108.
4. D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267-282.
5. R.T. Glassey, *Blow-up theorems for nonlinear wave equations*, Math. Z. **132** (1973), 183-203.
6. R.T. Glassey, *Finite-time blow-up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323-340.
7. R.T. Glassey, *Existence in the large for* $\Box u = F(u)$ *in two space dimensions*, Math. Z. **178** (1981), 233-261.
8. F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235-268.
9. H. Kubo, *Slowly decaying solutions for semilinear wave equations in odd space dimensions* , Hokkaido Univ. Preprint Ser. in Math. #245 (1994).
10. H. Kubo and K. Kubota, *Asymptotic behaviors of radially symmetric solutions of* $\Box u = |u|^p$ *for super critical values p in odd space dimensions*, to appear in Hokkaido Math. J.
11. K. Kubota, *Existence of a global solution to a semi-linear wave equation with initial data of non-compact support in low space dimensions*, Hokkaido Math. J. **22** (1993), 123-180.
12. Li Ta-tsien and Chen Yun-mei, *Initial value problems for nonlinear wave equations*, Comm. in PDE **13** (1988), 383-422.
13. Li Ta-tsien and Yu-Xin, *Life-span of classical solutions to fully nonlinear wave equations*, Comm. in PDE **26** (1991), 909-940.
14. J. Schaeffer, *The equation* $u_{tt} - \Delta u = |u|^p$ *for the critical value of p*, Proc. Roy. Soc. Edinburgh **101 A** (1985), 31-44.
15. T.C. Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Equations **52** (1984), 378-406.
16. H. Takamura, *Blow-up for semilinear wave equations with slowly decaying data in high dimensions*, to appear in J. Differential and Integral Equations.
17. K. Tsutaya, *A global existence theorem for semilinear wave equations with data of non compact support in two space dimensions*, Comm. in PDE **17(11 & 12)** (1992), 1925-1954.
18. K. Tsutaya, *Global existence theorem for semilinear wave equations with noncompact data in two space dimensions*, J. Differential Equations **104** (1993), 332-360.
19. K. Tsutaya, *Global existence and the life span of solutions of semilinear wave equations with data of non compact support in three space dimensions*, Funkcialaj Ekvacioj **37(1)** (1994), 1-18.
20. Y. Zhou, *Cauchy problem for semilinear wave equations in four space dimensions with small initial data*, IDMF PREPRINT 9217.