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Naoki Terai Takayuki Hibi

A simplicial complex Δ on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$. Given a subset W of V, the restriction of Δ to W is the subcomplex $\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$ of Δ. Let $\tilde{H}_i(\Delta; k)$ denote the i-th reduced simplicial homology group of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$, $\tilde{H}_{-1}(\{\emptyset\}; k) \cong k$, and $\tilde{H}_i(\{\emptyset\}; k) = 0$ for each $i \geq 0$. We write $\mid \Delta \mid$ for the geometric realization of Δ.

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k. In what follows, we consider A to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0}(k[\Delta])_n$ as a graded module over A with the quotient grading. Let $\mathbf{Z}$ denote the set of integers. We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}}[A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$.

We study a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

of $k[\Delta]$ over A. Here h is the homological dimension of $k[\Delta]$ over A and $\beta_i = \beta_i^A(k[\Delta]) := \sum_{j \in \mathbf{Z}} \beta_{i_j}$ is the i-th *Betti number* of $k[\Delta]$ over A. It is known [2, Theorem (5.1)] that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W) = j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k),$$

where $\sharp(W)$ is the cardinality of a finite set W. Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k). \tag{1}$$

LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = v$ and k a field. Then $\dim_k \tilde{H}_{v-3}(\Delta; k)$ is independent of k.*

Proof. Let 2^V denote the set of all subsets of V. Thus, the geometric realization X of the simplicial complex $2^V - \{V\}$ is the $(v-2)$-sphere. We may assume that $V \notin \Delta$; in particular, $|\Delta|$ is a subspace of X. Note that $H_{v-3}(|\Delta|; k) \cong H^{v-3}(|\Delta|; k)$ since k is a field. Now, the Alexander duality theorem of topology guarantees that $\tilde{H}^{v-3}(|\Delta|; k) \cong \tilde{H}_0(X - |\Delta|; k)$. On the other hand, $\dim_k \tilde{H}_0(X - |\Delta|; k) + 1$ is equal to the number of connected components of $X - |\Delta|$. Thus, $\dim_k \tilde{H}_{v-3}(\Delta; k) = \dim_k \tilde{H}_0(X - |\Delta|; k)$ is independent of the base field k as required. Q. E. D.

THEOREM. *The second Betti number of the Stanley–Reisner ring of a simplicial complex is independent of the base field.*

Proof. By virtue of Eq. (1), the second Betti number $\beta_2^A(k[\Delta])$ of $k[\Delta]$ over A is equal to $\sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-3}(\Delta_W; k)$, which is independent of k by the above Lemma. Q. E. D.

A ring-theoretical proof of the above Theorem is also given in [1].

References

[1] W. Bruns and J. Herzog, *On multigraded resolutions*, to appear.

[2] M. Hochster, *Cohen–Macaulay rings, combinatorics, and simplicial complexes*, in "Ring Theory II" (B. R. McDonald and R. Morris, eds.), Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, 1977, pp.171 - 223.

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