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Computation of Betti numbers of monomial ideals associated with stacked polytopes

Naoki Terai   Takayuki Hibi

Abstract

Let $P(v, d)$ be a stacked d-polytope with v vertices, $\Delta(P(v, d))$ the boundary complex of $P(v, d)$, and $k[\Delta(P(v, d))] = A/I_{\Delta(P(v,d))}$ the Stanley–Reisner ring of $\Delta(P(v, d))$ over a field k. We compute the Betti numbers which appear in a minimal free resolution of $k[\Delta(P(v, d))]$ over A, and show that every Betti number depends only on v and d and is independent of the base field k.

Introduction

Let Δ be a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$. Thus Δ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. Set $d = \max\{\sharp(\sigma); \sigma \in \Delta\}$ and define the dimension of Δ to be $\dim \Delta = d - 1$. Here $\sharp(\sigma)$ is the cardinality of a finite set σ. A maximal face of Δ is also called a *facet* of Δ.

Let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in v-variables over a field k with the standard grading, i.e., each $\deg x_i = 1$. We identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. The quotient algebra $k[\Delta] := A/I_\Delta$ is called the *Stanley–Reisner ring* of Δ over k. We may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading. We refer the reader to [Bru–Her], [H], [Hoc], [Sta] for the detailed information about Stanley–Reisner rings.

We are interested in a minimal free resolution of $k[\Delta]$ over A. Let $A(j)$, $j \in \mathbf{Z}$, denote the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. Here $\mathbf{Z}$ is the set of integers.

A graded *finite free resolution* of $k[\Delta]$ over A is an exact sequence

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (1)$$

of graded modules over A, where each $\bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{i_j}}$ is a graded free module of rank $\sum_{j\in\mathbf{Z}} \beta_{i_j}$ ($< \infty$), and where every φ_i is degree-preserving. The *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the minimal h possible in (1). It is known (see, e.g., [Bru–Her, Theorem (1.3.3), Corollary (2.2.14)]) that $v - d \leq \mathrm{hd}_A(k[\Delta]) \leq v$. A finite free resolution (1) is called *minimal* if each β_{i_j} is smallest possible. A minimal free resolution of $k[\Delta]$ over A exists and is essentially unique. See, e.g., [Bru–Her, p. 35]. When a finite free resolution (1) is minimal with $h = \mathrm{hd}_A(k[\Delta])$, we say that $\beta_{i_j} = \beta^A_{i_j}(k[\Delta])$ is the *i_j-th Betti number* of $k[\Delta]$ over A. Let $\beta_i = \beta^A_i(k[\Delta])$ denote the rank of the i-th free module which appears in a minimal free resolution of $k[\Delta]$ over A; viz, $\beta_i = \sum_{j\in\mathbf{Z}} \beta_{i_j}$.

In the paper [T–H], we give a combinatorial formula to compute the Betti numbers of the Stanley–Reisner ring of the boundary complex of the cyclic d-polytope $C(v,d)$ with v vertices and show that these Betti numbers do not depend on the base field. The purpose of the present paper is to study the same problem for a stacked d-polytope $P(v,d)$ with v vertices. In the combinatorial theory of convex polytopes, the cyclic polytope appears in the upper bound theorem and the stacked polytope appears in the lower bound theorem. See, e.g., [Bay–Lee] for background information about cyclic polytopes and stacked polytopes.

§1. Stacked polytopes and Betti numbers

We refer the reader to, e.g., [Brø] for foundations on convex polytopes. Starting with a d-simplex, one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex d-polytope with v vertices, called a *stacked polytope*. Recall that the boundary complex $\Delta(P)$ of a simplicial d-polytope $P \subset \mathbf{R}^N$ with the vertex set V is the simplicial complex on V of dimension $d-1$ whose faces are those subsets σ of V such that the convex hull of σ in $\mathbf{R}^N$ is a face of P.

Our main result in this paper is to present a combinatorial formula for the computation of the Betti numbers of the Stanley–Reisner ring associated with the boundary complex of a stacked d-polytope $P(v,d)$ with v vertices.

(1.1) THEOREM. *Fix $v > d \geq 3$. Let $P(v,d)$ be a stacked d-polytope with v vertices and $\Delta(P(v,d))$ its boundary complex. Then, a minimal free* *resolution of the Stanley–Reisner ring* $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ *over* A *is of the form*

$$\begin{aligned}
0 &\longrightarrow A(-v) \longrightarrow A(-v+d)^{b_{v-d-1}} \bigoplus A(-v+2)^{b_1} \\
&\longrightarrow A(-v+d+1)^{b_{v-d-2}} \bigoplus A(-v+3)^{b_2} \longrightarrow \cdots \\
&\longrightarrow A(-3)^{b_2} \bigoplus A(-d-1)^{b_{v-d-2}} \\
&\longrightarrow A(-2)^{b_1} \bigoplus A(-d)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta(P(v,d))] \longrightarrow 0,
\end{aligned}$$

where

$$b_i = (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2}\binom{j}{i}$$

for each $1 \leq i \leq v-d-1$.

When $d = 2$, $\Delta(P(v,2))$ is the cycle C_v with v vertices. A minimal free resolution of $k[C_v] = A/I_{C_v}$ over A is a *pure resolution*, which is discussed in, e.g., [B–H$_1$] and [B–H$_2$].

(1.2) COROLLARY. *Every Betti number of* $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ *over* A *is independent of the base field* k *and of the combinatorial type of* $P(v,d)$.

§2. Proof of Theorem (1.1)

(2.1) Given a subset W of the vertex set V of a simplicial complex Δ, the subcomplex Δ_W is defined to be $\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. Let $\tilde{H}_i(\Delta;k)$ denote the i-th reduced simplicial homology group of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta;k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\};k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Suppose that a graded finite free resolution (1) of $k[\Delta]$ over A is minimal with $h = \mathrm{hd}_A(k[\Delta])$. Then, Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W;k). \qquad (2)$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W;k).$$

(2.2) Let $P = P(v,d)$ be a stacked d-polytope with the vertex set V, $\sharp(V) = v$, $\Delta = \Delta(P)$ the boundary complex of P, and $\mathcal{F}$ a facet of P with the vertex set X. Let P' denote a stacked d-polytope with $(v+1)$-vertices which is obtained by building a shallow pyramid over $\mathcal{F}$ with a new vertex α, and Δ' the boundary complex of P'. Let $V' = V \bigcup \{\alpha\}$ be the vertex set of Δ' and W a subset of V'. We fix a base field k.

(2.2.1) LEMMA (a) *If $\alpha \notin W$ and $X \not\subset W$, then*

$$\Delta'_W = \Delta_W.$$

(b) *If $\alpha \notin W$, $W \neq V$ and $X \subset W$, then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_W; k) & (i \neq d-2) \\ \dim_k \tilde{H}_i(\Delta_W; k) + 1 & (i = d-2). \end{cases}$$

(c) *If $\alpha \in W$ and $X \bigcap W \neq \emptyset$, then, for each i, we have*

$$\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k).$$

(d) *If $\alpha \in W$, $W \neq \{\alpha\}$ and $X \bigcap W = \emptyset$, then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) & (i \neq 0) \\ \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k) + 1 & (i = 0). \end{cases}$$

Proof. (a) In general, $\Delta' = (\Delta - \{X\}) \bigcup \{\sigma \subset V' \mid \sigma \subset X \bigcup \{\alpha\}, \sigma \neq X\}$. Hence, we have $\Delta'_W = \Delta_W$ if $\alpha \notin W$ and $X \not\subset W$.

(b) Let Γ denote the set of all subsets of X and set $\partial\Gamma = \Gamma - \{X\}$. Then $\Delta'_W \bigcup \Gamma = \Delta_W$ and $\Delta'_W \bigcap \Gamma = \partial\Gamma$. Since Γ is a simplicial $(d-1)$-ball, $\partial\Gamma$ is a simplicial $(d-2)$-sphere and $\tilde{H}_{d-1}(\Delta_W; k) = 0$, the required equalities follow from the reduced Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_i(\partial\Gamma; k) &\longrightarrow \tilde{H}_i(\Gamma; k) \bigoplus \tilde{H}_i(\Delta'_W; k) \longrightarrow \tilde{H}_i(\Delta_W; k) \\ \longrightarrow \tilde{H}_{i-1}(\partial\Gamma; k) &\longrightarrow \tilde{H}_{i-1}(\Gamma; k) \bigoplus \tilde{H}_{i-1}(\Delta'_W; k) \longrightarrow \tilde{H}_{i-1}(\Delta_W; k) \\ \longrightarrow \cdots & \end{aligned}$$

(c) If $X \subset W$, then the geometric realization of Δ'_W is homeomorphic to that of $\Delta_{W-\{\alpha\}}$. Thus $\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ for each i. On the other hand, if $X \bigcap W \neq X$, then $\Delta_{W-\{\alpha\}} \bigcup \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta'_W$ and $\Delta_{W-\{\alpha\}} \bigcap \Delta'_{W\cap(\{\alpha\}\cup X)} = \Delta_{W\cap X}$. Since both $\Delta'_{W\cap(\{\alpha\}\cup X)}$ and $\Delta_{W\cap X}$ are contractible, again the reduced Mayer–Vietoris exact sequence guarantees the desired equalities.

(d) Since Δ'_W is the disjoint union of $\Delta_{W-\{\alpha\}}$ and one point $\{\alpha\}$, we immediately have the required equalities. Q. E. D.

(2.2.2) COROLLARY. *Let $\Delta = \Delta(P)$ denote the boundary complex of a stacked d-polytope $P = P(v,d)$ with the vertex set V, $\sharp(V) = v$. Then, for every non-empty subset W of V with $W \neq V$ and for each $i \neq 0, d-2$, we have*

$$\tilde{H}_i(\Delta_W; k) = 0.$$

Proof. If $v = d+1$, i.e., P is a d-simplex, then Δ_W is contractible. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each i. We now work with the same situation as in the above Lemma (2.2.1) and suppose that $\tilde{H}_i(\Delta_W; k) = 0$ for every non-empty subset W of V with $W \neq V$ and for each $i \neq 0, d-2$. Let W be a non-empty subset of V' with $W \neq V'$. If $W = V' - \{\alpha\}$, then Δ'_W is a simplicial $(d-1)$-ball. Hence, $\tilde{H}_i(\Delta_W; k) = 0$ for each i. Moreover, if $W = \{\alpha\}$, then $\tilde{H}_i(\Delta_W; k) = 0$ for each i. On the other hand, if W is a non-empty subset of V' with $W \neq V'$ such that $W \neq V$ and $W \neq \{\alpha\}$, and if $i \neq 0, d-2$, then $\dim_k \tilde{H}_i(\Delta'_W; k) = \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$ by Lemma (2.2.1). Hence, $\tilde{H}_i(\Delta_W; k) = 0$ as desired. Q. E. D.

(**2.3**) Fix $d \geq 3$, and keep the notation P, P', Δ and Δ' in (2.2). Let β_{i_j} be the i_j-th Betti number of $k[\Delta]$ and β'_{i_j} the i_j-th Betti number of $k[\Delta']$.

(2.3.1) LEMMA. *For each $i \geq 1$ we have*

$$\beta'_{i_{i+1}} = \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}.$$

Proof. By virtue of Eq. (2) as well as Lemma (2.2.1), we have

$$\begin{aligned}
\beta'_{i_{i+1}} &= \sum_{W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{\alpha \notin W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) + \sum_{\alpha \in W \subset V',\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\
&= \sum_{W \subset V,\ \sharp(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) + \sum_{W \subset V,\ \sharp(W)=i} \dim_k \tilde{H}_0(\Delta_W; k) + \binom{v-d}{i} \\
&= \beta_{i_{i+1}} + \beta_{i-1_i} + \binom{v-d}{i}
\end{aligned}$$

as desired. Q. E. D.

(2.3.2) COROLLARY. *Let $\Delta = \Delta(P)$ denote the boundary complex of a stacked d-polytope $P = P(v,d)$, $d \geq 3$, with v vertices. Then, for each $1 \leq i \leq v-d-1$, the i_{i+1}-th Betti number of $k[\Delta] = A/I_\Delta$ over A is*

$$\beta^A_{i_{i+1}}(k[\Delta]) = (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2}\binom{j}{i}.$$

Proof. In this proof, we set $\binom{a}{0} = 0$ for every integer $a \geq 0$. Thanks to Lemma (2.3.1), we have

$$\begin{aligned}
\beta^A_{i_{i+1}}(k[\Delta]) &= (v-d-2)\binom{v-d-2}{i} - \sum_{j=i}^{v-d-3}\binom{j}{i} \\
&\quad +(v-d-2)\binom{v-d-2}{i-1} - \sum_{j=i-1}^{v-d-3}\binom{j}{i-1} + \binom{v-d-1}{i} \\
&= (v-d-2)(\binom{v-d-2}{i} + \binom{v-d-2}{i-1}) + \binom{v-d-1}{i} \\
&\quad -(\binom{i-1}{i-1} + \sum_{j=i}^{v-d-3}(\binom{j}{i} + \binom{j}{i-1})) \\
&= (v-d-2)\binom{v-d-1}{i} + \binom{v-d-1}{i} \\
&\quad -(\binom{i}{i} + \sum_{j=i}^{v-d-3}\binom{j+1}{i}) \\
&= (v-d-1)\binom{v-d-1}{i} - \sum_{j=i}^{v-d-2}\binom{j}{i}
\end{aligned}$$

as required. Q. E. D.

(**2.4**) We are now in the position to give a proof of Theorem (1.1). Since $\Delta = \Delta(P(v,d))$ is a simplicial $(d-1)$-sphere with v vertices, we know that the homological dimension of $k[\Delta] = A/I_\Delta$ over A is $\mathrm{hd}_A(k[\Delta]) = v-d$ and that $\beta^A_{i_j}(k[\Delta]) = \beta^A_{v-d-i_{v-j}}(k[\Delta])$ for every i and j. By Corollary (2.2.2), we have $\beta^A_{i_j}(k[\Delta]) = 0$ for each $1 \leq i \leq v-d-1$ and for each $j \neq i+1, i+d-1$. On the other hand, Corollary (2.3.2) enables us to compute $b_i = \beta^A_{i_{i+1}}(k[\Delta]) = \beta^A_{v-d-i_{v-i-1}}(k[\Delta])$ for each $1 \leq i \leq v-d-1$. Hence, we obtain a desired minimal free resolution of $k[\Delta]$ over A.

§3. Unimodality of Betti number sequences

Let β_i, $0 \le i \le v-d$, denote the rank of the i-th free module which appears in a minimal free resolution of $k[\Delta(P(v,d))] = A/I_{\Delta(P(v,d))}$ over A. Then, $\beta_0 = \beta_{v-d} = 1$ and $\beta_i = b_i + b_{v-d-i}$ for each $1 \le i \le v-d-1$ with the notation of Theorem (1.1). The sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ is called the *Betti number sequence* of $k[\Delta(P(v,d))]$ over A. This sequence is *symmetric*, i.e., $\beta_i = \beta_{v-d-i}$ for every $0 \le i \le v-d$. We now show that the symmetric sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ is *unimodal*, i.e., $\beta_0 \le \beta_1 \le \cdots \le \beta_{[(v-d)/2]}$.

The following Lemma (3.1) follows from a simple combinatorial argument based on Lemma (2.3.1).

(3.1) LEMMA. (a) *If $v-d$ is even, then*

$$b_{\frac{v-d}{2}} \ge b_{\frac{v-d}{2}-1} \ge b_{\frac{v-d}{2}+1} \ge \cdots \ge b_1 \ge b_{v-d-1}.$$

(b) *If $v-d$ is odd, then*

$$b_{\frac{v-d-1}{2}} \ge b_{\frac{v-d-1}{2}+1} \ge b_{\frac{v-d-1}{2}-1} \ge \cdots \ge b_1 \ge b_{v-d-1}.$$

(3.2) COROLLARY. *Fix $v > d \ge 3$. Let $P = P(v,d)$ a stacked d-polytope with v vertices and $\Delta = \Delta(P)$ its boundary complex. Then, the Betti number sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ of $k[\Delta] = A/I_\Delta$ over A is unimodal.*

Proof. By Lemma (3.1), we have

$$1 \le b_1 \le b_2 \le \cdots \le b_{[\frac{v-d}{2}]} \ge b_{[\frac{v-d}{2}]+1} \ge \cdots \ge b_{v-d-1} \ge 1.$$

Hence, the Betti number sequence $(\beta_0, \beta_1, \cdots, \beta_{v-d})$ of $k[\Delta] = A/I_\Delta$ over A is unimodal. Q. E. D.

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