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# Computation of Betti numbers of monomial ideals associated with stacked polytopes

Naoki Terai      Takayuki Hibi

## Abstract

Let  $P(v, d)$  be a stacked  $d$ -polytope with  $v$  vertices,  $\Delta(P(v, d))$  the boundary complex of  $P(v, d)$ , and  $k[\Delta(P(v, d))] = A/I_{\Delta(P(v, d))}$  the Stanley–Reisner ring of  $\Delta(P(v, d))$  over a field  $k$ . We compute the Betti numbers which appear in a minimal free resolution of  $k[\Delta(P(v, d))]$  over  $A$ , and show that every Betti number depends only on  $v$  and  $d$  and is independent of the base field  $k$ .

## Introduction

Let  $\Delta$  be a simplicial complex on the vertex set  $V = \{x_1, x_2, \dots, x_v\}$ . Thus  $\Delta$  is a collection of subsets of  $V$  such that (i)  $\{x_i\} \in \Delta$  for every  $1 \leq i \leq v$  and (ii)  $\sigma \in \Delta, \tau \subset \sigma \Rightarrow \tau \in \Delta$ . Each element  $\sigma$  of  $\Delta$  is called a *face* of  $\Delta$ . Set  $d = \max\{\#\sigma; \sigma \in \Delta\}$  and define the dimension of  $\Delta$  to be  $\dim \Delta = d - 1$ . Here  $\#\sigma$  is the cardinality of a finite set  $\sigma$ . A maximal face of  $\Delta$  is also called a *facet* of  $\Delta$ .

Let  $A = k[x_1, x_2, \dots, x_v]$  denote the polynomial ring in  $v$ -variables over a field  $k$  with the standard grading, i.e., each  $\deg x_i = 1$ . We identify each  $x_i \in V$  with the indeterminate  $x_i$  of  $A$ . Define  $I_{\Delta}$  to be the ideal of  $A$  which is generated by square-free monomials  $x_{i_1} x_{i_2} \cdots x_{i_r}, 1 \leq i_1 < i_2 < \cdots < i_r \leq v$ , with  $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} \notin \Delta$ . The quotient algebra  $k[\Delta] := A/I_{\Delta}$  is called the *Stanley–Reisner ring* of  $\Delta$  over  $k$ . We may regard  $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$  as a graded module over  $A$  with the quotient grading. We refer the reader to [Bru–Her], [H], [Hoc], [Sta] for the detailed information about Stanley–Reisner rings.

We are interested in a minimal free resolution of  $k[\Delta]$  over  $A$ . Let  $A(j), j \in \mathbf{Z}$ , denote the graded module  $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$  over  $A$  with  $[A(j)]_n := A_{n+j}$ . Here  $\mathbf{Z}$  is the set of integers.

A graded finite free resolution of  $k[\Delta]$  over  $A$  is an exact sequence

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{hj}} \xrightarrow{\varphi_h} \dots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{1j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0 \quad (1)$$

of graded modules over  $A$ , where each  $\bigoplus_{j \in \mathbb{Z}} A(-j)^{\beta_{ij}}$  is a graded free module of rank  $\sum_{j \in \mathbb{Z}} \beta_{ij}$  ( $< \infty$ ), and where every  $\varphi_i$  is degree-preserving. The homological dimension  $\text{hd}_A(k[\Delta])$  of  $k[\Delta]$  over  $A$  is the minimal  $h$  possible in (1). It is known (see, e.g., [Bru-Her, Theorem (1.3.3), Corollary (2.2.14)]) that  $v - d \leq \text{hd}_A(k[\Delta]) \leq v$ . A finite free resolution (1) is called *minimal* if each  $\beta_{ij}$  is smallest possible. A minimal free resolution of  $k[\Delta]$  over  $A$  exists and is essentially unique. See, e.g., [Bru-Her, p. 35]. When a finite free resolution (1) is minimal with  $h = \text{hd}_A(k[\Delta])$ , we say that  $\beta_{ij} = \beta_{ij}^A(k[\Delta])$  is the  $i_j$ -th Betti number of  $k[\Delta]$  over  $A$ . Let  $\beta_i = \beta_i^A(k[\Delta])$  denote the rank of the  $i$ -th free module which appears in a minimal free resolution of  $k[\Delta]$  over  $A$ ; viz,  $\beta_i = \sum_{j \in \mathbb{Z}} \beta_{ij}$ .

In the paper [T-H], we give a combinatorial formula to compute the Betti numbers of the Stanley-Reisner ring of the boundary complex of the cyclic  $d$ -polytope  $C(v, d)$  with  $v$  vertices and show that these Betti numbers do not depend on the base field. The purpose of the present paper is to study the same problem for a stacked  $d$ -polytope  $P(v, d)$  with  $v$  vertices. In the combinatorial theory of convex polytopes, the cyclic polytope appears in the upper bound theorem and the stacked polytope appears in the lower bound theorem. See, e.g., [Bay-Lee] for background information about cyclic polytopes and stacked polytopes.

## §1. Stacked polytopes and Betti numbers

We refer the reader to, e.g., [Brø] for foundations on convex polytopes. Starting with a  $d$ -simplex, one can add new vertices by building shallow pyramids over facets to obtain a simplicial convex  $d$ -polytope with  $v$  vertices, called a *stacked polytope*. Recall that the boundary complex  $\Delta(P)$  of a simplicial  $d$ -polytope  $P \subset \mathbb{R}^N$  with the vertex set  $V$  is the simplicial complex on  $V$  of dimension  $d - 1$  whose faces are those subsets  $\sigma$  of  $V$  such that the convex hull of  $\sigma$  in  $\mathbb{R}^N$  is a face of  $P$ .

Our main result in this paper is to present a combinatorial formula for the computation of the Betti numbers of the Stanley-Reisner ring associated with the boundary complex of a stacked  $d$ -polytope  $P(v, d)$  with  $v$  vertices.

(1.1) THEOREM. Fix  $v > d \geq 3$ . Let  $P(v, d)$  be a stacked  $d$ -polytope with  $v$  vertices and  $\Delta(P(v, d))$  its boundary complex. Then, a minimal free

resolution of the Stanley-Reisner ring  $k[\Delta(P(v, d))] = A/I_{\Delta(P(v, d))}$  over  $A$  is of the form

$$\begin{aligned} 0 &\longrightarrow A(-v) \longrightarrow A(-v+d)^{b_{v-d-1}} \oplus A(-v+2)^{b_1} \\ &\longrightarrow A(-v+d+1)^{b_{v-d-2}} \oplus A(-v+3)^{b_2} \longrightarrow \dots \\ &\longrightarrow A(-3)^{b_2} \oplus A(-d-1)^{b_{v-d-2}} \\ &\longrightarrow A(-2)^{b_1} \oplus A(-d)^{b_{v-d-1}} \longrightarrow A \longrightarrow k[\Delta(P(v, d))] \longrightarrow 0, \end{aligned}$$

where

$$b_i = (v-d-1) \binom{v-d-1}{i} - \sum_{j=i}^{v-d-2} \binom{j}{i}$$

for each  $1 \leq i \leq v-d-1$ .

When  $d=2$ ,  $\Delta(P(v, 2))$  is the cycle  $C_v$  with  $v$  vertices. A minimal free resolution of  $k[C_v] = A/I_{C_v}$  over  $A$  is a *pure resolution*, which is discussed in, e.g., [B-H<sub>1</sub>] and [B-H<sub>2</sub>].

(1.2) COROLLARY. *Every Betti number of  $k[\Delta(P(v, d))] = A/I_{\Delta(P(v, d))}$  over  $A$  is independent of the base field  $k$  and of the combinatorial type of  $P(v, d)$ .*

## §2. Proof of Theorem (1.1)

(2.1) Given a subset  $W$  of the vertex set  $V$  of a simplicial complex  $\Delta$ , the subcomplex  $\Delta_W$  is defined to be  $\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$ . In particular,  $\Delta_V = \Delta$  and  $\Delta_\emptyset = \{\emptyset\}$ . Let  $\tilde{H}_i(\Delta; k)$  denote the  $i$ -th reduced simplicial homology group of  $\Delta$  with the coefficient field  $k$ . Note that  $\tilde{H}_{-1}(\Delta; k) = 0$  if  $\Delta \neq \{\emptyset\}$  and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

Suppose that a graded finite free resolution (1) of  $k[\Delta]$  over  $A$  is minimal with  $h = \text{hd}_A(k[\Delta])$ . Then, Hochster's formula [Hoc, Theorem (5.1)] guarantees that

$$\beta_{i_j} = \sum_{W \subset V, \#(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \quad (2)$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\#(W)-i-1}(\Delta_W; k).$$

(2.2) Let  $P = P(v, d)$  be a stacked  $d$ -polytope with the vertex set  $V$ ,  $\sharp(V) = v$ ,  $\Delta = \Delta(P)$  the boundary complex of  $P$ , and  $\mathcal{F}$  a facet of  $P$  with the vertex set  $X$ . Let  $P'$  denote a stacked  $d$ -polytope with  $(v + 1)$ -vertices which is obtained by building a shallow pyramid over  $\mathcal{F}$  with a new vertex  $\alpha$ , and  $\Delta'$  the boundary complex of  $P'$ . Let  $V' = V \cup \{\alpha\}$  be the vertex set of  $\Delta'$  and  $W$  a subset of  $V'$ . We fix a base field  $k$ .

(2.2.1) LEMMA (a) *If  $\alpha \notin W$  and  $X \not\subset W$ , then*

$$\Delta'_W = \Delta_W.$$

(b) *If  $\alpha \notin W$ ,  $W \neq V$  and  $X \subset W$ , then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_W; k) & (i \neq d - 2) \\ \dim_k \tilde{H}_i(\Delta_W; k) + 1 & (i = d - 2). \end{cases}$$

(c) *If  $\alpha \in W$  and  $X \cap W \neq \emptyset$ , then, for each  $i$ , we have*

$$\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W - \{\alpha\}}; k).$$

(d) *If  $\alpha \in W$ ,  $W \neq \{\alpha\}$  and  $X \cap W = \emptyset$ , then*

$$\dim_k \tilde{H}_i(\Delta'_W; k) = \begin{cases} \dim_k \tilde{H}_i(\Delta_{W - \{\alpha\}}; k) & (i \neq 0) \\ \dim_k \tilde{H}_i(\Delta_{W - \{\alpha\}}; k) + 1 & (i = 0). \end{cases}$$

*Proof.* (a) In general,  $\Delta' = (\Delta - \{X\}) \cup \{\sigma \subset V' \mid \sigma \subset X \cup \{\alpha\}, \sigma \neq X\}$ . Hence, we have  $\Delta'_W = \Delta_W$  if  $\alpha \notin W$  and  $X \not\subset W$ .

(b) Let  $\Gamma$  denote the set of all subsets of  $X$  and set  $\partial\Gamma = \Gamma - \{X\}$ . Then  $\Delta'_W \cup \Gamma = \Delta_W$  and  $\Delta'_W \cap \Gamma = \partial\Gamma$ . Since  $\Gamma$  is a simplicial  $(d - 1)$ -ball,  $\partial\Gamma$  is a simplicial  $(d - 2)$ -sphere and  $\tilde{H}_{d-1}(\Delta_W; k) = 0$ , the required equalities follow from the reduced Mayer-Vietoris exact sequence

$$\begin{aligned} \dots &\longrightarrow \tilde{H}_i(\partial\Gamma; k) \longrightarrow \tilde{H}_i(\Gamma; k) \oplus \tilde{H}_i(\Delta'_W; k) \longrightarrow \tilde{H}_i(\Delta_W; k) \\ &\longrightarrow \tilde{H}_{i-1}(\partial\Gamma; k) \longrightarrow \tilde{H}_{i-1}(\Gamma; k) \oplus \tilde{H}_{i-1}(\Delta'_W; k) \longrightarrow \tilde{H}_{i-1}(\Delta_W; k) \\ &\longrightarrow \dots \end{aligned}$$

(c) If  $X \subset W$ , then the geometric realization of  $\Delta'_W$  is homeomorphic to that of  $\Delta_{W - \{\alpha\}}$ . Thus  $\tilde{H}_i(\Delta'_W; k) \cong \tilde{H}_i(\Delta_{W - \{\alpha\}}; k)$  for each  $i$ . On the other hand, if  $X \cap W \neq X$ , then  $\Delta_{W - \{\alpha\}} \cup \Delta'_{W \cap (\{\alpha\} \cup X)} = \Delta'_W$  and  $\Delta_{W - \{\alpha\}} \cap \Delta'_{W \cap (\{\alpha\} \cup X)} = \Delta_{W \cap X}$ . Since both  $\Delta'_{W \cap (\{\alpha\} \cup X)}$  and  $\Delta_{W \cap X}$  are contractible, again the reduced Mayer-Vietoris exact sequence guarantees the desired equalities.

(d) Since  $\Delta'_W$  is the disjoint union of  $\Delta_{W-\{\alpha\}}$  and one point  $\{\alpha\}$ , we immediately have the required equalities. Q. E. D.

(2.2.2) COROLLARY. Let  $\Delta = \Delta(P)$  denote the boundary complex of a stacked  $d$ -polytope  $P = P(v, d)$  with the vertex set  $V$ ,  $\#(V) = v$ . Then, for every non-empty subset  $W$  of  $V$  with  $W \neq V$  and for each  $i \neq 0, d-2$ , we have

$$\tilde{H}_i(\Delta_W; k) = 0.$$

*Proof.* If  $v = d + 1$ , i.e.,  $P$  is a  $d$ -simplex, then  $\Delta_W$  is contractible. Hence,  $\tilde{H}_i(\Delta_W; k) = 0$  for each  $i$ . We now work with the same situation as in the above Lemma (2.2.1) and suppose that  $\tilde{H}_i(\Delta_W; k) = 0$  for every non-empty subset  $W$  of  $V$  with  $W \neq V$  and for each  $i \neq 0, d-2$ . Let  $W$  be a non-empty subset of  $V'$  with  $W \neq V'$ . If  $W = V' - \{\alpha\}$ , then  $\Delta'_W$  is a simplicial  $(d-1)$ -ball. Hence,  $\tilde{H}_i(\Delta'_W; k) = 0$  for each  $i$ . Moreover, if  $W = \{\alpha\}$ , then  $\tilde{H}_i(\Delta'_W; k) = 0$  for each  $i$ . On the other hand, if  $W$  is a non-empty subset of  $V'$  with  $W \neq V'$  such that  $W \neq V$  and  $W \neq \{\alpha\}$ , and if  $i \neq 0, d-2$ , then  $\dim_k \tilde{H}_i(\Delta'_W; k) = \dim_k \tilde{H}_i(\Delta_{W-\{\alpha\}}; k)$  by Lemma (2.2.1). Hence,  $\tilde{H}_i(\Delta'_W; k) = 0$  as desired. Q. E. D.

(2.3) Fix  $d \geq 3$ , and keep the notation  $P, P', \Delta$  and  $\Delta'$  in (2.2). Let  $\beta_i$  be the  $i_j$ -th Betti number of  $k[\Delta]$  and  $\beta'_i$  the  $i_j$ -th Betti number of  $k[\Delta']$ .

(2.3.1) LEMMA. For each  $i \geq 1$  we have

$$\beta'_{i+1} = \beta_{i+1} + \beta_{i-1} + \binom{v-d}{i}.$$

*Proof.* By virtue of Eq. (2) as well as Lemma (2.2.1), we have

$$\begin{aligned} \beta'_{i+1} &= \sum_{W \subset V', \#(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\ &= \sum_{\alpha \notin W \subset V', \#(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) + \sum_{\alpha \in W \subset V', \#(W)=i+1} \dim_k \tilde{H}_0(\Delta'_W; k) \\ &= \sum_{W \subset V, \#(W)=i+1} \dim_k \tilde{H}_0(\Delta_W; k) + \sum_{W \subset V, \#(W)=i} \dim_k \tilde{H}_0(\Delta_W; k) + \binom{v-d}{i} \\ &= \beta_{i+1} + \beta_{i-1} + \binom{v-d}{i} \end{aligned}$$

as desired.

Q. E. D.



(2.3.2) COROLLARY. Let  $\Delta = \Delta(P)$  denote the boundary complex of a stacked  $d$ -polytope  $P = P(v, d)$ ,  $d \geq 3$ , with  $v$  vertices. Then, for each  $1 \leq i \leq v - d - 1$ , the  $i_{i+1}$ -th Betti number of  $k[\Delta] = A/I_\Delta$  over  $A$  is

$$\beta_{i_{i+1}}^A(k[\Delta]) = (v - d - 1) \binom{v - d - 1}{i} - \sum_{j=i}^{v-d-2} \binom{j}{i}.$$

*Proof.* In this proof, we set  $\binom{a}{0} = 0$  for every integer  $a \geq 0$ . Thanks to Lemma (2.3.1), we have

$$\begin{aligned} \beta_{i_{i+1}}^A(k[\Delta]) &= (v - d - 2) \binom{v - d - 2}{i} - \sum_{j=i}^{v-d-3} \binom{j}{i} \\ &\quad + (v - d - 2) \binom{v - d - 2}{i-1} - \sum_{j=i-1}^{v-d-3} \binom{j}{i-1} + \binom{v - d - 1}{i} \\ &= (v - d - 2) \left( \binom{v - d - 2}{i} + \binom{v - d - 2}{i-1} \right) + \binom{v - d - 1}{i} \\ &\quad - \left( \binom{i-1}{i-1} + \sum_{j=i}^{v-d-3} \left( \binom{j}{i} + \binom{j}{i-1} \right) \right) \\ &= (v - d - 2) \binom{v - d - 1}{i} + \binom{v - d - 1}{i} \\ &\quad - \left( \binom{i}{i} + \sum_{j=i}^{v-d-3} \binom{j+1}{i} \right) \\ &= (v - d - 1) \binom{v - d - 1}{i} - \sum_{j=i}^{v-d-2} \binom{j}{i} \end{aligned}$$

as required.

Q. E. D.

(2.4) We are now in the position to give a proof of Theorem (1.1). Since  $\Delta = \Delta(P(v, d))$  is a simplicial  $(d-1)$ -sphere with  $v$  vertices, we know that the homological dimension of  $k[\Delta] = A/I_\Delta$  over  $A$  is  $\text{hd}_A(k[\Delta]) = v - d$  and that  $\beta_{i_j}^A(k[\Delta]) = \beta_{v-d-i_{v-j}}^A(k[\Delta])$  for every  $i$  and  $j$ . By Corollary (2.2.2), we have  $\beta_{i_j}^A(k[\Delta]) = 0$  for each  $1 \leq i \leq v - d - 1$  and for each  $j \neq i + 1, i + d - 1$ . On the other hand, Corollary (2.3.2) enables us to compute  $b_i = \beta_{i_{i+1}}^A(k[\Delta]) = \beta_{v-d-i_{v-i-1}}^A(k[\Delta])$  for each  $1 \leq i \leq v - d - 1$ . Hence, we obtain a desired minimal free resolution of  $k[\Delta]$  over  $A$ .

### §3. Unimodality of Betti number sequences

Let  $\beta_i$ ,  $0 \leq i \leq v - d$ , denote the rank of the  $i$ -th free module which appears in a minimal free resolution of  $k[\Delta(P(v, d))] = A/I_{\Delta(P(v, d))}$  over  $A$ . Then,  $\beta_0 = \beta_{v-d} = 1$  and  $\beta_i = b_i + b_{v-d-i}$  for each  $1 \leq i \leq v - d - 1$  with the notation of Theorem (1.1). The sequence  $(\beta_0, \beta_1, \dots, \beta_{v-d})$  is called the *Betti number sequence* of  $k[\Delta(P(v, d))]$  over  $A$ . This sequence is *symmetric*, i.e.,  $\beta_i = \beta_{v-d-i}$  for every  $0 \leq i \leq v - d$ . We now show that the symmetric sequence  $(\beta_0, \beta_1, \dots, \beta_{v-d})$  is *unimodal*, i.e.,  $\beta_0 \leq \beta_1 \leq \dots \leq \beta_{\lfloor (v-d)/2 \rfloor}$ .

The following Lemma (3.1) follows from a simple combinatorial argument based on Lemma (2.3.1).

(3.1) LEMMA. (a) *If  $v - d$  is even, then*

$$b_{\frac{v-d}{2}} \geq b_{\frac{v-d}{2}-1} \geq b_{\frac{v-d}{2}+1} \geq \dots \geq b_1 \geq b_{v-d-1}.$$

(b) *If  $v - d$  is odd, then*

$$b_{\frac{v-d-1}{2}} \geq b_{\frac{v-d-1}{2}+1} \geq b_{\frac{v-d-1}{2}-1} \geq \dots \geq b_1 \geq b_{v-d-1}.$$

(3.2) COROLLARY. *Fix  $v > d \geq 3$ . Let  $P = P(v, d)$  a stacked  $d$ -polytope with  $v$  vertices and  $\Delta = \Delta(P)$  its boundary complex. Then, the Betti number sequence  $(\beta_0, \beta_1, \dots, \beta_{v-d})$  of  $k[\Delta] = A/I_{\Delta}$  over  $A$  is unimodal.*

*Proof.* By Lemma (3.1), we have

$$1 \leq b_1 \leq b_2 \leq \dots \leq b_{\lfloor \frac{v-d}{2} \rfloor} \geq b_{\lfloor \frac{v-d}{2} \rfloor + 1} \geq \dots \geq b_{v-d-1} \geq 1.$$

Hence, the Betti number sequence  $(\beta_0, \beta_1, \dots, \beta_{v-d})$  of  $k[\Delta] = A/I_{\Delta}$  over  $A$  is unimodal. Q. E. D.

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