Finite free resolutions and 1-skeletons of simplicial$(d-1)$-spheres

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Finite free resolutions and 1-skeletons of simplicial $(d-1)$-spheres

Naoki Terai Takayuki Hibi

A simplicial complex Δ on the vertex set V is a collection of subsets of V such that (i) $\{x\} \in \Delta$ for every $x \in V$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. The 1-*skeleton* $\Delta^{(1)}$ of Δ is a graph on V whose edges are all 2-element subsets $\{x, y\}$ of V with $\{x, y\} \in \Delta$. We say that a simplicial complex Δ is a *simplicial* $(d-1)$-*sphere* if the geometric realization of Δ is homeomorphic to the $(d-1)$-sphere.

The purpose of the present paper is to give a ring-theoretical short proof of the following result, which was first proved by Barnette [1].

THEOREM. *The* 1-*skeleton of a simplicial* $(d-1)$-*sphere is d-connected.*

Given a subset W of the vertex set $V = \{x_1, x_2, \ldots, x_v\}$ of a simplicial complex Δ, the restriction of Δ to W is the subcomplex $\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$ of Δ. Let $\tilde{H}_i(\Delta; k)$ denote the i-th reduced simplicial homology group of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$, $\tilde{H}_{-1}(\{\emptyset\}; k) \cong k$, and $\tilde{H}_i(\{\emptyset\}; k) = 0$ for each $i \geq 0$.

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley–Reisner ring* of Δ over k. In what follows, we consider A to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading. Let $\mathbf{Z}$ denote the set of integers. We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$.

We study a graded minimal free resolution

$$0 \longrightarrow \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j\in\mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0$$

of $k[\Delta]$ over A. Here $h = \mathrm{hd}_A(k[\Delta])$ is the homological dimension of $k[\Delta]$ over A. It is known [3, Theorem (5.1)] that

$$\beta_{i_j} = \sum_{W\subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k), \tag{1}$$

where $\sharp(W)$ is the cardinality of a finite set W.

Now, suppose that Δ is a simplicial $(d-1)$-sphere on the vertex set V with $\sharp(V) = v$. Then $k[\Delta]$ is Cohen–Macaulay, i.e., $\mathrm{hd}_A(k[\Delta]) = v - d$. Moreover, since $k[\Delta]$ is Gorenstein, $\beta_{h_j} = 0$ if $j \neq v$ and $\beta_{h_v} = 1$. In particular, $\beta_{(v-d)_{v-(d-1)}} = 0$. Hence, it follows from Eq. (1) that $\tilde{H}_0(\Delta_W; k) = 0$ for every subset W of V with $\sharp(W) = v - (d-1)$. Thus $\mid \Delta_{V-W} \mid$ is connected for every subset W of V with $\sharp(W) = d-1$. Hence, the 1-skeleton $\Delta^{(1)}$ of Δ is d-connected.

The above ring-theoretical technique enables us to show that the 1-skeleton of a level complex Δ ([3], [6]) of dimension $d-1$ with v vertices is d-connected if $\sharp(\{\sigma \in \Delta \mid \sharp(\sigma) = d\}) \neq v - d + 1$.

We refer the reader to [2], [4], [5] and [7] for the detailed information about algebra and combinatorics on Stanley–Reisner rings.

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