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Series ♯282. March 1995

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# Asymptotic behavior of the first exit time of randomly perturbed dynamical systems with a repulsive equilibrium point

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Summary

We consider small random perturbations of dynamical systems $\{X^\varepsilon(t,x)\}_{t\geq 0, x\in R^d}$ ($\varepsilon > 0$) on a d-dimensional Euclidean space $R^d$ when the origin $o \in R^d$ is a repulsive equilibrium point of unperturbed dynamical systems and when $o$ is not exponentially unstable. The weak law of large numbers, as $\varepsilon \to 0$, on the first exit time $\tau_D^\varepsilon(x)$ of $\{X^\varepsilon(t,x)\}_{t\geq 0}$ from a bounded domain $D(\ni o)$ of $R^d$ and the asymptotic behavior of $E[\tau_D^\varepsilon(x)]$, as $\varepsilon \to 0$, are given.

## 0. Introduction.

Let $X^\varepsilon(t,x)$ ($t \geq 0, x \in R^d, \varepsilon > 0$) be the solution of the following stochastic differential equation:

$$(0.1).\qquad \begin{aligned} dX^\varepsilon(t,x) &= b(X^\varepsilon(t,x))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t,x))dW(t), \\ X^\varepsilon(0,x) &= x, \end{aligned}$$

where $b(\cdot) = (b^i(\cdot))_{i=1}^d : R^d \mapsto R^d$ is Lipschitz continuous, where $\sigma(\cdot) = (\sigma^{ij}(\cdot))_{i,j=1}^d : R^d \mapsto M_d(R)$ is bounded, Lipschitz continuous, and uniformly nondegenerate, and where $W(\cdot)$ is a d-dimensional Wiener process (see [11]). $X^\varepsilon(t,x)$ can be considered as the small random perturbations of $X^0(t,x)$ for small $\varepsilon$ (see [8]).

Let $D(\subset R^d)$ be a bounded domain which contains the origin $o$ and suppose that $b(x) = o$ if and only if $x = o$. The asymptotic behavior of the first exit time $\tau_D^\varepsilon(x)$ of $X^\varepsilon(t,x)$ from $D$ defined by

$$(0.2).\qquad \tau_D^\varepsilon(x) \equiv \inf\{t > 0; X^\varepsilon(t,x) \notin D\}$$

has been studied by many authors.

The first result on the asymptotic behavior of $\tau_D^\varepsilon(x)$ as $\varepsilon \to 0$ was given by M. I. Freidlin and A. D. Wentzell.

**Theorem 0.1.** *([8], p. 127, Theorems 4.1 and 4.2). Suppose that the origin $o$ is an asymptotically stable equilibrium point, and that $X^0(t,x) \in D$ $(t > 0)$ and $\lim_{t\to\infty} X^0(t,x) = o$ for all $x \in \overline{D}$. Then the following holds; for any $x \in D$ and $\delta > 0$,*

$$(0.3). \quad \begin{aligned} &\lim_{\varepsilon\to 0} P(\exp((V_D - \delta)/\varepsilon) < \tau_D^\varepsilon(x) < \exp((V_D + \delta)/\varepsilon)) = 1, \\ &\lim_{\varepsilon\to 0} \varepsilon \log E[\tau_D^\varepsilon(x)] = V_D, \end{aligned}$$

where we put

$$(0.4). \quad \begin{aligned} V_D \equiv \inf\{ &\int_0^t |\sigma(\varphi(s))^{-1}(d\varphi(s)/ds - b(\varphi(s)))|^2 ds/2; \varphi(0) = o, \varphi(t) \in \partial D, \\ &\{\varphi(s); 0 \le s < t\} \subset D, t > 0\} \end{aligned}$$

(see also [5], [7], [18]).

If the origin $o$ is not asymptotically stable, then the asymptotic behavior of $\tau_D^\varepsilon(x)$ as $\varepsilon \to 0$ was first studied by Y. Kifer (see [12]). To explain his result, let us give some notation. Put $A_1 \equiv \{x \in \overline{D};$ there exists $s = s(x) \le 0$ such that $X^0(t,x) \notin \overline{D}$ for $t < s$ and such that $X^0(t,x) \in D$ for $t > s$. $X^0(t,x) \to o$ as $t \to \infty\}$; $A_2 \equiv \{x \in \overline{D};$ there exists $s = s(x) \ge 0$ such that $X^0(t,x) \notin \overline{D}$ for $t > s$ and such that $X^0(t,x) \in D$ for $t < s$. $X^0(t,x) \to o$ as $t \to -\infty\}$; $A_3 \equiv \{x \in \overline{D};$ there exist $s_1 = s_1(x) \ge 0 \ge s_2 = s_2(x)$ such that $X^0(t,x) \notin \overline{D}$ for $s_2 > t$ and $t > s_1$ and such that $X^0(t,x) \in D$ for $s_2 < t < s_1\}$.

The following is the assumption in [12].

(A.D). $D$ has a $C^2$-boundary $\partial D$. $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$ and $A_2 \cup A_3$ is not empty. $b(\cdot)$ is differentiable at $o$. The eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ have non-zero real parts.

Denote by $\lambda$ the maximum of the real parts of the eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ which is positive under (A.D). Then the following is known.

**Theorem 0.2.** *([12], Theorems 2.1 and 2.2). Suppose that (A.D) holds. Then for any $\delta > 0$ and $x \in A_1 \cup \{o\} \backslash \partial D$,*

$$(0.5). \quad \begin{aligned} &\lim_{\varepsilon\to 0} P(|\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)}) - 1| < \delta) = 1, \\ &\lim_{\varepsilon\to 0} E[\tau_D^\varepsilon(x)]/\log(\varepsilon^{-1/2}) = \lambda, \end{aligned}$$

and for any $\delta > 0$ and $x \in A_2 \cup A_3 \backslash \partial D$,

$$(0.6). \quad \begin{aligned} &\lim_{\varepsilon\to 0} P(|\tau_D^\varepsilon(x)/\tau_D^0(x) - 1| < \delta) = 1, \\ &\lim_{\varepsilon\to 0} E[\tau_D^\varepsilon(x)] = \tau_D^0(x). \end{aligned}$$

The reader can find the large deviations results for $\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)})$ in [16], and the central limit theorem for $\tau_D^\varepsilon(x) - \log(\varepsilon^{-1/(2\lambda)})$ in [4].

The condition that $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$ is a topological one, and it does not imply that the eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ have non-zero real parts. In fact, in Theorem 0.1, they do not suppose it. For instance, in Theorem 0.1, $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ can be a zero matrix in which case the origin is not exponentially stable (see [10]). As a typical example, put $d = 1$, $D = (-1,1)$ and $b(x) = -x^3$ for $|x| \leq 1$ in which case $db(o)/dx = o$ and $X^0(t,x) = x(1+2tx^2)^{-1/2}$ satisfies the assumption in Theorem 0.1.

Therefore the following problem comes out natually; study the asymptotic behavior of $\tau_D^\varepsilon(x)$ as $\varepsilon \to 0$ when $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$ with $A_2 \cup A_3 \neq \emptyset$, and when $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ is a zero matrix.

In this paper we study the asymptotic behavior of $\tau_D^\varepsilon(x)$ when $\overline{D} = \{o\} \cup A_2$ in which case the origin $o$ is a repulsive equilibrium point, and when $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ is a zero matrix.

In section 1, we state our result. In section 2, we give lemmas which are necessary for the proof of our result. In section 3, we prove our result. In section 4, we consider the special class of the case $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$ as an application of the results in section 1.

## 1. Main result.

In this section we give our main result.

Let us first introduce our assumptions.

(A.0). $D(\subset R^d)$ is a bounded domain which contains $o$. $b(x) = o$ if and only if $x = o$.

(A.1). There exist positive constants $\ell$ and $C_1$ such that for $x \in D$

$$|b(x)| \leq C_1 |x|^{\ell+1}. \tag{1.1}$$

(A.2). $\overline{D} = \{o\} \cup A_2$, and there exist positive constants $\delta_o$, $\ell$ and $C_2$ such that for $x$ for which $|x| < \delta_o$,

$$< x, b(x) > \geq C_2 |x|^{\ell+2}. \tag{1.2}$$

The following is our first result.

**Theorem 1.1.**

(I). Suppose that (A.0)-(A.1) hold. Then for any $T \in (0,1)$,

$$\lim_{\varepsilon\to 0}\{\sup_{|y|\le\varepsilon^{1/(\ell+2)}} P(\tau_D^\varepsilon(y)\le \varepsilon^{-T\ell/(\ell+2)})\}=0. \tag{1.3}$$

(II). Suppose that (A.0)-(A.2) hold. Then for any $T>1$,

$$\lim_{\varepsilon\to 0}\{\inf_{y\in D} P(\tau_D^\varepsilon(y)\le \varepsilon^{-T\ell/(\ell+2)})\}=1. \tag{1.4}$$

(III). Suppose that $\overline{D}=\{o\}\cup A_2$. Then for any $x\in D\backslash\{o\}$ and $\delta>0$,

$$\lim_{\varepsilon\to 0} P(|\tau_D^\varepsilon(x)-\tau_D^0(x)|<\delta)=1. \tag{1.5}$$

The following corollary can be easily obtained from Theorem 1.1.

**Corollary 1.2.**

Suppose that (A.0)-(A.2) hold. Then for any $\delta>0$,

$$\lim_{\varepsilon\to 0} P(\varepsilon^{-(1-\delta)\ell/(\ell+2)}<\tau_D^\varepsilon(o)<\varepsilon^{-(1+\delta)\ell/(\ell+2)})=1. \tag{1.6}$$

*Remark 1.1.* Corollary 1.2 implies that $X^\varepsilon(t,o)$ exit $D$ at time of order $\varepsilon^{-\ell/(\ell+2)}$. This is a big difference between the case $\ell=0$ (see Theorem 0.2) and the case $\ell>0$. $\tau_D^\varepsilon(o)$ is, as $\varepsilon\to 0$, of order $\log(1/\varepsilon)$ when $\ell=0$, but it is, as $\varepsilon\to 0$, of polynomial order of $\varepsilon^{-1}$ when $\ell>0$. Moreover the asymptotic behavior of $\tau_D^\varepsilon(o)$ as $\varepsilon\to 0$ depends on the first derivatives of $b$ when $\ell=0$, but it does not depend on the derivatives of $b$ when $\ell>0$.

We also have the following result.

**Theorem 1.3.**

(I). Suppose that (A.0)-(A.1) hold. Then for any $\delta>0$,

$$\liminf_{\varepsilon\to 0}\{\varepsilon^{(1-\delta)\ell/(\ell+2)}\inf_{|y|\le\varepsilon^{1/(\ell+2)}} E[\tau_D^\varepsilon(y)]\}\ge 1. \tag{1.7}$$

(II). Suppose that (A.0)-(A.2) hold. Then for any $\delta>0$,

$$\limsup_{\varepsilon\to 0}\{\varepsilon^{(1+\delta)\ell/(\ell+2)}\sup_{y\in D} E[\tau_D^\varepsilon(y)]\}\le 1. \tag{1.8}$$

(III). Suppose that (A.0)-(A.2) hold. Then for any $x\in D\backslash\{o\}$,

$$\lim_{\varepsilon\to 0} E[\tau_D^\varepsilon(x)]=\tau_D^0(x). \tag{1.9}$$

From Theorem 1.3, (I) and (II), we get the following corollary.

**Corollary 1.4.**

Suppose that (A.0)-(A.2) hold. Then

$$\lim_{\varepsilon\to 0}\{\log E[\tau_D^\varepsilon(o)]\}/\log(\varepsilon^{-\ell/(\ell+2)}) = 1. \tag{1.10}$$

If $a(x) \equiv \sigma(x)\sigma(x)^*$, $b(x)$ and $\partial D$ are sufficiently smooth, then $u^\varepsilon(t,x) \equiv P(\tau_D^\varepsilon(x) \leq t) \in C^{1,2}((0,\infty)\times D; R)$ and $v^\varepsilon(x) \equiv E[\tau_D^\varepsilon(x)] \in C^2(D;R)$ satisfy the following, respectively (see [6]);

$$\begin{aligned}\partial u^\varepsilon(t,x)/\partial t = \varepsilon\{\sum_{i,j=1}^d a^{ij}(x)\partial^2 u^\varepsilon(t,x)/\partial x_i\partial x_j\}/2&\\ +\sum_{i=1}^d b^i(x)\partial u^\varepsilon(t,x)/\partial x_i \quad (t>0, x\in D),&\\ u^\varepsilon(0,x) = 0 \quad (x\in D),&\\ u^\varepsilon(t,x) = 1 \quad (t\geq 0, x\in\partial D).&\end{aligned} \tag{1.11}$$

$$\begin{aligned}\varepsilon\{\sum_{i,j=1}^d a^{ij}(x)\partial^2 v^\varepsilon(x)/\partial x_i\partial x_j\}/2 + \sum_{i=1}^d b^i(x)\partial v^\varepsilon(x)/\partial x_i = -1 \quad (x\in D),&\\ v^\varepsilon(x) = 0 \quad (x\in\partial D).&\end{aligned} \tag{1.12}$$

Put $\tilde{u}^\varepsilon(t,x) \equiv u^\varepsilon(\varepsilon^{-t\ell/(\ell+2)}, x)$. Then Corollary 1.2 can be rewritten as follows.

**Corollary 1.5.**

Suppose that (A.0)-(A.2) hold. Then

$$\lim_{\varepsilon\to 0} v^\varepsilon(t,o) = \begin{cases} 0 & \text{if } t<1, \\ 1 & \text{if } t>1. \end{cases} \tag{1.13}$$

## 2. Lemmas.

In this section, we give lemmas which are necessary for the proof of our results. The following lemma plays a crucial role in this paper.

**Lemma 2.1.**

Let $f(t)$ and $g(t)$ $(t \geq 0)$ be positive continuous functions.
(I). Suppose that there exists a positive constant $C$ such that the following holds for all $t \geq 0$;

$$f(t) \leq \exp(C \int_0^t f(s)ds)g(t). \tag{2.1}$$

Then

$$f(t) \leq g(t)/(1 - C \int_0^t g(s)ds) \tag{2.2}$$

as far as $1 > C \int_0^t g(s)ds$.
(II). Suppose that there exists a positive constant $\tilde{C}$ such that the following holds for all $t \geq 0$;

$$f(t) \geq \exp(\tilde{C} \int_0^t f(s)ds)g(t). \tag{2.3}$$

Then for all $t \geq 0$,

$$\begin{aligned} 1 &> \tilde{C} \int_0^t g(s)ds, \\ f(t) &\geq g(t)/(1 - \tilde{C} \int_0^t g(s)ds). \end{aligned} \tag{2.4}$$

*Proof.* We first prove (I). From (2.1),

$$-d[\exp(-C \int_0^t f(s)ds)]/dt \leq Cg(t). \tag{2.5}$$

Integrating both sides of (2.5) in $t$, we get for $t \geq 0$,

$$(2.6). \qquad 1 - C\int_0^t g(s)ds \le \exp(-C\int_0^t f(s)ds),$$

from which we get (2.2).

Let us prove (II). In the same way as in (2.6), we get for $t \ge 0$,

$$(2.7). \qquad 1 - \tilde{C}\int_0^t g(s)ds \ge \exp(-\tilde{C}\int_0^t f(s)ds) > 0,$$

which implies (2.4).

Q.E.D.

The following lemma is given the proof for the sake of completeness.

**Lemma 2.2.** *(see [10]).*

Let $f(t)$, $u(t)$, $g(t)$, and $v(t)$ $(t \ge 0)$ be positive continuous functions.
(I). Suppose that there exists a positive constant $C$ such that the following holds for all $t \ge 0$;

$$(2.8). \qquad f(t) \le C + \int_0^t u(s)f(s)ds.$$

Then for all $t \ge 0$,

$$(2.9). \qquad f(t) \le C\exp(\int_0^t u(s)ds).$$

(II). Suppose that there exists a positive constant $\tilde{C}$ such that the following holds for all $t \ge 0$;

$$(2.10). \qquad f(t) \ge \tilde{C} + \int_0^t v(s)f(s)ds.$$

Then for all $t \ge 0$,

$$(2.11). \qquad f(t) \ge \tilde{C}\exp(\int_0^t v(s)ds).$$

*Proof.* Since (I) and (II) can be proved similarly, we only prove (I).

From (2.8), for all $t \geq 0$

(2.12).
$$d[\int_0^t u(s)f(s)ds \exp(-\int_0^t u(s)ds)]/dt$$
$$\leq Cd[-\exp(-\int_0^t u(s)ds)]/dt.$$

From (2.12), we get (2.9) easily.

Q. E. D.

The following lemma plays a crucial role in the proof of Theorem 1.1, (I).

**Lemma 2.3.**

Suppose that (A.0)-(A.1) holds. Then for any $T > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

(2.13).
$$P(\tau_D^\varepsilon(x) \leq \varepsilon^{-T\ell/(\ell+2)})$$
$$\leq P(C_1\ell\varepsilon^{-T\ell/(\ell+2)}((|x|^2 + \varepsilon^{2/(\ell+2)})^{1/2}$$
$$+ \sup_{0\leq s\leq\varepsilon^{-T\ell/(\ell+2)}} \varepsilon^{1/2}|\int_0^t (|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2} < X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) > |$$
$$+\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}} Trace[a(X^\varepsilon(s,x))](|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2}ds/2)^\ell \geq 1/2),$$

uniformly in $x \in D$.
*Proof.* By the Ito formula (see [11]), for $t \in [0, \min(\tau_D^\varepsilon(x), \varepsilon^{-T\ell/(\ell+2)})]$,

(2.14). $$(|X^\varepsilon(t,x)|^2 + \varepsilon^{2/(\ell+2)})^{1/2}$$
$$= (|x|^2 + \varepsilon^{2/(\ell+2)})^{1/2}$$
$$+ \int_0^t < X^\varepsilon(s,x), b(X^\varepsilon(s,x)) > (|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2}ds$$
$$+ \varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2} < X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) >$$
$$+ \varepsilon\int_0^t [Trace[a(X^\varepsilon(s,x))] - < a(X^\varepsilon(s,x))X^\varepsilon(s,x), X^\varepsilon(s,x) >$$
$$/(|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})](|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2}ds/2$$

$$\leq (|x|^2+\varepsilon^{2/(\ell+2)})^{1/2} + \sup_{0\leq u\leq t} \varepsilon^{1/2}\Big|\int_0^u (|X^\varepsilon(s,x)|^2+\varepsilon^{2/(\ell+2)})^{-1/2}$$
$$\times < X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) > \Big|$$
$$+\varepsilon\int_0^t Trace[a(X^\varepsilon(s,x))](|X^\varepsilon(s,x)|^2+\varepsilon^{2/(\ell+2)})^{-1/2}ds/2$$
$$+\int_0^t C_1|X^\varepsilon(s,x)|^\ell(|X^\varepsilon(s,x)|^2+\varepsilon^{2/(\ell+2)})^{1/2}ds$$

from (A.1).

From (2.14) and Lemma 2.2 , for $t\in[0,\min(\tau_D^\varepsilon(x),\varepsilon^{-T\ell/(\ell+2)})]$,

(2.15). $(|X^\varepsilon(t,x)|^2+\varepsilon^{2/(\ell+2)})^{1/2}$

$$\leq ((|x|^2+\varepsilon^{2/(\ell+2)})^{1/2} + \sup_{0\leq u\leq \varepsilon^{-T\ell/(\ell+2)}} \varepsilon^{1/2}\Big|\int_0^u (|X^\varepsilon(s,x)|^2+\varepsilon^{2/(\ell+2)})^{-1/2}$$
$$\times < X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) > \Big|$$
$$+\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}} Trace[a(X^\varepsilon(s,x))](|X^\varepsilon(s,x)|^2+\varepsilon^{2/(\ell+2)})^{-1/2}ds/2)$$
$$\times \exp(\int_0^t C_1|X^\varepsilon(s,x)|^\ell ds)$$
$$\equiv F(\varepsilon)\exp(\int_0^t C_1|X^\varepsilon(s,x)|^\ell ds).$$

From (2.15), we get

$$|X^\varepsilon(t,x)|^\ell \leq F(\varepsilon)^\ell \exp(\ell C_1\int_0^t |X^\varepsilon(s,x)|^\ell ds). \tag{2.16}$$

Suppose that

$$C_1\ell\varepsilon^{-T\ell/(\ell+2)}F(\varepsilon)^\ell < 1/2. \tag{2.17}$$

Then from Lemma 2.1, for $t\in[0,\min(\tau_D^\varepsilon(x),\varepsilon^{-T\ell/(\ell+2)})]$,

$$|X^\varepsilon(t,x)| \leq 2^{1/\ell}F(\varepsilon) \leq (C_1\ell)^{-1/\ell}\varepsilon^{T/(\ell+2)}, \tag{2.18}$$

which means that $\tau_D^\varepsilon(x) > \varepsilon^{-T\ell/(\ell+2)}$, if $\varepsilon$ is sufficienlty small, depending on $D$.

Q.E.D.

Put

$$\tau_o^\varepsilon(x) \equiv \inf\{t>0; |X^\varepsilon(t,x)|\geq \delta_o\} \tag{2.19}$$

(see (A.2) for notation). Then the following lemma is used in the proof of Theorem 3.1.

**Lemma 2.4.**

Suppose that (A.0) and (A.2) hold. Then for any $\alpha > 0$ and $T > 1$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$
\begin{aligned}
(2.20).\ & P(\tau_o^\varepsilon(y) > \varepsilon^{-\ell/(\ell+2)}/3) \\
& \leq P(\sup\{|\varepsilon^{1/2} \int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > | \\
& \quad ; 0 \leq t \leq \varepsilon^{-\ell/(\ell+2)}/3, n \geq 1\} \geq |y|/2),
\end{aligned}
$$

for any $y$ for which $|y| \geq \varepsilon^{1/(\ell+2)-\alpha(T-1)}$.

*Proof.*

$$
\begin{aligned}
(2.21).\quad & P(\tau_o^\varepsilon(y) > \varepsilon^{-\ell/(\ell+2)}/3) \\
& \leq P(\sup\{|\varepsilon^{1/2} \int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y) \\
& \quad , \sigma(X^\varepsilon(s,y))dW(s) > |; 0 \leq t \leq \varepsilon^{-\ell/(\ell+2)}/3, n \geq 1\} \geq |y|/2) \\
& \quad + P(\tau_o^\varepsilon(y) > \varepsilon^{-\ell/(\ell+2)}/3, (|X^\varepsilon(t,y)|^2 + 1/n)^{1/2} \geq (|y|^2 + 1/n)^{1/2} \\
& \quad - |y|/2 + \int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} ds \\
& \quad \text{for all } n \geq 1, t \in [0, \varepsilon^{-\ell/(\ell+2)}/3]),
\end{aligned}
$$

since for $n \geq 1$ and $t \in [0, \min(\tau_o^\varepsilon(y), \varepsilon^{-\ell/(\ell+2)}/3)]$, by the Ito formula (see [11]),

$$
\begin{aligned}
(2.22).\quad & (|X^\varepsilon(t,y)|^2 + 1/n)^{1/2} \\
& = (|y|^2 + 1/n)^{1/2} \\
& \quad + \int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} ds \\
& \quad + \varepsilon^{1/2} \int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
& \quad + \varepsilon \int_0^t [Trace[a(X^\varepsilon(s,y))] - < a(X^\varepsilon(s,y))X^\varepsilon(s,y), X^\varepsilon(s,y) > \\
& \quad /(|X^\varepsilon(s,y)|^2 + 1/n)](|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} ds/2 \\
& \geq (|y|^2 + 1/n)^{1/2} \\
& \quad - |\varepsilon^{1/2} \int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > | \\
& \quad + \int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} ds.
\end{aligned}
$$

Let us show that the second probability on the right hand side of (2.21) is zero for sufficiently small $\varepsilon$. Suppose that $\tau_o^\varepsilon(y) > \varepsilon^{-\ell/(\ell+2)}/3$ and that the following holds for all $n \geq 1$ and $t \in [0, \varepsilon^{-\ell/(\ell+2)}/3]$;

$$
\begin{aligned}
(2.23). \qquad & (|X^\varepsilon(t,y)|^2 + 1/n)^{1/2} \\
& \geq (|y|^2 + 1/n)^{1/2} - |y|/2 \\
& \quad + \int_0^t < X^\varepsilon(s,y), b(X^\varepsilon(s,y)) > (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} ds.
\end{aligned}
$$

Then we get for $t \in [0, \varepsilon^{-\ell/(\ell+2)}/3]$

$$
(2.24). \qquad |X^\varepsilon(t,y)| \geq |y|/2 + \int_0^t C_2 |X^\varepsilon(s,y)|^{\ell+1} ds
$$

from (A.2), and henceforth

$$
(2.25). \qquad |X^\varepsilon(t,y)| \geq |y| \exp(\int_0^t C_2 |X^\varepsilon(s,y)|^\ell ds)/2,
$$

from Lemma 2.2.

From (2.25), we get for $t \in [0, \varepsilon^{-\ell/(\ell+2)}/3]$,

$$
(2.26). \qquad |X^\varepsilon(t,y)|^\ell \geq (|y|/2)^\ell \exp(\ell C_2 \int_0^t |X^\varepsilon(s,y)|^\ell ds),
$$

and from Lemma 2.1, for $y(|y| \geq \varepsilon^{1/(\ell+2)-\alpha(T-1)})$,

$$
(2.27). \qquad 1 > \ell C_2 [\varepsilon^{-\ell/(\ell+2)}/3](|y|/2)^\ell = [\ell C_2/2^\ell]\varepsilon^{-\alpha\ell(T-1)}/3.
$$

This does not holds if $\varepsilon$ is sufficiently small, depending on $T$ and $\alpha$.

Q.E.D.

Put $||a(x)|| \equiv (\sum_{i,j=1}^d a^{ij}(x)^2)^{1/2}$ and $C \equiv (C_1)^2 \sup_{x \in R^d} ||a(x)^{-1}||$. The following lemma is used in the proof of Theorem 3.1.

**Lemma 2.5.**

Suppose that (A.0)-(A.2) hold. Then for any $\alpha > 0$ and $\beta > 0$ for which $\beta > \alpha(\ell+2)$, and any $T \in (1, 1 + (\alpha(\ell+2))^{-1})$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$(2.28).\ P(\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(X^\varepsilon(s,x))^{-1}b(X^\varepsilon(s,x)), b(X^\varepsilon(s,x)) > ds \geq \varepsilon^{1-\beta(T-1)}/2$$
$$, \sup_{0 \leq t \leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,x)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha})$$
$$\leq P(C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^{2(\ell+1)} ds \geq \varepsilon^{1-\beta(T-1)}/2,$$
$$\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,x)|^\ell X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) >$$
$$< -C_2\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^{2(\ell+1)} ds/2),$$

for all $x \in D$.

*Proof.* Suppose that the following holds;

$$(2.29).\quad \sup_{0 \leq t \leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,x)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha},$$

$$(2.30).\quad \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(X^\varepsilon(s,x))^{-1}b(X^\varepsilon(s,x)), b(X^\varepsilon(s,x)) > ds \geq \varepsilon^{1-\beta(T-1)}/2.$$

Then from (A.1) and (2.30), we get

$$(2.31).\quad C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^{2(\ell+1)} ds \geq \varepsilon^{1-\beta(T-1)}/2.$$

Suppose also that the following holds;

$$(2.32).\quad \varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,x)|^\ell X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) >$$
$$\geq -C_2\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^{2(\ell+1)} ds/2.$$

Then we get the following contradiction if $\varepsilon$ is sufficiently small. From (2.29), $\tau_D^\varepsilon(x) > \varepsilon^{-T\ell/(\ell+2)}/3$ for sufficiently small $\varepsilon > 0$, depending on $D$, since $1/(\ell+2) > (T-1)\alpha$. By the Ito formula (see [11]), putting $y_i y_j |y|^{\ell-2} \equiv 0$ if $y = o$ $(i,j = 1,\cdots,d)$,

$$
\begin{aligned}
(2.33).\quad & \varepsilon^{1-(\ell+2)(T-1)\alpha}/(\ell+2) \\
& > |X^\varepsilon(\varepsilon^{-T\ell/(\ell+2)}/3,x)|^{\ell+2}/(\ell+2) \\
& = |x|^{\ell+2}/(\ell+2) + \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^\ell < X^\varepsilon(s,x), b(X^\varepsilon(s,x))ds \\
& \quad + \varepsilon^{1/2}\sigma(X^\varepsilon(s,x))dW(s) > \\
& \quad + \varepsilon \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^\ell [Trace[a(X^\varepsilon(s,x))] \\
& \quad + \ell < a(X^\varepsilon(s,x))X^\varepsilon(s,x), X^\varepsilon(s,x) > /|X^\varepsilon(s,x)|^2]ds/2 \\
& \geq \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^\ell < X^\varepsilon(s,x), b(X^\varepsilon(s,x))ds \\
& \quad + \varepsilon^{1/2}\sigma(X^\varepsilon(s,x))dW(s) > \\
& \geq \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} C_2|X^\varepsilon(s,x)|^{2(\ell+1)}ds + \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,x)|^\ell \\
& \quad \times < X^\varepsilon(s,x), \varepsilon^{1/2}\sigma(X^\varepsilon(s,x))dW(s) > \quad \text{(from (A.2))} \\
& \geq \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} C_2|X^\varepsilon(s,x)|^{2(\ell+1)}ds/2.
\end{aligned}
$$

From (2.31) and (2.33),

$$
(2.34).\qquad \varepsilon^{1-(\ell+2)(T-1)\alpha}/(\ell+2) > (C_2/(4C))\varepsilon^{1-\beta(T-1)},
$$

which does not hold for sufficiently small $\varepsilon$, since $\beta > \alpha(\ell+2)$.

Q. E. D.

## 3. Proof of Main result.

In this section we prove Theorems 1.1 and 1.3. Since (III) in Theorem 1.1 is an easy consequence of Freidlin-Wentzell theory (see [8]), we omit the proof.

Let us first prove (I) in Theorem 1.1.

*Proof of (I) in Theorem 1.1.* Put $C(\sigma) \equiv \sup_{x \in R^d}(\sum_{i,j=1}^d \sigma^{ij}(x)^2)^{1/2}$. Then from Lemma 2.3, by the time change (see [11], Chap. 4, section 4), there exists a one dimensional Wiener process $\tilde{W}$ such that the following holds; for $T \in (0,1)$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and $x(|x| \le \varepsilon^{1/(\ell+2)})$,

(3.1).

$$\begin{aligned}
&P(\tau_D^\varepsilon(x) \le \varepsilon^{-T\ell/(\ell+2)}) \\
&\le P(C_1 \ell \varepsilon^{-T\ell/(\ell+2)}(2\varepsilon^{1/(\ell+2)} + \sup_{0 \le s \le \varepsilon^{-T\ell/(\ell+2)}} \varepsilon^{1/2} | \int_0^t (|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2} \\
&\quad \times < X^\varepsilon(s,x), \sigma(X^\varepsilon(s,x))dW(s) > | + \varepsilon \int_0^{\varepsilon^{-T\ell/(\ell+2)}} Trace[a(X^\varepsilon(s,x))] \\
&\quad \times (|X^\varepsilon(s,x)|^2 + \varepsilon^{2/(\ell+2)})^{-1/2} ds/2)^\ell \ge 1/2) \\
&\le P(C_1 \ell \varepsilon^{-T\ell/(\ell+2)}(2\varepsilon^{1/(\ell+2)} + C(\sigma)\varepsilon^{[(1-T)\ell+2]/[2(\ell+2)]} \sup_{0 \le s \le 1} |\tilde{W}(s)| \\
&\quad + C(\sigma)^2 \varepsilon^{[(1-T)\ell+1]/(\ell+2)}/2)^\ell \ge 1/2) \\
&\le P(C(\sigma)\varepsilon^{(1-T)/2} \sup_{0 \le s \le 1} |\tilde{W}(s)| \ge (2C_1\ell)^{-1/\ell}/2),
\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$ (see [8], Chap. 3, sections 2, 3).

Q.E.D.

Next we prove (II) in Theorem 1.1.

*Proof of (II) in Theorem 1.1.* Take $\gamma > 0$ and $T_o > 0$ so that

$$(3.2). \qquad \inf\{\sup\{dist(X^0(t,x), D); 0 \le t \le T_o\}; |x| \ge \delta_o\} = \gamma,$$

which is possible from (A.2).

Since we only have to prove Theorem 1.1, (II), for $T > 1$ for which $T-1$ is sufficiently small, we assume that $T < (\ell+3)^2/[\ell(\ell+2)] + 1$. Then for sufficiently small $\varepsilon > 0$, we have the following (see (2.19) for notation);

$$
\begin{aligned}
(3.3).\ & P(\tau_D^\varepsilon(x) \le \varepsilon^{-T\ell/(\ell+2)}) \\
& \ge P(\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,x)| \ge \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}, \tau_o^\varepsilon(x) \le 2\varepsilon^{-T\ell/(\ell+2)}/3, \\
& \quad \sup_{0\le t\le T_o} |X^\varepsilon(t, X^\varepsilon(\tau_o^\varepsilon(x),x)) - X^0(t, X^\varepsilon(\tau_o^\varepsilon(x),x))| < \gamma) \quad \text{(from (A.2))} \\
& \ge \inf\{P(\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| \ge \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}) \\
& \quad ; |y| < \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}\} \\
& \quad \times \inf\{P(\tau_o^\varepsilon(y) \le \varepsilon^{-T\ell/(\ell+2)}/3); |y| \ge \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}\} \\
& \quad \times \inf\{P(\sup_{0\le t\le T_o} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma); |y| \ge \delta_o, y \in D\},
\end{aligned}
$$

uniformly in $x \in D$, by the strong Markov property of $X^\varepsilon$ (see [11]).

Since the last probability in the last part of (3.3) converges to 1 as $\varepsilon \to 0$, which is a fundamental fact in Freidlin-Wentzell theory (see [8], Chap. 3, sections 2, 3), we only have to prove the following theorem to complete the proof.

Q. E. D.

**Theorem 3.1.**

Suppose that (A.0)-(A.2) holds. Then for any $T \in (1, 1 + (\ell+3)^2/[\ell(\ell+2)])$,

$$
\begin{aligned}
(3.4).\quad & \lim_{\varepsilon\to 0}(\sup\{P(\tau_o^\varepsilon(y) > \varepsilon^{-T\ell/(\ell+2)}/3); |y| \ge \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}\}) = 0, \\
(3.5).\quad & \lim_{\varepsilon\to 0}(\sup\{P(\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}) \\
& \qquad ; |y| < \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2}\}) = 0.
\end{aligned}
$$

*Proof.* Let us first prove (3.4). Put $\alpha = \ell/(\ell+3)^2$. Then for any $y$ for which $|y| \ge \varepsilon^{1/(\ell+2)-\alpha(T-1)}$,

$$
\begin{aligned}
(3.6).\quad & P(\tau_o^\varepsilon(y) > \varepsilon^{-\ell/(\ell+2)}/3) \\
& \le P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2 + 1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > | \\
& \quad ; 0 \le t \le \varepsilon^{-\ell/(\ell+2)}/3, n \ge 1\} \ge |y|/2),
\end{aligned}
$$

provided that $\varepsilon > 0$ is sufficiently small (from Lemma 2.4).

The probability on the right hand side of (3.6) converges to 0, as $\varepsilon \to 0$, uniformly in $y(|y| \geq \varepsilon^{1/(\ell+2)-\alpha(T-1)})$. In fact, by the time change (see [11], Chap. 4, section 4), there exists a one dimensional Wiener process $\tilde{W}$ such that

$$
\begin{aligned}
(3.7).\quad & P(\sup\{|\varepsilon^{1/2}\int_0^t (|X^\varepsilon(s,y)|^2+1/n)^{-1/2} < X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > | \\
&\qquad ;0 \leq t \leq \varepsilon^{-\ell/(\ell+2)}/3, n \geq 1\} \geq |y|/2) \\
&\leq P(C(\sigma)\sup\{\varepsilon^{\alpha(T-1)}|\tilde{W}(s)|; 0 \leq s \leq 1\} \geq 3^{1/2}/2\}) \\
&\to 0 \quad \text{as } \varepsilon \to 0
\end{aligned}
$$

(see [8], Chap. 3, sections 2, 3).

Next we prove (3.5). Put $\beta = \alpha((\ell+2)^2+1/2)/(\ell+2)$, then for $y$ for which $(|y| < \varepsilon^{1/(\ell+2)-(T-1)\ell/(\ell+3)^2})$

$$
\begin{aligned}
(3.8).\quad & P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-\alpha(T-1)}) \\
&\leq P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha}, \\
&\qquad \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)), dX^\varepsilon(s,y) > \leq \varepsilon^{1-\beta(T-1)}) \\
&\quad + P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha}, \\
&\qquad \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)), dX^\varepsilon(s,y) > \geq \varepsilon^{1-\beta(T-1)}).
\end{aligned}
$$

Let us show that the first probability on the right hand side of (3.8) converges to 0, as $\varepsilon \to 0$, uniformly in $y(|y| \leq \varepsilon^{1/(\ell+2)-\alpha(T-1)})$.

Let $Y^\varepsilon(t,x)$ $(t \geq 0, x \in R^d, \varepsilon > 0)$ be the solution of the following stochastic differential equation:

$$
(3.9).\qquad \begin{aligned}
dY^\varepsilon(t,x) &= \varepsilon^{1/2}\sigma(Y^\varepsilon(t,x))dW(t), \\
Y^\varepsilon(0,x) &= x.
\end{aligned}
$$

Then by Maruyama-Girsanov formula (see [11]),

$$
(3.10).\quad P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha},
$$

$$\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)), dX^\varepsilon(s,y) > \leq \varepsilon^{1-\beta(T-1)})$$

$$= E[\exp([\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(Y^\varepsilon(s,y))^{-1}b(Y^\varepsilon(s,y)), dY^\varepsilon(s,y) >$$

$$- \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |\sigma(Y^\varepsilon(s,y))^{-1}b(Y^\varepsilon(s,y))|^2 ds/2]/\varepsilon)$$

$$; \sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |Y^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha},$$

$$\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < a(Y^\varepsilon(s,y))^{-1}b(Y^\varepsilon(s,y)), dY^\varepsilon(s,y) > \leq \varepsilon^{1-\beta(T-1)}]$$

$$\leq \exp(\varepsilon^{-\beta(T-1)})P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |Y^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha}),$$

and by the time change (see [11], Chap. 4, section 4), there exist a one dimensional Wiener process $\tilde{W}$ and a positive constant $C_3$ such that for $y(|y| \leq \varepsilon^{1/(\ell+2)-(T-1)\alpha})$,

(3.11).
$$P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |Y^\varepsilon(t,y)| < \varepsilon^{1/(\ell+2)-(T-1)\alpha})$$
$$\leq P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |Y^\varepsilon(t,y)^1| < \varepsilon^{1/(\ell+2)-(T-1)\alpha})$$
$$\leq P(\sup_{0\leq t\leq \varepsilon^{-T\ell/(\ell+2)}/3} |Y^\varepsilon(t,y)^1 - y^1| < \varepsilon^{1/(\ell+2)-(T-1)\alpha} + |y^1|)$$
$$\leq P(\sup_{0\leq t\leq 1} |\tilde{W}(t)| \leq 2\cdot 3^{1/2}(\inf\{\sum_{k=1}^d \sigma^{1k}(z)^2; z\in R^d\})^{-1/2}$$
$$\times\, \varepsilon^{(T-1)\alpha((\ell+2)^2+1)/[2(\ell+2)]})$$
$$\leq \exp(-C_3\varepsilon^{-(T-1)\alpha((\ell+2)^2+1)/(\ell+2)})$$

for sufficiently small $\varepsilon > 0$ (see [11], Chap 6, section 9). Here we used that $\alpha = \ell/(\ell+3)^2$. (3.10)-(3.11) completes the proof, since

(3.12).
$$\beta < \alpha((\ell+2)^2+1)/(\ell+2),$$

from the construction (see below (3.7)).

Next we show that the second probability on the right hand side of (3.8) converges to 0, as $\varepsilon \to 0$, uniformly in $y(|y| \leq \varepsilon^{1/(\ell+2)-\alpha(T-1)})$.

(3.13).

$$P\Big(\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3}|X^\varepsilon(t,y)|<\varepsilon^{1/(\ell+2)-(T-1)\alpha},$$
$$\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),dX^\varepsilon(s,y)>\ge\varepsilon^{1-\beta(T-1)}\Big)$$
$$\le P\Big(\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),b(X^\varepsilon(s,y))>ds\le\varepsilon^{1-\beta(T-1)}/2,$$
$$\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<\sigma(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),dW(s)>\ge\varepsilon^{1-\beta(T-1)}/2\Big)$$
$$+P\Big(\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3}|X^\varepsilon(t,y)|<\varepsilon^{1/(\ell+2)-(T-1)\alpha},$$
$$\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),b(X^\varepsilon(s,y))>ds\ge\varepsilon^{1-\beta(T-1)}/2\Big).$$

The first probability on the right hand side of (3.13) can be shown to converge to 0, as $\varepsilon\to 0$, uniformly in $y(|y|\le\varepsilon^{1/(\ell+2)-\alpha(T-1)})$ as follows; by the time change (see [11], Chap. 4, section 4), there exists a one dimensional Wiener process $\tilde W$ such that

$$(3.14).\ P\Big(\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),b(X^\varepsilon(s,y))>ds\le\varepsilon^{1-\beta(T-1)}/2,$$
$$\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<\sigma(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),dW(s)>\ge\varepsilon^{1-\beta(T-1)}/2\Big)$$
$$\le P\big(\varepsilon^{\beta(T-1)/2}\sup_{0\le t\le 1}|\tilde W(t)|\ge 2^{-1/2}\big)$$
$$\to 0\quad \text{as } \varepsilon\to 0$$

(see [8], Chap. 3, sections 2, 3).

Putting $R=C_2/(2C(\sigma)^2\varepsilon)$, the second probability on the right hand side of (3.13) can be shown to converge to 0, as $\varepsilon\to 0$, uniformly in $y(|y|\le\varepsilon^{1/(\ell+2)-\alpha(T-1)})$, as follows; for sufficiently small $\varepsilon>0$,

(3.15).
$$P\Big(\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}<a(X^\varepsilon(s,y))^{-1}b(X^\varepsilon(s,y)),b(X^\varepsilon(s,y))>ds\ge\varepsilon^{1-\beta(T-1)}/2$$
$$,\sup_{0\le t\le \varepsilon^{-T\ell/(\ell+2)}/3}|X^\varepsilon(t,y)|<\varepsilon^{1/(\ell+2)-(T-1)\alpha}\Big)$$
$$\le P\Big(C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3}|X^\varepsilon(s,y)|^{2(\ell+1)}ds\ge\varepsilon^{1-\beta(T-1)}/2,$$

$$
\begin{aligned}
&\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,y)|^\ell X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
&\qquad < -C_2\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds/2) \quad \text{(from Lemma 2.5)} \\
&\le E[\exp(-R\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,y)|^\ell X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
&\quad - RC_2\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds/2) \\
&\quad ; C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds \ge \varepsilon^{1-\beta(T-1)}/2] \\
&= E[\exp(-R\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,y)|^\ell X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
&\quad - R^2\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |\sigma(X^\varepsilon(s,y))^* X^\varepsilon(s,y)|^2 |X^\varepsilon(s,y)|^{2\ell}ds/2 \\
&\quad + R^2\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |\sigma(X^\varepsilon(s,y))^* X^\varepsilon(s,y)|^2 |X^\varepsilon(s,y)|^{2\ell}ds/2 \\
&\quad - RC_2\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds/2) \\
&\quad ; C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds \ge \varepsilon^{1-\beta(T-1)}/2] \\
&\le E[\exp(-R\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,y)|^\ell X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
&\quad - R^2\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |\sigma(X^\varepsilon(s,y))^* X^\varepsilon(s,y)|^2 |X^\varepsilon(s,y)|^{2\ell}ds/2 \\
&\quad + \int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds(R^2C(\sigma)^2\varepsilon - RC_2)/2) \\
&\quad ; C\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |X^\varepsilon(s,y)|^{2(\ell+1)}ds \ge \varepsilon^{1-\beta(T-1)}/2] \\
&\le \exp([-(C_2)^2/(8C(\sigma)^2\varepsilon)]\varepsilon^{1-\beta(T-1)}/(2C)) \\
&\quad \times E[\exp(-R\varepsilon^{1/2}\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} < |X^\varepsilon(s,y)|^{2\ell} X^\varepsilon(s,y), \sigma(X^\varepsilon(s,y))dW(s) > \\
&\quad - R^2\varepsilon\int_0^{\varepsilon^{-T\ell/(\ell+2)}/3} |\sigma(X^\varepsilon(s,y))^* X^\varepsilon(s,y)|^2 |X^\varepsilon(s,y)|^{2\ell}ds/2)] \\
&\le \exp(-(C_2)^2\varepsilon^{-\beta(T-1)}/(16C(\sigma)^2C)) \quad \text{(see [11] for exponential martingales )}
\end{aligned}
$$

$$\to 0 \qquad \text{as } \varepsilon \to 0.$$

Q.E.D.

*Remark 3.1.* It turns out that there exists a positive constant $C'$ such that the rate of convergence in Theorem 1.1 is less than $\exp(-\varepsilon^{-C'|T-1|})$ for $T$ for which $|T-1|$ is sufficiently small. This is also a big difference between the case $\ell = 0$ and that $\ell > 0$ (see [16]). In fact when $\ell = 0$, in Theorem 0.2, (0.5), there exists $C'' > 0$ such that

$$(3.16). \qquad P(\tau_D^\varepsilon(o) > (1+\delta)\log(\varepsilon^{-1/(2\lambda)})) \geq \varepsilon^{C''}$$

(see [16]).

Let us prove Theorem 1.3.

*Proof of Theorem 1.3, (I).* For $y$ for which $|y| \leq \varepsilon^{1/(\ell+2)}$ and $\delta > 0$,

$$\begin{aligned}(3.17). \qquad E[\tau_D^\varepsilon(y)] &\geq E[\tau_D^\varepsilon(y); \tau_D^\varepsilon(y) \geq \varepsilon^{-(1-\delta)\ell/(\ell+2)}] \\ &\geq \varepsilon^{-(1-\delta)\ell/(\ell+2)} P(\tau_D^\varepsilon(y) \geq \varepsilon^{-(1-\delta)\ell/(\ell+2)}) \\ &\geq \varepsilon^{-(1-\delta)\ell/(\ell+2)} \inf_{|z| \leq \varepsilon^{1/(\ell+2)}} P(\tau_D^\varepsilon(z) \geq \varepsilon^{-(1-\delta)\ell/(\ell+2)}).\end{aligned}$$

(3.17) and Theorem 1.1, (I) completes the proof.

Q. E. D.

Next we prove Theorem 1.3, (II).

*Proof of Theorem 1.3, (II).* For $y \in D$ and $\delta > 0$,

$$\begin{aligned}(3.18). \qquad E[\tau_D^\varepsilon(y)] &\leq \sum_{k=0}^{\infty} \varepsilon^{-(1+\delta)\ell/(\ell+2)}(k+1) \\ &\qquad \times P(\varepsilon^{-(1+\delta)\ell/(\ell+2)}k \leq \tau_D^\varepsilon(y) \leq \varepsilon^{-(1+\delta)\ell/(\ell+2)}(k+1)) \\ &\leq \varepsilon^{-(1+\delta)\ell/(\ell+2)} \sum_{k=0}^{\infty} P(\varepsilon^{-(1+\delta)\ell/(\ell+2)}k \leq \tau_D^\varepsilon(y)) \\ &\leq \varepsilon^{-(1+\delta)\ell/(\ell+2)} \{\inf_{z \in D} P(\tau_D^\varepsilon(z) \leq \varepsilon^{-(1+\delta)\ell/(\ell+2)})\}^{-1}.\end{aligned}$$

(3.18) and Theorem 1.1, (II) completes the proof.

Q. E. D.

Finally we prove Theorem 1.3, (III).

*Proof of Theorem 1.3, (III).* We first prove the following; for $x \in D\backslash\{o\}$

$$(3.19). \qquad \liminf_{\varepsilon\to 0} E[\tau_D^\varepsilon(x)] \geq \tau_D^0(x).$$

For $\delta \in (0, \tau_D^0(x))$, put

$$(3.20). \qquad \inf\{dist(X^0(t,x), D^c); 0 \leq t \leq \tau_D^0(x) - \delta\} \equiv \gamma(x,\delta),$$

which is positive from (A.2). Then

$$(3.21). \qquad \begin{aligned} E[\tau_D^\varepsilon(x)] &\geq E[\tau_D^\varepsilon(x); \sup_{0\leq t\leq \tau_D^0(x)-\delta} |X^\varepsilon(t,x) - X^0(t,x)| < \gamma(x,\delta)] \\ &\geq (\tau_D^0(x) - \delta) P(\sup_{0\leq t\leq \tau_D^0(x)-\delta} |X^\varepsilon(t,x) - X^0(t,x)| < \gamma(x,\delta)). \end{aligned}$$

The probability in the last part of (3.21) converges to 1 as $\varepsilon \to 0$, which is a fundamental fact in Freidlin-Wentzell theory (see [8], Chap. 3, sections 2, 3). Since $\delta > 0$ can be arbitrary small, we get (3.19).

Next we prove the following; for $x \in D\backslash\{o\}$

$$(3.22). \qquad \limsup_{\varepsilon\to 0} E[\tau_D^\varepsilon(x)] \leq \tau_D^0(x).$$

Put

$$(3.23). \qquad S(x) \equiv \{y \in D; \tau_D^0(y) > \tau_D^0(x)\}.$$

Take $r > 0$ so that

$$(3.24). \qquad U_r(o) \cap S(x)^c = \emptyset.$$

Then

$$(3.25). \qquad \begin{aligned} &(1 - \sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y))) \sup_{y\in\partial S(x)} E[\tau_D^\varepsilon(y)] \\ &\leq \sup_{y\in\partial S(x)} E[\min(\tau_D^\varepsilon(y), \tau_{U_r(o)^c}^\varepsilon(y))] \\ &\quad + \sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y)) \sup_{|y|=r} E[\tau_{S(x)}^\varepsilon(y)], \end{aligned}$$

since

(3.26).
$$\sup_{y\in\partial S(x)} E[\tau_D^\varepsilon(y)] \leq \sup_{y\in\partial S(x)} E[\min(\tau_D^\varepsilon(y), \tau_{U_r(o)^c}^\varepsilon(y))] + \sup_{y\in\partial S(x)} E[\tau_D^\varepsilon(y) - \tau_{U_r(o)^c}^\varepsilon(y); \tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y)]$$
$$\leq \sup_{y\in\partial S(x)} E[\min(\tau_D^\varepsilon(y), \tau_{U_r(o)^c}^\varepsilon(y))] + \sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y)) \sup_{|y|=r} E[\tau_D^\varepsilon(y)],$$

and since

(3.27).
$$\sup_{|y|=r} E[\tau_D^\varepsilon(y)] \leq \sup_{|y|=r} E[\tau_{S(x)}^\varepsilon(y)] + \sup_{y\in\partial S(x)} E[\tau_D^\varepsilon(y)].$$

To complete the proof, let us prove the following;

(3.28).
$$\lim_{\varepsilon\to 0}( \sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y))) = 0,$$

(3.29).
$$\lim_{\varepsilon\to 0}( \sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y)) \sup_{|y|=r} E[\tau_{S(x)}^\varepsilon(y)]) = 0,$$

(3.30).
$$\limsup_{\varepsilon\to 0}( \sup_{y\in\partial S(x)} E[\min(\tau_D^\varepsilon(y), \tau_{U_r(o)^c}^\varepsilon(y))]) \leq \tau_D^0(x).$$

Let us first prove (3.28). For $\delta > 0$, put

(3.31).
$$\eta(=\eta(x,\delta)) = \min(\inf\{\sup\{dist(X^0(t,y), D); 0 \leq t \leq \tau_D^0(x)+\delta\}; y \in \partial S(x)\}, \inf\{\inf\{dist(X^0(t,y), U_r(o)); 0 \leq t \leq \tau_D^0(x)+\delta\}; y \in \partial S(x)\}),$$

which is positive from (A.2). Then there exists $C(x,\delta) > 0$ such that for sufficiently small $\varepsilon$

(3.32).
$$\sup_{y\in\partial S(x)} P(\tau_D^\varepsilon(y) > \tau_{U_r(o)^c}^\varepsilon(y)) \leq \sup_{y\in\partial S(x)} P( \sup_{0\leq t\leq \tau_D^0(x)+\delta} |X^\varepsilon(t,y) - X^0(t,y)| \geq \eta) \leq \exp(-C(x,\delta)/\varepsilon)$$

(see [8], Chap. 3, sections 2,3), which shows that (3.28) is true.

Next we prove (3.29). From Theorem 1.3, (II), for sufficiently small $\varepsilon$,

$$(3.33). \qquad \sup_{|y|=r} E[\tau^{\varepsilon}_{S(x)}(y)] \le \sup_{y\in D} E[\tau^{\varepsilon}_{D}(y)] \le \varepsilon^{-2\ell/(\ell+2)}.$$

(3.32)-(3.33) shows that (3.29) is true.

Finally, let us prove (3.30). Put

$$(3.34). \qquad T(r) \equiv \sup\{\tau^0_D(z); |z| \ge r\} (= \sup\{\tau^0_D(z); |z| = r\}),$$

and for $\delta > 0$, put

$$(3.35). \quad \tilde{\eta} = \tilde{\eta}(x,\delta) \equiv \inf\{\sup\{dist(X^0(t,y),D); 0 \le t \le T(r)+\delta\}; y \in D\backslash U_r(o)\}$$
$$(= \inf\{\sup\{dist(X^0(t,y),D); 0 \le t \le T(r)+\delta\}; |y| = r\})$$

which is positive from (A.2). For any $y \in \partial S(x)$,

$$\begin{aligned}(3.36). \quad & E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y))] \\ &= E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)); \min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)) < T(r)+\delta] \\ &\quad + E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)); \min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)) \ge T(r)+\delta].\end{aligned}$$

The first quantity on the right hand side of (3.36) can be considered as follows (see (3.31) for notation);

$$\begin{aligned}(3.37). \quad & E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)); \min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)) < T(r)+\delta] \\ &\le E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)); \sup_{0\le t\le \tau^0_D(x)+\delta} |X^{\varepsilon}(t,y) - X^0(t,y)| < \eta] \\ &\quad + E[\min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)); \sup_{0\le t\le \tau^0_D(x)+\delta} |X^{\varepsilon}(t,y) - X^0(t,y)| \ge \eta \\ &\quad , \min(\tau^{\varepsilon}_D(y), \tau^{\varepsilon}_{U_r(o)^c}(y)) < T(r)+\delta] \\ &\le (\tau^0_D(x)+\delta) P(\sup_{0\le t\le \tau^0_D(x)+\delta} |X^{\varepsilon}(t,y) - X^0(t,y)| < \eta) \\ &\quad + (T(r)+\delta) P(\sup_{0\le t\le \tau^0_D(x)+\delta} |X^{\varepsilon}(t,y) - X^0(t,y)| \ge \eta) \\ &\to \tau^0_D(x) + \delta \quad \text{as } \varepsilon \to 0,\end{aligned}$$

uniformly in $y \in \partial S(x)$ (see [8], Chap. 3, sections 2, 3).

The second quantity on the right hand side of (3.36) can be shown to converge to 0, as $\varepsilon \to 0$, uniformly in $y \in \partial S(x)$ as follows;

$$
\begin{aligned}
(3.38). \qquad & E[\min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y)); \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y)) \geq T(r)+\delta] \\
& \leq \sum_{k=1}^{\infty}(T(r)+\delta)(k+1)P((T(r)+\delta)k \\
& \qquad \leq \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y)) \leq (T(r)+\delta)(k+1)) \\
& \leq 2(T(r)+\delta)\sum_{k=1}^{\infty} P((T(r)+\delta)k \leq \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y))) \\
& \leq 2(T(r)+\delta) \sup_{y \in D \cap U_r(o)^c} P(T(r)+\delta \leq \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y))) \\
& \qquad \times (1 - \sup_{y \in D \cap U_r(o)^c} P(T(r)+\delta \leq \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y))))^{-1} \\
& \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

uniformly in $y \in \partial S(x)$ (see [8], Chap. 3, sections 2, 3). This is true, since for $y \in \partial S(x)$

$$
\begin{aligned}
(3.39). \qquad & P(T(r)+\delta \leq \min(\tau^\varepsilon_D(y), \tau^\varepsilon_{U_r(o)^c}(y))) \\
& \leq P(\sup_{0 \leq t \leq T(r)+\delta} |X^\varepsilon(t,y) - X^0(t,y)| \geq \tilde{\eta}) \\
& \to 0 \quad \text{as } \varepsilon \to 0,
\end{aligned}
$$

uniformly in $y \in \partial S(x)$ (see [8], Chap. 3, sections 2, 3).

(3.37)-(3.38) completes the proof, since $\delta > 0$ can be taken arbitrary small.

Q. E. D.

## 4. Case $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$.

In this section we consider the special class of the case $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$, as an application of the results in section 1.

Let us first introduce the assumptions.

(H.1). $\sigma(x)$ is an identity matrix. There exist $d_1$ and $d_2$ for which $d_1 + d_2 = d$, and $b_1 : R^{d_1} \mapsto R^{d_1}$ and $b_2 : R^{d_2} \mapsto R^{d_2}$ such that $b(x_1, x_2) = (b_1(x_1), b_2(x_2))$.

For $x = (x_1, x_2) \in R^d = R^{d_1} \times R^{d_2}$, let $\{X_1^\varepsilon(t, x_1)\}_{t\geq 0}$ and $\{X_2^\varepsilon(t, x_2)\}_{t\geq 0}$ be the solutions to the following stochastic differential equations;

$$\text{(4.1).} \qquad \begin{aligned} dX_1^\varepsilon(t, x_1) &= b_1(X_1^\varepsilon(t, x_1))dt + \varepsilon^{1/2} dW_1(t), \\ X_1^\varepsilon(0, x_1) &= x_1, \end{aligned}$$

$$\text{(4.2).} \qquad \begin{aligned} dX_2^\varepsilon(t, x_2) &= b_2(X_2^\varepsilon(t, x_2))dt + \varepsilon^{1/2} dW_2(t), \\ X_2^\varepsilon(0, x_2) &= x_2, \end{aligned}$$

where we put $W(t) = (W_1(t), W_2(t))$ (see (0.1)). Then $X^\varepsilon(t, x) = (X_1^\varepsilon(t, x_1), X_2^\varepsilon(t, x_2))$.

We also assume the following.

(H.2). There exist the domains $D_1 \subset R^{d_1}$ and $D_2 \subset R^{d_2}$ such that $D_1 = \{o_1\} \cup \{x_1 \in R^{d_1};$ there exists $s = s(x_1) \leq 0$ such that $X_1^0(t, x_1) \notin \overline{D}_1$ for $t < s$ and such that $X_1^0(t, x_1) \in D_1$ for $t > s$. $X_1^0(t, x_1) \to o_1$ as $t \to \infty\}$ and that $D_2 = \{o_2\} \cup \{x_2 \in R^{d_2};$ there exists $s = s(x_2) \geq 0$ such that $X_2^0(t, x_2) \notin \overline{D}_2$ for $t > s$ and such that $X_2^0(t, x_2) \in D_2$ for $t < s$. $X_2^0(t, x_2) \to o_2$ as $t \to -\infty\}$. $D = D_1 \times D_2$.

Under (H.2), $A_1 = \{(x_1, o_2); x_1 \in \overline{D}_1 \backslash \{o_1\}\}$, and $A_2 = \{(o_1, x_2); x_2 \in \overline{D}_2 \backslash \{o_2\}\}$, and $A_3 = \overline{D} \backslash (\{o\} \cup A_1 \cup A_2)$ (see section 0 for notation).

The following is the last assumption in this section.

(H.3). There exist positive constants $\ell$ and $\tilde{C}_1$ such that for $x \in A_2$

$$\text{(4.3).} \qquad |b(x)| \leq \tilde{C}_1 |x|^{\ell+1},$$

and there exist positive constants $\tilde{\delta}_o$ and $\tilde{C}_2$ such that for $x \in A_2$ for which $|x| < \tilde{\delta}_o$,

$$\text{(4.4).} \qquad < x, b(x) > \geq \tilde{C}_2 |x|^{\ell+2}.$$

Under (H.1)-(H.3), the following can be obtained from Theorems 0.1, 1.1, and 1.3.

**Theorem 4.1.**

Suppose that (H.1)-(H.3) hold. Then the following holds.
(I). For any $\delta > 0$ and any $x \in A_1 \cup \{o\} \backslash \partial D$,

$$(4.5). \qquad \lim_{\varepsilon \to 0} P(\varepsilon^{-(1-\delta)\ell/(\ell+2)} < \tau_D^\varepsilon(x) < \varepsilon^{-(1+\delta)\ell/(\ell+2)}) = 1,$$

and

$$(4.6). \qquad \limsup_{\varepsilon \to 0} \varepsilon^{(1+\delta)\ell/(\ell+2)} E[\tau_D^\varepsilon(x)] \le 1 \le \liminf_{\varepsilon \to 0} \varepsilon^{(1-\delta)\ell/(\ell+2)} E[\tau_D^\varepsilon(x)].$$

(II). For any $\delta > 0$ and any $x \in A_2 \cup A_3 \backslash \partial D$,

$$(4.7). \qquad \lim_{\varepsilon \to 0} P(|\tau_D^\varepsilon(x) - \tau_D^0(x)| < \delta) = 1,$$

and

$$(4.8). \qquad \lim_{\varepsilon \to 0} E[\tau_D^\varepsilon(x)] = \tau_D^0(x)].$$

*Proof.* Put

$$(4.9). \qquad \begin{aligned} \tau_{D_1}^\varepsilon(x_1) &\equiv \inf\{t > 0; X_1^\varepsilon(t, x_1) \notin D_1\}, \\ \tau_{D_2}^\varepsilon(x_2) &\equiv \inf\{t > 0; X_2^\varepsilon(t, x_2) \notin D_2\}. \end{aligned}$$

Then

$$(4.10). \qquad \tau_D^\varepsilon(x) = \min(\tau_{D_1}^\varepsilon(x_1), \tau_{D_2}^\varepsilon(x_2)).$$

(4.5) can be proved as follows; from (4.10), for $x = (x_1, o_2) \in A_1 \cup \{o\} \backslash \partial D$

$$(4.11). \qquad \begin{aligned} &P(\varepsilon^{-(1-\delta)\ell/(\ell+2)} < \tau_D^\varepsilon(x) < \varepsilon^{-(1+\delta)\ell/(\ell+2)}) \\ &\ge P(\varepsilon^{-(1-\delta)\ell/(\ell+2)} < \tau_{D_2}^\varepsilon(o_2) < \varepsilon^{-(1+\delta)\ell/(\ell+2)}, \varepsilon^{-(1-\delta)\ell/(\ell+2)} < \tau_{D_1}^\varepsilon(x_1)) \\ &\to 1 \quad \text{as } \varepsilon \to 0 \end{aligned}$$

from Theorem 0.1 and Corollary 1.2.

Next we prove (4.6). From (4.10), for $x = (x_1, o_2) \in A_1 \cup \{o\} \backslash \partial D$,

$$(4.12).\qquad \begin{aligned} E[\tau^{\varepsilon}_{D_2}(o_2)] &\geq E[\tau^{\varepsilon}_{D}(x)] \\ &\geq E[\tau^{\varepsilon}_{D}(x); \tau^{\varepsilon}_{D}(x) > \varepsilon^{-(1-\delta)\ell/(\ell+2)}] \\ &\geq \varepsilon^{-(1-\delta)\ell/(\ell+2)} P(\tau^{\varepsilon}_{D}(x) > \varepsilon^{-(1-\delta)\ell/(\ell+2)}), \end{aligned}$$

which completes the proof from Theorems 0.1, 1.3 and Corollary 1.2.

Since (4.7) can be proved in the routine manner (see [8], Chap. 3, sections 2,3), we omit the proof and proceed to the proof of (4.8).

For $x = (x_1, x_2) \in A_2 \cup A_3 \backslash \partial D$,

$$(4.13).\qquad \limsup_{\varepsilon \to 0} E[\tau^{\varepsilon}_{D}(x)] \leq \limsup_{\varepsilon \to 0} E[\tau^{\varepsilon}_{D_2}(x_2)] \leq \tau^{0}_{D_2}(x_2) = \tau^{0}_{D}(x)$$

from (4.10) and Theorem 1.3, (III).

We also have the following;

$$(4.14).\qquad \liminf_{\varepsilon \to 0} E[\tau^{\varepsilon}_{D}(x)] \geq \tau^{0}_{D}(x) (= \tau^{0}_{D_2}(x_2)).$$

Let us prove (4.14). For $\delta > 0$, put

$$(4.15).\qquad \inf\{dist(X^{0}_{2}(t, x_2), D^{c}_{2}); 0 \leq t \leq \tau^{0}_{D_2}(x_2) - \delta\} \equiv \gamma(x_2, \delta),$$

which is positive from (H.2). Then

$$(4.16).\ \begin{aligned} E[\tau^{\varepsilon}_{D}(x)] &\geq E[\tau^{\varepsilon}_{D}(x); \tau^{\varepsilon}_{D}(x) > \tau^{0}_{D_2}(x_2) - \delta] \\ &\geq E[\tau^{\varepsilon}_{D}(x); \sup_{0 \leq t \leq \tau^{0}_{D_2}(x_2) - \delta} |X^{\varepsilon}_{2}(t, x_2) - X^{0}_{2}(t, x_2)| < \gamma(x_2, \delta) \\ &\qquad , \tau^{\varepsilon}_{D_1}(x_1) > \tau^{0}_{D_2}(x_2) - \delta] \\ &\geq (\tau^{0}_{D_2}(x_2) - \delta)\{P(\sup_{0 \leq t \leq \tau^{0}_{D_2}(x_2) - \delta} |X^{\varepsilon}_{2}(t, x_2) - X^{0}_{2}(t, x_2)| < \gamma(x_2, \delta)) \\ &\qquad - P(\tau^{\varepsilon}_{D_1}(x_1) \leq \tau^{0}_{D_2}(x_2) - \delta)\}. \end{aligned}$$

The first probability in the last part of (4.16) converges to 1 as $\varepsilon \to 0$, which is a fundamental fact in Freidlin-Wentzell theory (see [8], Chap. 3, sections 2, 3), and the second probability in the last part of (4.16) converges to 0 as $\varepsilon \to 0$ from Theorem 0.1, which completes the proof, since $\delta$ can be taken arbitrary small.

Q. E. D.

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