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T. Nakazi and M. Yamada

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Riesz's Functions In Weighted Hardy And Bergman Spaces

by

Takahiko Nakazi *

and

Masahiro Yamada

Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

Department of Mathematics

Hokkaido University

Sapporo 060, Japan

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Abstract. Let μ be a finite positive Borel measure on the closed unit disc $\bar{D}$. For each a in $\bar{D}$, put

$$S(a) = \inf \int_{\bar{D}} |f|^p d\mu$$

where f ranges over all analytic polynomials with $f(a) = 1$. This upper semicontinuous function $S(a)$ is called a Riesz's function and studied in detail. Moreover several applications are given to weighted Bergman and Hardy spaces.

§1. Introduction

Let D be the open unit disc in the complex plane $\mathbb{C}$. P denotes a set of all analytic polynomials and H denotes a set of all analytic functions on D . Suppose $0 < p < \infty$. When μ is a finite positive Borel measure on $\bar{D}$ and $a \in \bar{D}$, put

$$S(\mu, a) = S(\mu, p, a) = \inf \left\{ \int_D |f|^p d\mu ; f \in P \text{ and } f(a) = 1 \right\}$$

and

$$R(\mu, a) = R(\mu, p, a) = \sup \{ |f(a)|^p ; f \in P \text{ and } \int_D |f|^p d\mu \leq 1 \}.$$

When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu, a) = s(\mu, p, a) = \inf \left\{ \int_D |f|^p d\mu ; f \in H \text{ and } f(a) = 1 \right\}$$

and

$$r(\mu, a) = r(\mu, p, a) = \sup \{ |f(a)|^p ; f \in H \text{ and } \int_D |f|^p d\mu \leq 1 \}.$$

The four functions S, R, s and r are called Riesz's functions. In this paper we study these four Riesz's functions. M. Riesz used such functions to solve the moment problem on the real line (cf. [6, Chapter 5]). T. Kriete and T. Trent [7] also investigated the relationship between μ and $R(\mu, 2, a)$. In the investigations of Riesz's functions, the most fundamental and important result is the following theorem by G. Szegő (cf. [5, Chapter 3]). He proved it only when $p = 2$ but it can be proved for arbitrary p .

Szegő's Theorem. Suppose $0 < p < \infty$, μ is a finite positive Borel measure on $\bar{D}$ with $\text{supp } \mu \subseteq \partial D$ and $d\mu/(d\theta/2\pi) = w(e^{i\theta})$.

Then,

$$S(\mu, p, a) = (1 - |a|^2) \exp(\log w)^\wedge(a) \quad (a \in D)$$

$$\text{where } (\log w)^\wedge(a) = \int_0^{2\pi} \log w(e^{i\theta}) \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta/2\pi.$$

It is most desirable to describe $S(\mu, p, a)$ using μ as in Szegő's Theorem, when μ is an arbitrary measure on $\bar{D}$. However such a problem is very difficult except for some special measures μ . In Section 2, we study the behaviour of $S(\mu, p, a)$ as $|a| \rightarrow 1$ for an arbitrary measure on $\bar{D}$. Moreover we note that $S(\mu, p, a) R(\mu, p, a) = 1$ ($a \in \bar{D}$). Thus we need to know only S or R . In this paper, the results and the proofs about s and r are very similar to those about S and R . Hence we concentrate on only S or R in

Sections 2, 3 and 4. Let m be the normalized area measure on D , that is, $dm = r dr d\theta / \pi$. In Section 3, we give the several lower estimates of S using $d\mu/dm$. It is more difficult to give the upper estimates of S . We do it only in very special cases. In Section 4, we show that $R(\mu, p, a)$ is not in $L^1(\mu)$ if $\text{supp } \mu$ is not a finite set

Suppose $0 < p < \infty$. $H^p(\mu)$ denotes the closure of P in $L^p(\mu)$ when μ is a finite positive Borel measure on $\bar{D}$. $H^p(\mu)$ is called a weighted Hardy space. If $d\mu = d\theta/2\pi$, $H^p(\mu) = H^p$ is the classical Hardy space. When μ is a finite positive Borel measure on D , then one defines $L_a^p(\mu) = H \cap L^p(\mu)$. $L_a^p(\mu)$ is called a weighted Bergman space. If $\mu = m$, $L_a^p(\mu) = L_a^p$ is the usual Bergman space. H^p can be embedded in H . $L_a^p = H^p(m)$ and hence L_a^p is closed. We are interested in the following questions : (1) When $H^p(\mu)$ can be embedded in H ? (2) When $L_a^p(\mu)$ is closed ? (3) When $H^p(\mu)$ can be embedded in $L_a^p(\mu)$? Of course it is very interesting to know when $L_a^p(\mu) = H^p(\mu)$, where μ is a measure on D . This problem is classical and important (cf. [2]). However, in this paper we are not going to consider this problem. The problem (2) was studied by M.Yamada [13]. If μ is a measure on D , the problem (1) is equivalent to (3). Note that the measure μ for (2) satisfies (3). In Section 5, we study the three problems above. For example, for some compact set K in D , if $\int_{D \setminus K} \log W dm > -\infty$ then $H^p(\mu)$ can be embedded in H where $W = d\mu/dm$. This result follows from the lower estimate of $S(\mu, p, a)$ in Section 3.

In this paper, we will use the following notations. For each $a \in D$, let ϕ_a be the Möbius function on D , that is,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (z \in D),$$

and put

$$\beta(a, z) = \frac{1}{2} \log \frac{1 + |\phi_a(z)|}{1 - |\phi_a(z)|} \quad (a, z \in D).$$

For $0 < r \leq \infty$ and $a \in D$,

$$D_r(a) = \{z \in D ; \beta(a, z) < r\}$$

be the Bergman disc with 'center' a and 'radius' r . For $u \in L^1(m)$,

$$\tilde{u}(a) = \int_D u \circ \phi_a(z) dm(z) \quad (a \in D).$$

Then $\tilde{u}$ may be bounded on D even if u is not bounded on D .

§2. Riesz's function

If $\mu = m$, then for $0 < p < \infty$ $S(m, p, a) = (1 - |a|^2)^2$. Hence $\mu = m$ or $\text{supp } \mu \subseteq \partial D$, by Szegő's Theorem $\lim_{r \rightarrow 1^-} S(\mu, p, re^{i\theta}) = 0$ a.e. θ . In this section, we show that this is true in general. In particular, R is not bounded on D . In fact, for arbitrary μ , we show that $\lim_{r \rightarrow 1^-} S(\mu, p, re^{i\theta}) = 0$ except a countable set of θ .

Proposition 1. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. Then the following are valid for $R(a) = R(\mu, p, a)$ and $S(a) = S(\mu, p, a)$.

- (1) $R(\mu, p, a) S(\mu, p, a) = 1$ for $a \in \bar{D}$, assuming $\infty \times 0 = 1$.
- (2) $R(\mu)$ is lower semicontinuous on $(0, \infty) \times D$, and $S(\mu)$ is upper semicontinuous on the same set. Moreover $R(\mu, p, a) \geq 1/\mu(\bar{D})$ and $S(\mu, p, a) \leq \mu(\bar{D})$.
- (3) If $\log R$ or R is in $L^1(m)$, then for $a \in D$

$$R(a) \leq \exp(\log R)^\sim(a) \leq \tilde{R}(a).$$

- (4) If $r < \infty$, then for $a \in D$

$$\log R(a) \leq \left(\frac{1 + s|a|}{1 - s|a|} \right)^2 \frac{1}{m(D_r(a))} \int_{D_r(a)} \log R dm$$

where $s = \tanh r$. Hence for $a \in D$

$$\log R(a) \leq \left(\frac{1 + |a|}{1 - |a|} \right)^2 \int_D \log R dm.$$

These inequalities are also valid for R instead of $\log R$.

- (5) For $a \in D$,

$$S(\mu, p, a) \geq S(S(\mu)dm, p, a).$$

- (6) R is not bounded on D and $\bar{D}$.

Proof. (1) It is easy to see that $1 \leq R(a)S(a)$ for $a \in \bar{D}$. If $1 < R(a)S(a)$, then there exists a positive constant γ such that $1 \leq \gamma S(a)$ and $\gamma < R(a)$. Hence $1 \leq \gamma \int |g|^p d\mu$ for any $g \in P$ with $g(a) = 1$ and so

$$|f(a)|^p \leq \gamma \int_{\bar{D}} |f|^p d\mu \text{ for any } f \in P.$$

This implies $\gamma \geq R(a)$. This contradiction shows that $1 = R(a)S(a)$. (2) is clear by (1). (3) If $f \in P$, then $\log |f|$ is subharmonic on D and hence for any $a \in D$,

$$\log |f(a)|^p \leq \int_D \log |f(z)|^p \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm(z).$$

Assuming $\int |f|^p d\mu \leq 1$, by definition on R

$$\log R(a) \leq \int_D \log R(z) \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm(z).$$

This implies $R(a) \leq \exp(\log R)^{\sim}(a) \leq \tilde{R}(a)$. (4) If $0 < r < \infty$, for any $a \in D_r(0)$ and any $f \in P$,

$$\log |f(a)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} \log |f(z)|^p \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm(z)$$

and hence

$$\log |f(a)|^p \leq \frac{1}{m(D_r(a))} \left(\frac{1 + s|a|}{1 - s|a|} \right)^2 \int_{D_r(a)} \log |f|^p dm$$

where $s = \tanh r$. This proof is same to that of [14, Proposition 4.3.8.]. Assuming $\int |f|^p d\mu \leq 1$, we get (4) as in (3). (5) By (1),

$$\int |f|^p d\mu \geq S(\mu, z) |f(z)|^p \quad (z \in D).$$

and hence $\int |f|^p d\mu \geq \int |f|^p S(\mu) dm$. Assuming $f(a) = 1$ and $a \in D$, we get $S(\mu, a) \geq S(S(\mu) dm, a)$. (6) If $R(\mu, p, a)$ is bounded on $\bar{D}$, then $H^p(\mu) \subset L^\infty(\mu)$. By [11, Theorem 5.2], $H^p(\mu)$ is finitely dimensional. It is easy to see that $\text{supp } \mu$ is a finite set. Then trivially $R(\mu, p, a) = \infty$ except $\text{supp } \mu$. The proof of the statement for D is same to that for $\bar{D}$, assuming $\mu = \mu|_D$.

Even if v is not bounded, $\tilde{v}$ may be bounded. However (3) and (6) of Proposition 1 show that $\tilde{R}$ is also not bounded. The following theorem gives a stronger result.

Theorem 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure. If $a \in \partial D$, then the following are valid.

- (1) $\mu(\{a\}) = 0$ if and only if $S(\mu, p, a) = 0$.
- (2) $\lim_{r \rightarrow 1^-} S(\mu, p, ra) = 0$ except a countable set of a in ∂D .
- (3) If $\mu(\{a\}) = 0$ and $\{a_n\}$ is a sequence in D with $\lim a_n = a$, then $\lim_{n \rightarrow \infty} S(\mu, p, a_n) = 0$.
- (4) If $\mu(\{a\}) > 0$, then for each n , the set $\{z \in D; |z - a| < 1/n\} \cap \{z \in D; S(\mu, p, z) < 1/n\}$ is a nonempty open set.
- (5) If $b < c$ and $E = \{z \in D; z = re^{i\theta}, 0 \leq r < 1 \text{ and } b \leq \theta \leq c\}$, then R is not bounded on E .

Proof. We may assume $a = 1$. (1) If $\mu(\{1\}) > 0$, then $|f(1)|^p \leq \int |f|^p d\mu / \mu(\{1\})$ and so $R(\mu, p, 1) \leq 1/\mu(\{1\})$. (1) of Proposition 1 implies $S(\mu, p, 1) > 0$. Conversely suppose $\mu(\{1\}) = 0$. If $z \in \bar{D}$ and $z \neq 1$, then $\lim_{t \rightarrow 1+} |(1-t)/(z-t)| = 0$ and

$$\left| \frac{z-1}{z-t} - 1 \right| = \left| \frac{1-t}{z-t} \right| < 1 \quad (t > 1).$$

For any $t > 1$,

$$S(\mu, p, 1) \leq \int_{\bar{D}} \left| 1 - \frac{z-1}{z-t} \right|^p d\mu(z) = \int_{\bar{D} \setminus \{1\}} \left| \frac{1-t}{z-t} \right|^p d\mu(z).$$

As $t \rightarrow 1$, by the Lebesgue's dominated convergence theorem, $S(\mu, p, 1) = 0$. (2) Suppose $\mu(\{1\}) = 0$. If there exist a sequence $\{r_n\}$ and a positive constant ε such that $0 < r_n < 1$ with $r_n \rightarrow 1$ and $S(\mu, p, r_n) \geq \varepsilon > 0$, then

$$|f(r_n)|^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} |f|^p d\mu \text{ and so } |f(1)|^p \leq \frac{1}{\varepsilon} \int_{\bar{D}} |f|^p d\mu.$$

This implies $S(\mu, p, 1) > 0$ and contradicts (1). Hence if $\mu(\{1\}) = 0$, then $\lim_{r \rightarrow 1^-} S(\mu, p, r) = 0$. This implies (2) because $\{a \in \partial D ; \mu(\{a\}) > 0\}$ is a countable set. (3) is clear by the proof of (2). (4) Suppose $\mu(\{1\}) > 0$ and for each n , put

$$G_n = \{z \in \bar{D} ; |z-1| < \frac{1}{n}\} \cap \{z \in \bar{D} ; S(\mu, p, z) < \frac{1}{n}\}.$$

Since $\{z \in \partial D ; \mu(\{z\}) > 0\}$ is a countable set, for each n there exists $b_n \in \{z \in \partial D ; |z-1| < \frac{1}{n}\}$ with $\mu(\{b_n\}) = 0$. Then $S(\mu, p, b_n) = 0$ by (1) and hence G_n is not empty. G_n is a relatively open set in $\bar{D}$ by (2) of Proposition 1 and so $G_n \cap D$ is a nonempty open set. (5) follows from (2).

If $R(\mu, 2, a) < \infty$, then there exists k_a in $H^2(\mu)$ such that $f(a) = \int f(z) \overline{k_a(z)} d\mu(z)$ for any f in $H^2(\mu)$ and hence $R(\mu, 2, a) = \int |k_a(z)|^2 d\mu(z)$. Thus the results in this section give the informations about the reproducing kernel k_a .

§3. Estimate of Riesz's function

In this section we give upper and lower estimates of S . The lower ones will be used later. The following proposition is a generalization of Szegő's Theorem in Introduction. In fact, if $\mu \ll D$ is a zero measure, then it gives Szegő's Theorem.

Proposition 3. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure such that $(d\mu \ll \partial D)/(d\theta/2\pi) = w(e^{i\theta})$, $\mu \ll D = \sum a_j \delta_{z_j}$ and $\sum (1 - |z_j|) < \infty$. Let b be a Blaschke product of $\{z_\ell\}$ and b_j a Blaschke product of $\{z_\ell\}_{\ell \neq j}$. Then for all $a \in D$, $(1 - |a|^2) \exp(\log w)^\wedge(a) \leq S(\mu, p, a)$. If $a \in D \setminus \{z_\ell\}$, then

$$S(\mu, p, a) \leq |b(a)|^{-p} (1 - |a|^2) \exp(\log w)^{\wedge}(a).$$

If $a = z_j$, then

$$S(\mu, p, a) \leq |b_j(a)|^{-p} (1 - |a|^2) \exp(\log w)^{\wedge}(a) + a_j.$$

In particular, $S(\mu, p, a) > 0$ if and only if $\log w \in L^1(d\theta)$.

Proof. Since $S(\mu, p, a) \geq S(wd\theta/2\pi, p, a)$ for all $a \in D$, by Szegő's Theorem $(1 - |a|^2) \exp(\log w)^{\wedge}(a) \leq S(\mu, p, a)$ for all $a \in D$. Let B_n be a finite Blaschke product of $\{z_1, z_2, \dots, z_n\}$. If $a \in D \setminus \{z_\ell\}$, then

$$\begin{aligned} S(\mu, p, a) &\leq \inf \left\{ \int \left| \frac{B_n}{B_n(a)} g \right|^p d\mu \mid \partial D + \sum_{j=1}^{\infty} a_j \left| \frac{B_n(z_j)}{B_n(a)} g(z_j) \right|^p ; g \in P \text{ and } g(a) = 1 \right\} \\ &= \frac{1}{|B_n(a)|^p} \inf \left\{ \int |B_n g|^p d\mu \mid \partial D + \sum_{j=n+1}^{\infty} a_j |B_n(z_j)|^p |g(z_j)|^p ; g \in P \right. \\ &\quad \left. \text{and } g(a) = 1 \right\}. \end{aligned}$$

As $n \rightarrow \infty$,

$$S(\mu, p, a) \leq \frac{1}{|b(a)|^p} \inf \left\{ \int |g|^p d\mu \mid \partial D ; g \in P \text{ and } g(a) = 1 \right\}.$$

Now by Szegő's Theorem, for each $a \in D$ $S(\mu, p, a) \leq |b(a)|^{-p} (1 - |a|^2) \exp(\log w)^{\wedge}(a)$. Let $B_{j,n}$ be a finite Blaschke product of $\{z_1, z_2, \dots, z_n\} \setminus \{z_j\}$. If $a = z_j$ and $n > j$, then

$$\begin{aligned} S(\mu, p, a) &\leq \inf \left\{ \int \left| \frac{B_{j,n}}{B_{j,n}(a)} g \right|^p d\mu ; g \in P \text{ and } g(a) = 1 \right\} \\ &= \frac{1}{|B_{j,n}(a)|^p} \inf \left\{ \int |B_{j,n} g|^p d\mu \mid \partial D + a_j |B_{j,n}(a)|^p \right. \\ &\quad \left. + \sum_{\ell \geq n+1} a_\ell |B_{j,n}(z_\ell)|^p |g(z_\ell)|^p ; g \in P \text{ and } g(a) = 1 \right\}. \end{aligned}$$

As $n \rightarrow \infty$, by Szegő's Theorem, for $a = z_j$,

$$S(\mu, p, a) \leq |b_j(a)|^{-p} (1 - |a|^2) \exp(\log w)^{\wedge}(a) + a_j.$$

The following proposition is related to Theorem 2 in this paper and Theorem in [7]. In fact, if $\tilde{W}$ is bounded on D , then $(1 - |a|^2)^{-2} S(Wdm, p, a)$ is bounded on D . Moreover if W is continuous on $\bar{D}$, then for all $e^{i\theta}$

$$\lim_{a \rightarrow e^{i\theta}} (1 - |a|^2)^2 R(W dm, p, a) = 1/W(e^{i\theta}).$$

$\bar{D}$.

Proposition 4. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on

- (1) $\tilde{\mu}(a) \geq (S(\mu))^\sim(a)$ ($a \in D$).
- (2) If $d\mu = W dm$ and $a \in D$, then

$$(1 - |a|^2)^2 \exp(\log W)^\sim(a) \leq S(\mu, p, a) \leq (1 - |a|^2)^2 \tilde{W}(a).$$

- (3) $S(W dm, a) = (1 - |a|^2)^2 S(W \circ \phi_a dm, 0)$ for $a \in D$.

Proof. (1) For all $z \in D$

$$\int |f|^p d\mu \geq |f(z)|^p S(z) \text{ and so } \int |f|^p d\mu \geq \int |f|^p S dm.$$

Assuming $f(z) = \{(1 - |a|^2)/(1 - \bar{a}z)^2\}^{2/p}$ for $a \in D$, $\tilde{\mu}(a) \geq \tilde{S}(a)$. (2) If $\log W \in L^1(m)$, then

$$\begin{aligned} S(W dm, p, a) &= \inf \left\{ \int |f|^p W dm ; f \in P \text{ and } f(a) = 1 \right\} \\ &= \inf \left\{ \int |g|^p W \circ \phi_a \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm ; g \in H^p(W \circ \phi_a dm) \text{ and } g(0) = 1 \right\} \\ &= (1 - |a|^2)^2 \inf \left\{ \int |k|^p W \circ \phi_a dm ; k \in H^p(W \circ \phi_a dm) \text{ and } k(0) = 1 \right\} \\ &\geq (1 - |a|^2)^2 \exp \int (\log W) \circ \phi_a dm = (1 - |a|^2)^2 \exp(\log W)^\sim(a). \end{aligned}$$

The inequality above is proved by two Jensen's inequalities. (3) is clear by the proof of (2).

In (2) of Proposition 4, we can get estimates of $S(\mu, p, a)$ as in Proposition 3 when $d\mu = W dm + \sum_{j=1}^{\infty} a_j \delta_{z_j}$, $\{z_j\} \subset D$ and $\sum (1 - |z_j|) < \infty$. The following theorem is important in this paper and the following lemma is used to prove it.

Lemma 1. Let $\Delta_s(a)$ be the set $\{z \in D ; |(a - z)/(1 - \bar{a}z)| < s\}$ where $a \in D$ and $s \in (0, 1)$. If $t \in (0, 1)$ and $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$, then $\Delta_t(0) \subset \Delta_s(a)$.

Proof. The Euclidean center and radius of $\Delta_s(a)$ are

$$C = \frac{1-s^2}{1-s^2|a|^2}a, \quad R = \frac{1-|a|^2}{1-s^2|a|^2}s$$

respectively. Hence to prove $\overline{\Delta_t(0)} \subset \Delta_s(a)$, it is sufficient to show that

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| \leq \frac{1-|a|^2}{1-s^2|a|^2}s.$$

If $1-s^2 = (1-|a|^2)(1-t^2)/5$, then

$$1-s^2 \leq \frac{(1-|a|^2)(1-t^2)}{5-|a|^2}$$

and hence $s^2 \geq \{4 + (1-|a|^2)t^2\}/(5-|a|^2)$. The last inequality is equivalent to

$$1-s^2 \leq \frac{(1-|a|^2)(s^2-t^2)}{4}.$$

Then

$$1-s^2 \leq \frac{(1-|a|^2)(s-t)}{2} \frac{s+t}{2} \leq \frac{(1-|a|^2)(s-t)}{|a|(t|a|+1)}$$

because $s+t \leq 2$ and $|a|(t|a|+1) \leq 2$. This implies that

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| \leq \frac{1-|a|^2}{1-s^2|a|^2}s.$$

Theorem 5. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on $\bar{D}$. Let $d\mu/dm = Wdm$, K an arbitrary compact set in D and $t = \max\{|z|; z \in K\}$. Then, for $a \in D$

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5} \exp \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)} \int_{K^c} \log(W \wedge 1) dm.$$

If $1 \leq p < \infty$ and $a \in D$, then

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^{3(2-\frac{1}{p})}(1-t^2)^{2-\frac{1}{p}}}{2^{4(1-\frac{1}{p})} \cdot 5^{2-\frac{1}{p}}} \left(\int_{K^c} W^{-\frac{1}{p-1}} dm \right)^{\frac{1}{p}-1}.$$

Proof. By two Jensen's inequalities, for $a \in D$

$$\begin{aligned} S(\mu, p, a) &\geq S(Wdm, p, a) \\ &= \inf \left\{ \int |g|^p W \circ \phi_a \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm ; g(0) = 1 \right\} \\ &= (1-|a|^2)^2 \inf \left\{ \int |k|^p W \circ \phi_a dm ; k(0) = 1 \right\} \end{aligned}$$

$$\begin{aligned}
&\geq (1-|a|^2)^2 \int_0^1 2r dr \exp \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&\geq (1-|a|^2)^2 (1-s^2) \int_s^1 \frac{2r}{1-s^2} dr \exp \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&\geq (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_s^1 2r dr \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&= (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_{D \setminus \Delta_s(0)} \log W \circ \phi_a dm \\
&= (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_{D \setminus \Delta_s(a)} \log W \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm \\
&\geq (1-|a|^2)^2 (1-s^2) \exp \frac{(1-|a|^2)^2}{(1-|a|^4)} \frac{1}{1-s^2} \int_{D \setminus \Delta_s(a)} \log(W \wedge 1) dm
\end{aligned}$$

where $s \in (0,1)$ and $\Delta_s(a) = \{z \in D ; |(a-z)/(1-\bar{a}z)| < s\}$. For each compact set $K \subset D$, if $t = \max\{|z| ; z \in K\}$ and $1-s^2 = (1-|a|^2)(1-t^2)/5$, then by Lemma 1 $\Delta_t(0) \subset \Delta_s(a)$. Hence $K \subset \Delta_s(a)$ and so $K^c \supset D \setminus \Delta_s(a)$. Thus, if $1-s^2 = (1-|a|^2)(1-t^2)/5$, then

$$\frac{(1-|a|^2)^2}{(1-|a|^4)} \frac{1}{1-s^2} = \frac{(1+|a|^2)^4}{(1-|a|^2)^2(1-s^2)} \leq \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)}$$

and hence for all $a \in D$

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5} \exp \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)} \int_{K^c} \log(W \wedge 1) dm.$$

Now we will prove the second inequality. Instead of two Jensen's inequalities, we will use Kolmogoroff's inequality (cf.[12, Theorem 4.3.1]). For $a \in D$, if $1 \leq p < \infty$ and $1/p + 1/q = 1$,

$$\begin{aligned}
&S(\mu, p, a) \\
&\geq (1-|a|^2)^2 \int_0^1 2r dr \left(\int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2) \int_s^1 \frac{2r}{1-s^2} dr \left(\int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2) \left(\frac{1}{1-s^2} \int_s^1 2r dr \int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left(\int_{D \setminus \Delta_s(0)} (W \circ \phi_a)^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left(\int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left\{ \frac{(1-|a|^2)^2}{(1-|a|^4)} \int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} dm \right\}^{-\frac{1}{q}}
\end{aligned}$$

$$\geq \frac{(1 - |a|^2)^{2(1+\frac{1}{q})}(1 - s^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}} \left(\int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}}$$

where $s \in (0, 1)$. As in the proof of the first inequality, for each compact set $K \subset D$, if $t = \max\{|z|; z \in K\}$ and $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$, then $K^c \supset D \setminus \Delta_s(a)$. Thus, if $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$, then for all $a \in D$

$$S(\mu, p, a) \geq \frac{(1 - |a|^2)^{3(1+\frac{1}{q})}(1 - t^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}} \cdot 5^{1+\frac{1}{q}}} \left(\int_{K^c} W^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}}.$$

The second inequality of Theorem 5 implies $S(\mu, 1, a) \geq (1 - |a|^2)^3 \times (1 - t^2)(1/5) \text{ess.inf}\{W(x); x \in K^c\}$. Let σ be a finite positive Borel measure on $[0, 1]$. $\mu(re^{i\theta}) = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ is more general than $Wdm = 2rdr \times W(re^{i\theta})d\theta/2\pi$. If $\sigma(r)$ is singular to the Lebesgue measure on $[0, 1]$, then μ is singular to m . However we can give an interesting lower estimate. It is different from that of Theorem 5 in case of $\mu = Wdm$.

Theorem 6. Suppose $0 < p < \infty$ and $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ where $\sigma(r)$ is a finite positive Borel measure on $[0, 1]$. If $W(e^{i\theta}) = \sup_r W(re^{i\theta})$ and $W_r(e^{i\theta}) = W(re^{i\theta})$, then for $a \in D$

$$\begin{aligned} (1 - |a|^2) \int_{|a|}^1 \exp(\log W_r)^{\wedge}(a) d\sigma(r) &\leq S(\mu, p, a) \\ &\leq \sigma([0, 1]) \inf \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &\leq \sigma([0, 1]) \inf \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(e^{i\theta}) d\theta/2\pi; f(a) = 1 \right\}. \end{aligned}$$

Proof. For $a \in D$,

$$\begin{aligned} S(\mu, p, a) &= \inf \left\{ \int_0^1 d\sigma(r) \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &\geq \int_0^1 d\sigma(r) \inf \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &= \int_{|a|}^1 d\sigma(r) \inf \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &= \int_{|a|}^1 (1 - |a|^2) \exp(\log W_r)^{\wedge}(a) d\sigma(r). \end{aligned}$$

We used Szegő's Theorem in the last equality. The upper estimates are trivial.

Corollary 1. Let $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$ as in Theorem 6 and $0 < p < \infty$.

(1) If $W(re^{i\theta}) \equiv 1$, then for $a \in D$

$$(1 - |a|^2) \sigma([|a|, 1]) \leq S(\mu, p, a) \leq (1 - |a|^2) \sigma([0, 1]).$$

In particular, $S(\mu, p, 0) = \sigma([0, 1])$.

(2) If $W(re^{i\theta}) = |h(re^{i\theta})|$ for some outer function h in $H^1(d\theta)$, then for $a \in D$

$$(1 - |a|^2) \int_{|a|}^1 W(ra) d\sigma(r) \leq S(\mu, p, a) \leq (1 - |a|^2) W(a) \sigma([0, 1]).$$

(3) If $1 < p < \infty$ and $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$ satisfies the A_p condition, then there exists a positive constant γ such that for $a \in D$

$$S(\mu, p, a) \leq \gamma(1 - |a|^2) \exp(\log \mathbf{W})^\wedge(a) \sigma([0, 1]).$$

Proof. (1) is a special case of (2). (2) Since h is an outer function in H^1 , for $a \in D$

$$\exp(\log W_r)^\wedge(a) = \exp(\log |h_r|)^\wedge(a) = |h(ra)| = W(ra)$$

and

$$\begin{aligned} & \inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta / 2\pi \right\} \\ &= \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p |h(e^{i\theta})| d\theta / 2\pi = (1 - |a|^2) |h(a)| = (1 - |a|^2) W(a). \end{aligned}$$

Now Theorem 6 implies (2). (3) By a theorem of M. Rosenblum (cf. [10] and [9, Theorem 2.2]), there exists a positive constant γ such that for any $f \in P$

$$\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \leq \gamma \int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi$$

because $\mathbf{W} \in A_p$. By Theorem 6 and Szegő's Theorem, for $a \in D$

$$\begin{aligned} & \inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \right\} \leq \gamma \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \\ &= \gamma(1 - |a|^2) \exp(\log \mathbf{W})^\wedge(a). \end{aligned}$$

This implies (3).

§4. Carleson inequality and Riesz's function

Let ν and μ be finite positive Borel measures on $\bar{D}$ and $1 \leq p < \infty$. We say that ν and μ satisfy the (ν, μ, p) - Carleson inequality, if there exists a constant $\gamma > 0$ such that

$$\int_{\bar{D}} |f|^p d\nu \leq \gamma \int_{\bar{D}} |f|^p d\mu$$

for all $f \in P$ (see [8]). ν and μ satisfy the (ν, μ, p) - Carleson inequality if and only if $H^p(\mu) \subset H^p(\nu)$ and the inclusion mapping $i_p : H^p(\mu) \rightarrow H^p(\nu)$ is bounded. We say that for $p > 1$, ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality if $H^p(\mu) \subset H^p(\nu)$ and $i_p : H^p(\mu) \rightarrow H^p(\nu)$ is compact. We say that for $p = 1$, ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality if i_p is star-compact. We could not prove Theorem 7 for $p = 1$ because we do not know anything about the predual of $H^1(\mu)$. Using Riesz's functions, we will show vanishing Carleson inequalities. As a result, we show that $R(\mu, p) \notin L^1(\mu)$ if $\text{supp } \mu$ is not a finite set. Moreover, from given a measure μ , we will show how to construct a measure ν such that the (ν, μ, p) -vanishing Carleson inequality is valid.

Theorem 7. Suppose $1 < p < \infty$, and ν and μ are finite positive Borel measures on $\bar{D}$.

(1) If $\int R(\mu, p) d\nu < \infty$, then ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality and

$$R(\mu, p, a) \leq \left(\int R(\mu, p) d\nu \right) R(\nu, p, a) \quad (a \in \bar{D}).$$

(2) If V is a Borel function such that $0 \leq V \leq S$ on $\bar{D}$, then $V |g|^p$ is bounded on $\bar{D}$ for each g in $H^p(\mu)$, and $V d\mu$ and μ satisfy the $(V d\mu, \mu, p)$ -vanishing Carleson inequality.

Proof. (1) By definition of $R(\mu, p, a)$, for $a \in \bar{D}$,

$$|f(a)|^p \leq R(\mu, p, a) \int |f|^p d\mu \quad (f \in P).$$

Hence if $\gamma = \int R(\mu, p) d\nu < \infty$, then $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$ ($f \in P$) and so $i_p : H^p(\mu) \rightarrow H^p(\nu)$ is bounded. We will show that i_p is compact. If $f_n \rightarrow f$ weakly in $H^p(\mu)$, then there exists a finite positive constant γ' such that

$$\int |f_n - f|^p d\mu \leq \gamma' \quad \text{for all } n.$$

By the hypothesis, $R(\mu, p, a) < \infty$ ν -a.e. on $\bar{D}$ and so $f_n \rightarrow f$ ν -a.e. on $\bar{D}$ because $f_n \rightarrow f$ weakly. Moreover by definition of $R(\mu, p, a)$, $|f_n(a) - f(a)|^p \leq \gamma' R(\mu, p, a)$ and by the hypothesis, $R(\mu, p, a) \in L^1(\nu)$. Thus

$$\int |f_n - f|^p d\nu \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

by the Lebesgue's dominated convergence theorem. This implies i_p is compact. Since $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$ and $\gamma = \int R(\mu, p) d\nu$, assuming $f(a) = 1$, we get $S(\nu, p, a) \leq \gamma S(\mu, p, a)$. Now by (1) of Proposition 1, we get the inequality of (1). (2) If $0 \leq V \leq S$, then $VR \leq 1$ and hence $V(a) |f(a)|^p$ is bounded on $\bar{D}$ by $\int |f|^p d\mu$, for each $f \in H^p(\mu)$. Moreover if $\nu = Vd\mu$ and $0 \leq V \leq S$, then $\int R(\mu, p) d\nu \leq \int d\mu = 1$ and hence by (1) ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality.

Corollary 2. If $0 < p < \infty$ and $\text{supp } \mu$ is not a finite set, then $R(\mu, p) \notin L^1(\mu)$.

Proof. Suppose $1 < p < \infty$. If $R(\mu, p) \in L^1(\mu)$, then the inclusion map $i_p : H^p(\mu) \rightarrow H^p(\mu)$ is compact. It is easy to see that i_p is an identity operator. Hence the unit ball of $H^p(\mu)$ is compact with respect to the norm. Therefore $H^p(\mu)$ is finitely dimensional. This contradicts that $\text{supp } \mu$ is not a finite set. This implies that $R(\mu, p) \notin L^1(\mu)$. For $0 < p \leq 1$, the proof is due to the referee. Choose n sufficiently large that $np > 1$. If $g(a) = 1$ then $g^n(a) = 1$ as well, and g^n is a polynomial if g is a polynomial. Thus,

$$\begin{aligned} S(\mu, p, a) &= \inf \left\{ \int_D |f|^p d\mu ; f \in P, f(a) = 1 \right\} \\ &\leq \inf \left\{ \int_D |g^n|^p d\mu ; g \in P, g(a) = 1 \right\} = S(\mu, np, a). \end{aligned}$$

This implies that $R(\mu, p) \notin L^1(\mu)$ for $0 < p \leq 1$.

Corollary 3. Suppose $1 < p < \infty$ and $d\mu/dm = W$

(1) If $\log W \in L^1(m)$ and $d\nu = (1 - |z|^2)^2 \exp(\log W) dm$, then ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality.

(2) If $\chi_K \log(W \wedge 1) \in L^1(m)$ for some compact set K in D , there exist positive constant a and nonpositive constant b such that $d\nu = a(1 - |z|^2)^3 \{\exp b(1 - |z|^2)^{-3}\} dm$ and μ satisfy the (ν, μ, p) -vanishing Carleson inequality.

(3) Suppose $\chi_K W^{-\frac{1}{p-1}} \in L^1(m)$ for some compact set K in D . If $d\nu = c(1 - |z|^2)^{3(2-\frac{1}{p})} dm$, then ν and μ satisfy the (ν, μ, p) -vanishing Carleson inequality.

Suppose $1 < p < \infty$ and $d\mu/dm = W$. If $\chi_K \log W \in L^1(m)$ for some compact set K in D , there exist positive constant a and nonpositive constant b such that

$$\{a(1 - |z|^2)^3 \exp b(1 - |z|^2)^{-3}\} |f(z)|^p$$

is bounded on D for each $f \in H^p(\mu)$. Here a and b do not depend on f . This is a corollary of (2) in Theorem 7.

§5. $H^p(\mu)$ and $L_a^p(\mu)$

The following is a result of Theorem 5. If $d\mu/dm = W$ and $\log W$ is integrable on the complement K^c of a compact set in D , then $H^p(\mu) \subseteq L_a^p(\mu)$. In this section, we show that if $\log W$ is locally integrable on K^c , then the same result is true. We give a necessary and sufficient condition for $H^p(\mu) \subset L_a^p(\mu)$ using Riesz's function. Theorem 8 is a joint work with K. Takahashi. A subset E of D is a uniqueness set if E satisfies the following : If f in H is zero on E , then $f \equiv 0$ on D .

Lemma 2. Suppose $0 < p < \infty$ and μ is a finite positive Borel measure on D . Then, the following (1) $\sim$ (3) are equivalent.

- (1) $\sup_{a \in K} R(\mu, p, a) < \infty$ for all compact set K in D .
- (2) $\int_K R(\mu, p) dm < \infty$ for all compact set K in D .
- (3) $\int_K \log R(\mu, p) dm < \infty$ for all compact set K in D .

Proof. Both (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We will show (3) $\Rightarrow$ (1). We may assume that $\mu(D) = 1$. For any $f \in P$,

$$\log |f(0)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(0)} \log |f|^p dm.$$

If $a \in D_r(0)$, then for all $f \in P$

$$\log |f(a)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} \log |f|^p \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm.$$

Assuming $\int |f|^p d\mu \leq 1$, we get

$$\log R(\mu, p, a) \leq \frac{1}{m(D_r(0))} \frac{(1 + |a|)^2}{(1 - |a|)^2} \int_{D_r(a)} \log R(\mu, p) dm.$$

Since $D_r(a) \subset D_{2r}(0)$ and $R(\mu, p, a) \geq 1$, for each $a \in D_r(0)$ there exists a finite positive constant γ_r such that

$$\log R(\mu, p, a) \leq \gamma_r \int_{D_{2r}(0)} \log R(\mu, p) dm.$$

This implies (1).

Lemma 3. Let X be a Banach space which consists of analytic functions on D and contains 1. Suppose there exists a dense subspace Y of X such that if f in Y , then $(f - f(a))/(z - a)$ belongs to Y for some a in D . If $(z - a)X$ is not dense in X , then the functional $f \rightarrow f(a)$ is bounded.

Proof. By the hypothesis, if $f \in Y$, then $f = f(a) + (z - a)g$ for some $g \in Y$. Since $(z - a)X$ is not dense in X , there exists $\phi \in X^*$ such that $\langle (z - a)h, \phi \rangle = 0$ for all $h \in X$ but $\langle 1, \phi \rangle \neq 0$. Hence, $\langle f, \phi \rangle = f(a) \langle 1, \phi \rangle$. Thus $|f(a)| \leq \gamma \|f\|$ for all $f \in X$ where $\gamma = |\langle 1, \phi \rangle|^{-1} \|\phi\|_*$.

Theorem 8. Suppose $1 \leq p < \infty$ and μ is a finite positive Borel measure on D such that $(\text{supp } \mu) \cap D$ is a uniqueness set for H .

(1) $L_a^p(\mu)$ is closed if and only if for all compact set K in D

$$\int_K \log r(\mu, p) dm < \infty \text{ or } \int_K \log s(\mu, p) dm > -\infty.$$

(2) $H^p(\mu) \subset L_a^p(\mu)$ if and only if for all compact set K in D

$$\int_K \log R(\mu, p) dm < \infty \text{ or } \int_K \log S(\mu, p) dm > -\infty.$$

Proof. (1) If $f \in L_a^p(\mu)$, then $(f - f(0))/z$ belongs to H . Since $(f - f(0))/z$ is bounded on $|z| \leq t < 1$ and $1/z$ is bounded on $|z| \geq t$, $(f - f(0))/z$ belongs to $L_a^p(\mu)$. This implies that $\{f \in L_a^p(\mu) ; f(0) = 0\} = zL_a^p(\mu)$ and hence $L_a^p(\mu) = \mathbb{C} \oplus zL_a^p(\mu)$. If $Af = zf$ for $f \in L_a^p(\mu)$, then A is a bounded operator on $L_a^p(\mu)$ and the range of A is complemented in $L_a^p(\mu)$ by what was just proved. By [4, Part III, Corollary 2.3], the range of A is closed and hence $zL_a^p(\mu)$ is not dense in $L_a^p(\mu)$. Applying Lemma 3 with $X = Y = L_a^p(\mu)$ and $a = 0$, $r(\mu, p, 0) < \infty$ follows. The same argument is true for all $a \in D \setminus \{0\}$ and hence $r(\mu, p, a) < \infty$ for all $a \in D$. By the boundedness of holomorphic functions on compact sets and the uniform boundedness principle, $\sup_{a \in K} r(\mu, p, a) < \infty$ for all compact set K in D . As Lemma 2 is also for $r(\mu, p, a)$,

$$\int_K \log r(\mu, p) dm < \infty \text{ or } \int_K \log s(\mu, p) dm > -\infty.$$

Conversely, suppose $\int_K \log r(\mu, p) dm < \infty$ for any compact set K . Then by the above lemma, $\sup_K r(\mu, p) < \infty$ for any compact set K . If f is in the $L^p(\mu)$ -norm closure of $L_a^p(\mu)$, then there exists a sequence $\{f_n\}$ in $L_a^p(\mu)$ such that $\int |f - f_n|^p d\mu \rightarrow 0$. Then by hypothesis on $r(\mu, p)$, $\sup\{|f_n(z)| ; z \in D_r(0)\} < \infty$ for each $r < \infty$. Hence, for each $r < \infty$ there exists a subsequence $\{f_{n_j}\}$ in $L_a^p(\mu)$ and an analytic function g_r on $D_r(0)$ such that $f_{n_j} \rightarrow g_r$ uniformly on $D_r(0)$. This implies that $f = g_r \mu - a.e.$ on $D_r(0)$ for all $r < \infty$. Thus $g = \lim_r g_r$ is analytic on D and $f = g \mu - a.e.$ on D .

(2) The 'if' part is same to (1) and hence we will show the 'only if' part. Put $M = \{f \in L^p(\mu) ; zf \in H^p(\mu)\}$, then M is a closed subspace of $L^p(\mu)$ such that

$$M \supseteq H^p(\mu) \supseteq zM \supseteq H^p(\mu)_0$$

where $H^p(\mu)_0 = \{f \in H^p(\mu) ; f(0) = 0\}$. $H^p(\mu)_0$ is well defined because $H^p(\mu) \subset L_a^p(\mu)$. Suppose $H^p(\mu) \neq zM$. Then $H^p(\mu) = \mathbb{C} + H^p(\mu)_0 = \mathbb{C} + zM$ and $\mathbb{C} \cap zM = \{0\}$. As in the proof of (1), by [4, Part III, Corollary 2.3], zM is closed and hence $zH^p(\mu)$ is not dense in $H^p(\mu)$. Applying Lemma 3 with $X = H^p(\mu)$, $Y = P$ and $a = 0$, $R(\mu, p, 0) < \infty$ follows. Suppose $H^p(\mu) = zM$. Then $z^{-1} \in L^p(\mu)$ and hence $\mu(\{0\}) = 0$. If $Af = zf$ for $f \in M$, then A is a one-one bounded operator from M onto $H^p(\mu)$. Therefore A is invertible and hence $A(zM) = zH^p(\mu)$ is closed. Since $H^p(\mu) \subset L_a^p(\mu)$, $zH^p(\mu) \neq H^p(\mu)$ and hence by Lemma 3, $R(\mu, p, 0) < \infty$ follows. The same argument implies that $R(\mu, p, a) < \infty$ for all $a \in D$. Now, as in the proof of (1), Lemma 2 implies the 'only if' part of (2).

Corollary 4. Suppose $1 \leq p < \infty$ and $d\mu/dm = W$. If $\log W$ is locally integrable on K_0^c for some compact set K_0 in D , then $L_a^p(\mu)$ is closed and $H^p(\mu) \subseteq L_a^p(\mu)$.

Proof. By (1) of Theorem 8, it is sufficient to prove that for any compact set K in D , $\inf_K s(\mu, p) > -\infty$. If $\log W$ is integrable on K_0^c , then by the proof of Theorem 5 $\inf_K s(\mu, p) > -\infty$. For a more general W in this corollary, we have to proceed as the following. Suppose $a \in D$ and $0 < \varepsilon < \delta < 1$. As in the proof of Theorem 5,

$$\begin{aligned} s(\mu, p, a) &\geq (1 - |a|^2)^2 \int_{\varepsilon}^{\delta} \exp \left(\int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \right) 2r dr \\ &\geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp \left(\frac{1}{\delta^2 - \varepsilon^2} \int_{\Delta_{\delta}(0) \setminus \Delta_{\varepsilon}(0)} \log W \circ \phi_a dm \right) \\ &\geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp \left(\frac{2^2}{(1 - |a|^2)^2 (\delta^2 - \varepsilon^2)} \int_{\Delta_{\delta}(a) \setminus \Delta_{\varepsilon}(a)} \log(W \wedge 1) dm \right). \end{aligned}$$

Suppose K is an arbitrary compact set in D . Put $t = \max\{|z| ; z \in K_0\}$ and $k = \max\{|z| ; z \in K\}$. The Euclidean center and radius of $\Delta_{\gamma}(k)$ ($0 < \gamma < 1$) are

$$C(\gamma) = \frac{1 - \gamma^2}{1 - \gamma^2 k^2} k, \quad R(\gamma) = \frac{1 - k^2}{1 - \gamma^2 k^2} \gamma$$

respectively. Put $\ell = R(\delta) + C(\delta)$ and $s = R(\varepsilon) - C(\varepsilon)$. There exist δ and ε such that

$$\Delta_{\ell}(0) \setminus \Delta_s(0) \subset D \setminus \Delta_t(0).$$

Then for all $a \in K$

$$\Delta_{\delta}(a) \setminus \Delta_{\varepsilon}(a) \subset \Delta_{\ell}(0) \setminus \Delta_s(0).$$

Hence for all $a \in K$

$$\Delta_{\delta}(a) \setminus \Delta_{\varepsilon}(a) \subset K_0^c$$

and so for all $a \in K$

$$s(\mu, p, a) \geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp \left(\frac{2^2}{(1 - |a|^2)^2 (\delta^2 - \varepsilon^2)} \int_{K_0^\varepsilon} \log(W \wedge 1) dm \right).$$

This shows the corollary.

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