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Riesz's Functions In Weighted Hardy And Bergman Spaces

by

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Dedicated to Professor Fumi-Yuki Maeda on his sixtieth birthday

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Abstract. Let  $\mu$  be a finite positive Borel measure on the closed unit disc  $\bar{D}$ . For each  $a$  in  $\bar{D}$ , put

$$S(a) = \inf \int_{\bar{D}} |f|^p d\mu$$

where  $f$  ranges over all analytic polynomials with  $f(a) = 1$ . This upper semicontinuous function  $S(a)$  is called a Riesz's function and studied in detail. Moreover several applications are given to weighted Bergman and Hardy spaces.

## §1. Introduction

Let  $D$  be the open unit disc in the complex plane  $\mathbb{C}$ .  $P$  denotes a set of all analytic polynomials and  $H$  denotes a set of all analytic functions on  $D$ . Suppose  $0 < p < \infty$ . When  $\mu$  is a finite positive Borel measure on  $\bar{D}$  and  $a \in \bar{D}$ , put

$$S(\mu, a) = S(\mu, p, a) = \inf \left\{ \int_D |f|^p d\mu ; f \in P \text{ and } f(a) = 1 \right\}$$

and

$$R(\mu, a) = R(\mu, p, a) = \sup \{ |f(a)|^p ; f \in P \text{ and } \int_D |f|^p d\mu \leq 1 \}.$$

When  $\mu$  is a finite positive Borel measure on  $D$  and  $a \in D$ , put

$$s(\mu, a) = s(\mu, p, a) = \inf \left\{ \int_D |f|^p d\mu ; f \in H \text{ and } f(a) = 1 \right\}$$

and

$$r(\mu, a) = r(\mu, p, a) = \sup \{ |f(a)|^p ; f \in H \text{ and } \int_D |f|^p d\mu \leq 1 \}.$$

The four functions  $S, R, s$  and  $r$  are called Riesz's functions. In this paper we study these four Riesz's functions. M. Riesz used such functions to solve the moment problem on the real line (cf. [6, Chapter 5]). T. Kriete and T. Trent [7] also investigated the relationship between  $\mu$  and  $R(\mu, 2, a)$ . In the investigations of Riesz's functions, the most fundamental and important result is the following theorem by G. Szegő (cf. [5, Chapter 3]). He proved it only when  $p = 2$  but it can be proved for arbitrary  $p$ .

**Szegő's Theorem.** Suppose  $0 < p < \infty$ ,  $\mu$  is a finite positive Borel measure on  $\bar{D}$  with  $\text{supp } \mu \subseteq \partial D$  and  $d\mu/(d\theta/2\pi) = w(e^{i\theta})$ .

Then,

$$S(\mu, p, a) = (1 - |a|^2) \exp(\log w)^{\wedge}(a) \quad (a \in D)$$

$$\text{where } (\log w)^{\wedge}(a) = \int_0^{2\pi} \log w(e^{i\theta}) \frac{1 - |a|^2}{|1 - \bar{a}e^{i\theta}|^2} d\theta/2\pi.$$

It is most desirable to describe  $S(\mu, p, a)$  using  $\mu$  as in Szegő's Theorem, when  $\mu$  is an arbitrary measure on  $\bar{D}$ . However such a problem is very difficult except for some special measures  $\mu$ . In Section 2, we study the behaviour of  $S(\mu, p, a)$  as  $|a| \rightarrow 1$  for an arbitrary measure on  $\bar{D}$ . Moreover we note that  $S(\mu, p, a) R(\mu, p, a) = 1$  ( $a \in \bar{D}$ ). Thus we need to know only  $S$  or  $R$ . In this paper, the results and the proofs about  $s$  and  $r$  are very similar to those about  $S$  and  $R$ . Hence we concentrate on only  $S$  or  $R$  in

Sections 2, 3 and 4. Let  $m$  be the normalized area measure on  $D$ , that is,  $dm = r dr d\theta / \pi$ . In Section 3, we give the several lower estimates of  $S$  using  $d\mu/dm$ . It is more difficult to give the upper estimates of  $S$ . We do it only in very special cases. In Section 4, we show that  $R(\mu, p, a)$  is not in  $L^1(\mu)$  if  $\text{supp } \mu$  is not a finite set

Suppose  $0 < p < \infty$ .  $H^p(\mu)$  denotes the closure of  $P$  in  $L^p(\mu)$  when  $\mu$  is a finite positive Borel measure on  $\bar{D}$ .  $H^p(\mu)$  is called a weighted Hardy space. If  $d\mu = d\theta/2\pi$ ,  $H^p(\mu) = H^p$  is the classical Hardy space. When  $\mu$  is a finite positive Borel measure on  $D$ , then one defines  $L_a^p(\mu) = H \cap L^p(\mu)$ .  $L_a^p(\mu)$  is called a weighted Bergman space. If  $\mu = m$ ,  $L_a^p(\mu) = L_a^p$  is the usual Bergman space.  $H^p$  can be embedded in  $H$ .  $L_a^p = H^p(m)$  and hence  $L_a^p$  is closed. We are interested in the following questions : (1) When  $H^p(\mu)$  can be embedded in  $H$  ? (2) When  $L_a^p(\mu)$  is closed ? (3) When  $H^p(\mu)$  can be embedded in  $L_a^p(\mu)$  ? Of course it is very interesting to know when  $L_a^p(\mu) = H^p(\mu)$ , where  $\mu$  is a measure on  $D$ . This problem is classical and important (cf. [2]). However, in this paper we are not going to consider this problem. The problem (2) was studied by M.Yamada [13]. If  $\mu$  is a measure on  $D$ , the problem (1) is equivalent to (3). Note that the measure  $\mu$  for (2) satisfies (3). In Section 5, we study the three problems above. For example, for some compact set  $K$  in  $D$ , if  $\int_{D \setminus K} \log W dm > -\infty$  then  $H^p(\mu)$  can be embedded in  $H$  where  $W = d\mu/dm$ . This result follows from the lower estimate of  $S(\mu, p, a)$  in Section 3.

In this paper, we will use the following notations. For each  $a \in D$ , let  $\phi_a$  be the Möbius function on  $D$ , that is,

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad (z \in D),$$

and put

$$\beta(a, z) = \frac{1}{2} \log \frac{1 + |\phi_a(z)|}{1 - |\phi_a(z)|} \quad (a, z \in D).$$

For  $0 < r \leq \infty$  and  $a \in D$ ,

$$D_r(a) = \{z \in D ; \beta(a, z) < r\}$$

be the Bergman disc with 'center'  $a$  and 'radius'  $r$ . For  $u \in L^1(m)$ ,

$$\tilde{u}(a) = \int_D u \circ \phi_a(z) dm(z) \quad (a \in D).$$

Then  $\tilde{u}$  may be bounded on  $D$  even if  $u$  is not bounded on  $D$ .

## §2. Riesz's function

If  $\mu = m$ , then for  $0 < p < \infty$   $S(m, p, a) = (1 - |a|^2)^2$ . Hence  $\mu = m$  or  $\text{supp } \mu \subseteq \partial D$ , by Szegő's Theorem  $\lim_{r \rightarrow 1^-} S(\mu, p, re^{i\theta}) = 0$  a.e.  $\theta$ . In this section, we show that this is true in general. In particular,  $R$  is not bounded on  $D$ . In fact, for arbitrary  $\mu$ , we show that  $\lim_{r \rightarrow 1^-} S(\mu, p, re^{i\theta}) = 0$  except a countable set of  $\theta$ .

Proposition 1. Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure. Then the following are valid for  $R(a) = R(\mu, p, a)$  and  $S(a) = S(\mu, p, a)$ .

- (1)  $R(\mu, p, a) S(\mu, p, a) = 1$  for  $a \in \bar{D}$ , assuming  $\infty \times 0 = 1$ .
- (2)  $R(\mu)$  is lower semicontinuous on  $(0, \infty) \times D$ , and  $S(\mu)$  is upper semicontinuous on the same set. Moreover  $R(\mu, p, a) \geq 1/\mu(\bar{D})$  and  $S(\mu, p, a) \leq \mu(\bar{D})$ .
- (3) If  $\log R$  or  $R$  is in  $L^1(m)$ , then for  $a \in D$

$$R(a) \leq \exp(\log R)^\sim(a) \leq \tilde{R}(a).$$

- (4) If  $r < \infty$ , then for  $a \in D$

$$\log R(a) \leq \left( \frac{1+s|a|}{1-s|a|} \right)^2 \frac{1}{m(D_r(a))} \int_{D_r(a)} \log R dm$$

where  $s = \tanh r$ . Hence for  $a \in D$

$$\log R(a) \leq \left( \frac{1+|a|}{1-|a|} \right)^2 \int_D \log R dm.$$

These inequalities are also valid for  $R$  instead of  $\log R$ .

- (5) For  $a \in D$ ,

$$S(\mu, p, a) \geq S(S(\mu)dm, p, a).$$

- (6)  $R$  is not bounded on  $D$  and  $\bar{D}$ .

Proof. (1) It is easy to see that  $1 \leq R(a)S(a)$  for  $a \in \bar{D}$ . If  $1 < R(a)S(a)$ , then there exists a positive constant  $\gamma$  such that  $1 \leq \gamma S(a)$  and  $\gamma < R(a)$ . Hence  $1 \leq \gamma \int |g|^p d\mu$  for any  $g \in P$  with  $g(a) = 1$  and so

$$|f(a)|^p \leq \gamma \int_{\bar{D}} |f|^p d\mu \text{ for any } f \in P.$$

This implies  $\gamma \geq R(a)$ . This contradiction shows that  $1 = R(a)S(a)$ . (2) is clear by (1). (3) If  $f \in P$ , then  $\log |f|$  is subharmonic on  $D$  and hence for any  $a \in D$ ,

$$\log |f(a)|^p \leq \int_D \log |f(z)|^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm(z).$$



Assuming  $\int |f|^p d\mu \leq 1$ , by definition on  $R$

$$\log R(a) \leq \int_D \log R(z) \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm(z).$$

This implies  $R(a) \leq \exp(\log R)^{\sim}(a) \leq \tilde{R}(a)$ . (4) If  $0 < r < \infty$ , for any  $a \in D_r(0)$  and any  $f \in P$ ,

$$\log |f(a)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} \log |f(z)|^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm(z)$$

and hence

$$\log |f(a)|^p \leq \frac{1}{m(D_r(a))} \left( \frac{1+s|a|}{1-s|a|} \right)^2 \int_{D_r(a)} \log |f|^p dm$$

where  $s = \tanh r$ . This proof is same to that of [14, Proposition 4.3.8.]. Assuming  $\int |f|^p d\mu \leq 1$ , we get (4) as in (3). (5) By (1),

$$\int |f|^p d\mu \geq S(\mu, z) |f(z)|^p \quad (z \in D).$$

and hence  $\int |f|^p d\mu \geq \int |f|^p S(\mu) dm$ . Assuming  $f(a) = 1$  and  $a \in D$ , we get  $S(\mu, a) \geq S(S(\mu) dm, a)$ . (6) If  $R(\mu, p, a)$  is bounded on  $\bar{D}$ , then  $H^p(\mu) \subset L^\infty(\mu)$ . By [11, Theorem 5.2],  $H^p(\mu)$  is finitely dimensional. It is easy to see that  $\text{supp } \mu$  is a finite set. Then trivially  $R(\mu, p, a) = \infty$  except  $\text{supp } \mu$ . The proof of the statement for  $D$  is same to that for  $\bar{D}$ , assuming  $\mu = \mu|_D$ .

Even if  $v$  is not bounded,  $\tilde{v}$  may be bounded. However (3) and (6) of Proposition 1 show that  $\tilde{R}$  is also not bounded. The following theorem gives a stronger result.

**Theorem 2.** Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure. If  $a \in \partial D$ , then the following are valid.

- (1)  $\mu(\{a\}) = 0$  if and only if  $S(\mu, p, a) = 0$ .
- (2)  $\lim_{r \rightarrow 1^-} S(\mu, p, ra) = 0$  except a countable set of  $a$  in  $\partial D$ .
- (3) If  $\mu(\{a\}) = 0$  and  $\{a_n\}$  is a sequence in  $D$  with  $\lim a_n = a$ , then  $\lim_{n \rightarrow \infty} S(\mu, p, a_n) = 0$ .
- (4) If  $\mu(\{a\}) > 0$ , then for each  $n$ , the set  $\{z \in D; |z-a| < 1/n\} \cap \{z \in D; S(\mu, p, z) < 1/n\}$  is a nonempty open set.
- (5) If  $b < c$  and  $E = \{z \in D; z = re^{i\theta}, 0 \leq r < 1 \text{ and } b \leq \theta \leq c\}$ , then  $R$  is not bounded on  $E$ .

**Proof.** We may assume  $a = 1$ . (1) If  $\mu(\{1\}) > 0$ , then  $|f(1)|^p \leq \int |f|^p d\mu / \mu(\{1\})$  and so  $R(\mu, p, 1) \leq 1/\mu(\{1\})$ . (1) of Proposition 1 implies  $S(\mu, p, 1) > 0$ . Conversely suppose  $\mu(\{1\}) = 0$ . If  $z \in \bar{D}$  and  $z \neq 1$ , then  $\lim_{t \rightarrow 1^+} |(1-t)/(z-t)| = 0$  and

$$\left| \frac{z-1}{z-t} - 1 \right| = \left| \frac{1-t}{z-t} \right| < 1 \quad (t > 1).$$

For any  $t > 1$ ,

$$S(\mu, p, 1) \leq \int_D \left| 1 - \frac{z-1}{z-t} \right|^p d\mu(z) = \int_{\bar{D} \setminus \{1\}} \left| \frac{1-t}{z-t} \right|^p d\mu(z).$$

As  $t \rightarrow 1$ , by the Lebesgue's dominated convergence theorem,  $S(\mu, p, 1) = 0$ . (2) Suppose  $\mu(\{1\}) = 0$ . If there exist a sequence  $\{r_n\}$  and a positive constant  $\varepsilon$  such that  $0 < r_n < 1$  with  $r_n \rightarrow 1$  and  $S(\mu, p, r_n) \geq \varepsilon > 0$ , then

$$|f(r_n)|^p \leq \frac{1}{\varepsilon} \int_D |f|^p d\mu \text{ and so } |f(1)|^p \leq \frac{1}{\varepsilon} \int_D |f|^p d\mu.$$

This implies  $S(\mu, p, 1) > 0$  and contradicts (1). Hence if  $\mu(\{1\}) = 0$ , then  $\lim_{r \rightarrow 1^-} S(\mu, p, r) = 0$ . This implies (2) because  $\{a \in \partial D ; \mu(\{a\}) > 0\}$  is a countable set. (3) is clear by the proof of (2). (4) Suppose  $\mu(\{1\}) > 0$  and for each  $n$ , put

$$G_n = \left\{ z \in \bar{D} ; \left| z-1 \right| < \frac{1}{n} \right\} \cap \left\{ z \in \bar{D} ; S(\mu, p, z) < \frac{1}{n} \right\}.$$

Since  $\{z \in \partial D ; \mu(\{z\}) > 0\}$  is a countable set, for each  $n$  there exists  $b_n \in \{z \in \partial D ; |z-1| < \frac{1}{n}\}$  with  $\mu(\{b_n\}) = 0$ . Then  $S(\mu, p, b_n) = 0$  by (1) and hence  $G_n$  is not empty.  $G_n$  is a relatively open set in  $\bar{D}$  by (2) of Proposition 1 and so  $G_n \cap D$  is a nonempty open set. (5) follows from (2).

If  $R(\mu, 2, a) < \infty$ , then there exists  $k_a$  in  $H^2(\mu)$  such that  $f(a) = \int f(z) \overline{k_a(z)} d\mu(z)$  for any  $f$  in  $H^2(\mu)$  and hence  $R(\mu, 2, a) = \int |k_a(z)|^2 d\mu(z)$ . Thus the results in this section give the informations about the reproducing kernel  $k_a$ .

### §3. Estimate of Riesz's function

In this section we give upper and lower estimates of  $S$ . The lower ones will be used later. The following proposition is a generalization of Szegő's Theorem in Introduction. In fact, if  $\mu \ll D$  is a zero measure, then it gives Szegő's Theorem.

Proposition 3. Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure such that  $(d\mu \ll \partial D)/(d\theta/2\pi) = w(e^{i\theta})$ ,  $\mu \ll D = \sum a_j \delta_{z_j}$  and  $\sum (1 - |z_j|) < \infty$ . Let  $b$  be a Blaschke product of  $\{z_\ell\}$  and  $b_j$  a Blaschke product of  $\{z_\ell\}_{\ell \neq j}$ . Then for all  $a \in D$ ,  $(1 - |a|^2) \exp(\log w)^{\wedge}(a) \leq S(\mu, p, a)$ . If  $a \in D \setminus \{z_\ell\}$ , then

$$S(\mu, p, a) \leq |b(a)|^{-p} (1 - |a|^2) \exp(\log w)^{(a)}.$$

If  $a = z_j$ , then

$$S(\mu, p, a) \leq |b_j(a)|^{-p} (1 - |a|^2) \exp(\log w)^{(a)} + a_j.$$

In particular,  $S(\mu, p, a) > 0$  if and only if  $\log w \in L^1(d\theta)$ .

Proof. Since  $S(\mu, p, a) \geq S(wd\theta/2\pi, p, a)$  for all  $a \in D$ , by Szegő's Theorem  $(1 - |a|^2) \exp(\log w)^{(a)} \leq S(\mu, p, a)$  for all  $a \in D$ . Let  $B_n$  be a finite Blaschke product of  $\{z_1, z_2, \dots, z_n\}$ . If  $a \in D \setminus \{z_\ell\}$ , then

$$\begin{aligned} S(\mu, p, a) &\leq \inf \left\{ \int \left| \frac{B_n}{B_n(a)} g \right|^p d\mu \mid \partial D + \sum_{j=1}^{\infty} a_j \left| \frac{B_n(z_j)}{B_n(a)} g(z_j) \right|^p ; g \in P \text{ and } g(a) = 1 \right\} \\ &= \frac{1}{|B_n(a)|^p} \inf \left\{ \int |B_n g|^p d\mu \mid \partial D + \sum_{j=n+1}^{\infty} a_j |B_n(z_j)|^p |g(z_j)|^p ; g \in P \right. \\ &\quad \left. \text{and } g(a) = 1 \right\}. \end{aligned}$$

As  $n \rightarrow \infty$ ,

$$S(\mu, p, a) \leq \frac{1}{|b(a)|^p} \inf \left\{ \int |g|^p d\mu \mid \partial D ; g \in P \text{ and } g(a) = 1 \right\}.$$

Now by Szegő's Theorem, for each  $a \in D$   $S(\mu, p, a) \leq |b(a)|^{-p} (1 - |a|^2) \exp(\log w)^{(a)}$ . Let  $B_{j,n}$  be a finite Blaschke product of  $\{z_1, z_2, \dots, z_n\} \setminus \{z_j\}$ . If  $a = z_j$  and  $n > j$ , then

$$\begin{aligned} S(\mu, p, a) &\leq \inf \left\{ \int \left| \frac{B_{j,n}}{B_{j,n}(a)} g \right|^p d\mu ; g \in P \text{ and } g(a) = 1 \right\} \\ &= \frac{1}{|B_{j,n}(a)|^p} \inf \left\{ \int |B_{j,n} g|^p d\mu \mid \partial D + a_j |B_{j,n}(a)|^p \right. \\ &\quad \left. + \sum_{\ell \geq n+1} a_\ell |B_{j,n}(z_\ell)|^p |g(z_\ell)|^p ; g \in P \text{ and } g(a) = 1 \right\}. \end{aligned}$$

As  $n \rightarrow \infty$ , by Szegő's Theorem, for  $a = z_j$ ,

$$S(\mu, p, a) \leq |b_j(a)|^{-p} (1 - |a|^2) \exp(\log w)^{(a)} + a_j.$$

The following proposition is related to Theorem 2 in this paper and Theorem in [7]. In fact, if  $\tilde{W}$  is bounded on  $D$ , then  $(1 - |a|^2)^{-2} S(Wdm, p, a)$  is bounded on  $D$ . Moreover if  $W$  is continuous on  $\bar{D}$ , then for all  $e^{i\theta}$

$$\lim_{a \rightarrow e^{i\theta}} (1 - |a|^2)^2 R(W dm, p, a) = 1/W(e^{i\theta}).$$

$\bar{D}$ .

Proposition 4. Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure on

- (1)  $\tilde{\mu}(a) \geq (S(\mu))^\sim(a)$  ( $a \in D$ ).
- (2) If  $d\mu = W dm$  and  $a \in D$ , then

$$(1 - |a|^2)^2 \exp(\log W)^\sim(a) \leq S(\mu, p, a) \leq (1 - |a|^2)^2 \tilde{W}(a).$$

- (3)  $S(W dm, a) = (1 - |a|^2)^2 S(W \circ \phi_a dm, 0)$  for  $a \in D$ .

Proof. (1) For all  $z \in D$

$$\int |f|^p d\mu \geq |f(z)|^p S(z) \text{ and so } \int |f|^p d\mu \geq \int |f|^p S dm.$$

Assuming  $f(z) = \{(1 - |a|^2)/(1 - \bar{a}z)^2\}^{2/p}$  for  $a \in D$ ,  $\tilde{\mu}(a) \geq \tilde{S}(a)$ . (2) If  $\log W \in L^1(m)$ , then

$$\begin{aligned} S(W dm, p, a) &= \inf \left\{ \int |f|^p W dm ; f \in P \text{ and } f(a) = 1 \right\} \\ &= \inf \left\{ \int |g|^p W \circ \phi_a \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm ; g \in H^p(W \circ \phi_a dm) \text{ and } g(0) = 1 \right\} \\ &= (1 - |a|^2)^2 \inf \left\{ \int |k|^p W \circ \phi_a dm ; k \in H^p(W \circ \phi_a dm) \text{ and } k(0) = 1 \right\} \\ &\geq (1 - |a|^2)^2 \exp \int (\log W) \circ \phi_a dm = (1 - |a|^2)^2 \exp(\log W)^\sim(a). \end{aligned}$$

The inequality above is proved by two Jensen's inequalities. (3) is clear by the proof of (2).

In (2) of Proposition 4, we can get estimates of  $S(\mu, p, a)$  as in Proposition 3 when  $d\mu = W dm + \sum_{j=1}^{\infty} a_j \delta_{z_j}$ ,  $\{z_j\} \subset D$  and  $\sum (1 - |z_j|) < \infty$ . The following theorem is important in this paper and the following lemma is used to prove it.

Lemma 1. Let  $\Delta_s(a)$  be the set  $\{z \in D ; |(a - z)/(1 - \bar{a}z)| < s\}$  where  $a \in D$  and  $s \in (0, 1)$ . If  $t \in (0, 1)$  and  $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$ , then  $\Delta_t(0) \subset \Delta_s(a)$ .

Proof. The Euclidean center and radius of  $\Delta_s(a)$  are

$$C = \frac{1-s^2}{1-s^2|a|^2}a, \quad R = \frac{1-|a|^2}{1-s^2|a|^2}s$$

respectively. Hence to prove  $\overline{\Delta_t(0)} \subset \Delta_s(a)$ , it is sufficient to show that

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| \leq \frac{1-|a|^2}{1-s^2|a|^2}s.$$

If  $1-s^2 = (1-|a|^2)(1-t^2)/5$ , then

$$1-s^2 \leq \frac{(1-|a|^2)(1-t^2)}{5-|a|^2}$$

and hence  $s^2 \geq \{4 + (1-|a|^2)t^2\}/(5-|a|^2)$ . The last inequality is equivalent to

$$1-s^2 \leq \frac{(1-|a|^2)(s^2-t^2)}{4}.$$

Then

$$1-s^2 \leq \frac{(1-|a|^2)(s-t)}{2} \frac{s+t}{2} \leq \frac{(1-|a|^2)(s-t)}{|a|(t|a|+1)}$$

because  $s+t \leq 2$  and  $|a|(t|a|+1) \leq 2$ . This implies that

$$t + \frac{1-s^2}{1-s^2|a|^2}|a| \leq \frac{1-|a|^2}{1-s^2|a|^2}s.$$

**Theorem 5.** Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure on  $\bar{D}$ . Let  $d\mu/dm = Wdm$ ,  $K$  an arbitrary compact set in  $D$  and  $t = \max\{|z|; z \in K\}$ . Then, for  $a \in D$

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5} \exp \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)} \int_{K^c} \log(W \wedge 1) dm.$$

If  $1 \leq p < \infty$  and  $a \in D$ , then

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^{3(2-\frac{1}{p})}(1-t^2)^{2-\frac{1}{p}}}{2^{4(1-\frac{1}{p})} \cdot 5^{2-\frac{1}{p}}} \left( \int_{K^c} W^{-\frac{1}{p-1}} dm \right)^{\frac{1}{p}-1}.$$

**Proof.** By two Jensen's inequalities, for  $a \in D$

$$\begin{aligned} S(\mu, p, a) &\geq S(Wdm, p, a) \\ &= \inf \left\{ \int |g|^p W \circ \phi_a \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm ; g(0) = 1 \right\} \\ &= (1-|a|^2)^2 \inf \left\{ \int |k|^p W \circ \phi_a dm ; k(0) = 1 \right\} \end{aligned}$$

$$\begin{aligned}
&\geq (1-|a|^2)^2 \int_0^1 2r dr \exp \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&\geq (1-|a|^2)^2 (1-s^2) \int_s^1 \frac{2r}{1-s^2} dr \exp \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&\geq (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_s^1 2r dr \int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi \\
&= (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_{D \setminus \Delta_s(0)} \log W \circ \phi_a dm \\
&= (1-|a|^2)^2 (1-s^2) \exp \frac{1}{1-s^2} \int_{D \setminus \Delta_s(a)} \log W \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm \\
&\geq (1-|a|^2)^2 (1-s^2) \exp \frac{(1-|a|^2)^2}{(1-|a|^2)^4} \frac{1}{1-s^2} \int_{D \setminus \Delta_s(a)} \log(W \wedge 1) dm
\end{aligned}$$

where  $s \in (0,1)$  and  $\Delta_s(a) = \{z \in D ; |(a-z)/(1-\bar{a}z)| < s\}$ . For each compact set  $K \subset D$ , if  $t = \max\{|z| ; z \in K\}$  and  $1-s^2 = (1-|a|^2)(1-t^2)/5$ , then by Lemma 1  $\Delta_t(0) \subset \Delta_s(a)$ . Hence  $K \subset \Delta_s(a)$  and so  $K^c \supset D \setminus \Delta_s(a)$ . Thus, if  $1-s^2 = (1-|a|^2)(1-t^2)/5$ , then

$$\frac{(1-|a|^2)^2}{(1-|a|^2)^4} \frac{1}{1-s^2} = \frac{(1+|a|^2)^4}{(1-|a|^2)^2(1-s^2)} \leq \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)}$$

and hence for all  $a \in D$

$$S(\mu, p, a) \geq \frac{(1-|a|^2)^3(1-t^2)}{5} \exp \frac{2^4 \cdot 5}{(1-|a|^2)^3(1-t^2)} \int_{K^c} \log(W \wedge 1) dm.$$

Now we will prove the second inequality. Instead of two Jensen's inequalities, we will use Kolmogoroff's inequality (cf.[12, Theorem 4.3.1]). For  $a \in D$ , if  $1 \leq p < \infty$  and  $1/p + 1/q = 1$ ,

$$\begin{aligned}
&S(\mu, p, a) \\
&\geq (1-|a|^2)^2 \int_0^1 2r dr \left( \int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2) \int_s^1 \frac{2r}{1-s^2} dr \left( \int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2) \left( \frac{1}{1-s^2} \int_s^1 2r dr \int_0^{2\pi} (W \circ \phi_a)^{-\frac{1}{p-1}} d\theta / 2\pi \right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left( \int_{D \setminus \Delta_s(0)} (W \circ \phi_a)^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}} \\
&= (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left( \int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dm \right)^{-\frac{1}{q}} \\
&\geq (1-|a|^2)^2 (1-s^2)^{1+\frac{1}{q}} \left\{ \frac{(1-|a|^2)^2}{(1-|a|^2)^4} \int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} dm \right\}^{-\frac{1}{q}}
\end{aligned}$$

$$\geq \frac{(1 - |a|^2)^{2(1+\frac{1}{q})}(1 - s^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}}} \left( \int_{D \setminus \Delta_s(a)} W^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}}$$

where  $s \in (0, 1)$ . As in the proof of the first inequality, for each compact set  $K \subset D$ , if  $t = \max\{|z|; z \in K\}$  and  $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$ , then  $K^c \supset D \setminus \Delta_s(a)$ . Thus, if  $1 - s^2 = (1 - |a|^2)(1 - t^2)/5$ , then for all  $a \in D$

$$S(\mu, p, a) \geq \frac{(1 - |a|^2)^{3(1+\frac{1}{q})}(1 - t^2)^{1+\frac{1}{q}}}{2^{\frac{4}{q}} \cdot 5^{1+\frac{1}{q}}} \left( \int_{K^c} W^{-\frac{1}{p-1}} dm \right)^{-\frac{1}{q}}.$$

The second inequality of Theorem 5 implies  $S(\mu, 1, a) \geq (1 - |a|^2)^3 \times (1 - t^2)(1/5) \text{ess.inf}\{W(x); x \in K^c\}$ . Let  $\sigma$  be a finite positive Borel measure on  $[0, 1]$ .  $\mu(re^{i\theta}) = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$  is more general than  $Wdm = 2rdr \times W(re^{i\theta})d\theta/2\pi$ . If  $\sigma(r)$  is singular to the Lebesgue measure on  $[0, 1]$ , then  $\mu$  is singular to  $m$ . However we can give an interesting lower estimate. It is different from that of Theorem 5 in case of  $\mu = Wdm$ .

Theorem 6. Suppose  $0 < p < \infty$  and  $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$  where  $\sigma(r)$  is a finite positive Borel measure on  $[0, 1]$ . If  $W(e^{i\theta}) = \sup_r W(re^{i\theta})$  and  $W_r(e^{i\theta}) = W(re^{i\theta})$ , then for  $a \in D$

$$\begin{aligned} (1 - |a|^2) \int_{|a|}^1 \exp(\log W_r)^{\wedge}(a) d\sigma(r) &\leq S(\mu, p, a) \\ &\leq \sigma([0, 1]) \inf \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &\leq \sigma([0, 1]) \inf \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(e^{i\theta}) d\theta/2\pi; f(a) = 1 \right\}. \end{aligned}$$

Proof. For  $a \in D$ ,

$$\begin{aligned} S(\mu, p, a) &= \inf \left\{ \int_0^1 d\sigma(r) \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &\geq \int_0^1 d\sigma(r) \inf \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &= \int_{|a|}^1 d\sigma(r) \inf \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta/2\pi; f(a) = 1 \right\} \\ &= \int_{|a|}^1 (1 - |a|^2) \exp(\log W_r)^{\wedge}(a) d\sigma(r). \end{aligned}$$

We used Szegő's Theorem in the last equality. The upper estimates are trivial.

Corollary 1. Let  $d\mu = \sigma(r) \times W(re^{i\theta})d\theta/2\pi$  as in Theorem 6 and  $0 < p < \infty$ .  
(1) If  $W(re^{i\theta}) \equiv 1$ , then for  $a \in D$

$$(1 - |a|^2) \sigma(|a|, 1) \leq S(\mu, p, a) \leq (1 - |a|^2) \sigma([0, 1]).$$

In particular,  $S(\mu, p, 0) = \sigma([0, 1])$ .

(2) If  $W(re^{i\theta}) = |h(re^{i\theta})|$  for some outer function  $h$  in  $H^1(d\theta)$ , then for  $a \in D$

$$(1 - |a|^2) \int_{|a|}^1 W(ra) d\sigma(r) \leq S(\mu, p, a) \leq (1 - |a|^2) W(a) \sigma([0, 1]).$$

(3) If  $1 < p < \infty$  and  $\mathbf{W}(e^{i\theta}) = \sup_r W(re^{i\theta})$  satisfies the  $A_p$  condition, then there exists a positive constant  $\gamma$  such that for  $a \in D$

$$S(\mu, p, a) \leq \gamma (1 - |a|^2) \exp(\log \mathbf{W})^{\wedge}(a) \sigma([0, 1]).$$

Proof. (1) is a special case of (2). (2) Since  $h$  is an outer function in  $H^1$ , for  $a \in D$

$$\exp(\log W_r)^{\wedge}(a) = \exp(\log |h_r|)^{\wedge}(a) = |h(ra)| = W(ra)$$

and

$$\begin{aligned} & \inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p W(re^{i\theta}) d\theta / 2\pi \right\} \\ &= \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p |h(e^{i\theta})| d\theta / 2\pi = (1 - |a|^2) |h(a)| = (1 - |a|^2) W(a). \end{aligned}$$

Now Theorem 6 implies (2). (3) By a theorem of M. Rosenblum (cf. [10] and [9, Theorem 2.2]), there exists a positive constant  $\gamma$  such that for any  $f \in P$

$$\sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \leq \gamma \int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi$$

because  $\mathbf{W} \in A_p$ . By Theorem 6 and Szegő's Theorem, for  $a \in D$

$$\begin{aligned} & \inf_f \left\{ \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \right\} \leq \gamma \inf_f \int_0^{2\pi} |f(e^{i\theta})|^p \mathbf{W}(e^{i\theta}) d\theta / 2\pi \\ &= \gamma (1 - |a|^2) \exp(\log \mathbf{W})^{\wedge}(a). \end{aligned}$$

This implies (3).

#### §4. Carleson inequality and Riesz's function

Let  $\nu$  and  $\mu$  be finite positive Borel measures on  $\bar{D}$  and  $1 \leq p < \infty$ . We say that  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$  - Carleson inequality, if there exists a constant  $\gamma > 0$  such that



$$\int_{\bar{D}} |f|^p d\nu \leq \gamma \int_{\bar{D}} |f|^p d\mu$$

for all  $f \in P$  (see [8]).  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$  - Carleson inequality if and only if  $H^p(\mu) \subset H^p(\nu)$  and the inclusion mapping  $i_p : H^p(\mu) \rightarrow H^p(\nu)$  is bounded. We say that for  $p > 1$ ,  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality if  $H^p(\mu) \subset H^p(\nu)$  and  $i_p : H^p(\mu) \rightarrow H^p(\nu)$  is compact. We say that for  $p = 1$ ,  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality if  $i_p$  is star-compact. We could not prove Theorem 7 for  $p = 1$  because we do not know anything about the predual of  $H^1(\mu)$ . Using Riesz's functions, we will show vanishing Carleson inequalities. As a result, we show that  $R(\mu, p) \notin L^1(\mu)$  if  $\text{supp } \mu$  is not a finite set. Moreover, from given a measure  $\mu$ , we will show how to construct a measure  $\nu$  such that the  $(\nu, \mu, p)$  -vanishing Carleson inequality is valid.

Theorem 7. Suppose  $1 < p < \infty$ , and  $\nu$  and  $\mu$  are finite positive Borel measures on  $\bar{D}$ .

(1) If  $\int R(\mu, p) d\nu < \infty$ , then  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality and

$$R(\mu, p, a) \leq \left( \int R(\mu, p) d\nu \right) R(\nu, p, a) \quad (a \in \bar{D}).$$

(2) If  $V$  is a Borel function such that  $0 \leq V \leq S$  on  $\bar{D}$ , then  $V |g|^p$  is bounded on  $\bar{D}$  for each  $g$  in  $H^p(\mu)$ , and  $V d\mu$  and  $\mu$  satisfy the  $(V d\mu, \mu, p)$ -vanishing Carleson inequality.

Proof. (1) By definition of  $R(\mu, p, a)$ , for  $a \in \bar{D}$ ,

$$|f(a)|^p \leq R(\mu, p, a) \int |f|^p d\mu \quad (f \in P).$$

Hence if  $\gamma = \int R(\mu, p) d\nu < \infty$ , then  $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$  ( $f \in P$ ) and so  $i_p : H^p(\mu) \rightarrow H^p(\nu)$  is bounded. We will show that  $i_p$  is compact. If  $f_n \rightarrow f$  weakly in  $H^p(\mu)$ , then there exists a finite positive constant  $\gamma'$  such that

$$\int |f_n - f|^p d\mu \leq \gamma' \quad \text{for all } n.$$

By the hypothesis,  $R(\mu, p, a) < \infty$   $\nu$  - a.e. on  $\bar{D}$  and so  $f_n \rightarrow f$   $\nu$  - a.e. on  $\bar{D}$  because  $f_n \rightarrow f$  weakly. Moreover by definition of  $R(\mu, p, a)$ ,  $|f_n(a) - f(a)|^p \leq \gamma' R(\mu, p, a)$  and by the hypothesis,  $R(\mu, p, a) \in L^1(\nu)$ . Thus

$$\int |f_n - f|^p d\nu \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

by the Lebesgue's dominated convergence theorem. This implies  $i_p$  is compact. Since  $\int |f|^p d\nu \leq \gamma \int |f|^p d\mu$  and  $\gamma = \int R(\mu, p) d\nu$ , assuming  $f(a) = 1$ , we get  $S(\nu, p, a) \leq \gamma S(\mu, p, a)$ . Now by (1) of Proposition 1, we get the inequality of (1). (2) If  $0 \leq V \leq S$ , then  $VR \leq 1$  and hence  $V(a) |f(a)|^p$  is bounded on  $\bar{D}$  by  $\int |f|^p d\mu$ , for each  $f \in H^p(\mu)$ . Moreover if  $\nu = Vdm$  and  $0 \leq V \leq S$ , then  $\int R(\mu, p) d\nu \leq \int dm = 1$  and hence by (1)  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality.

Corollary 2. If  $0 < p < \infty$  and  $\text{supp } \mu$  is not a finite set, then  $R(\mu, p) \notin L^1(\mu)$ .

Proof. Suppose  $1 < p < \infty$ . If  $R(\mu, p) \in L^1(\mu)$ , then the inclusion map  $i_p : H^p(\mu) \rightarrow H^p(\mu)$  is compact. It is easy to see that  $i_p$  is an identity operator. Hence the unit ball of  $H^p(\mu)$  is compact with respect to the norm. Therefore  $H^p(\mu)$  is finitely dimensional. This contradicts that  $\text{supp } \mu$  is not a finite set. This implies that  $R(\mu, p) \notin L^1(\mu)$ . For  $0 < p \leq 1$ , the proof is due to the referee. Choose  $n$  sufficiently large that  $np > 1$ . If  $g(a) = 1$  then  $g^n(a) = 1$  as well, and  $g^n$  is a polynomial if  $g$  is a polynomial. Thus,

$$\begin{aligned} S(\mu, p, a) &= \inf \left\{ \int_D |f|^p d\mu ; f \in P, f(a) = 1 \right\} \\ &\leq \inf \left\{ \int_D |g^n|^p d\mu ; g \in P, g(a) = 1 \right\} = S(\mu, np, a). \end{aligned}$$

This implies that  $R(\mu, p) \notin L^1(\mu)$  for  $0 < p \leq 1$ .

Corollary 3. Suppose  $1 < p < \infty$  and  $d\mu/dm = W$

(1) If  $\log W \in L^1(m)$  and  $d\nu = (1 - |z|^2)^2 \exp(\log W) dm$ , then  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality.

(2) If  $\chi_K \log(W \wedge 1) \in L^1(m)$  for some compact set  $K$  in  $D$ , there exist positive constant  $a$  and nonpositive constant  $b$  such that  $d\nu = a(1 - |z|^2)^3 \{\exp b(1 - |z|^2)^{-3}\} dm$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality.

(3) Suppose  $\chi_K W^{-\frac{1}{p-1}} \in L^1(m)$  for some compact set  $K$  in  $D$ . If  $d\nu = c(1 - |z|^2)^{3(2-\frac{1}{p})} dm$ , then  $\nu$  and  $\mu$  satisfy the  $(\nu, \mu, p)$ -vanishing Carleson inequality.

Suppose  $1 < p < \infty$  and  $d\mu/dm = W$ . If  $\chi_K \log W \in L^1(m)$  for some compact set  $K$  in  $D$ , there exist positive constant  $a$  and nonpositive constant  $b$  such that

$$\{a(1 - |z|^2)^3 \exp b(1 - |z|^2)^{-3}\} |f(z)|^p$$

is bounded on  $D$  for each  $f \in H^p(\mu)$ . Here  $a$  and  $b$  do not depend on  $f$ . This is a corollary of (2) in Theorem 7.

§5.  $H^p(\mu)$  and  $L_a^p(\mu)$

The following is a result of Theorem 5. If  $d\mu/dm = W$  and  $\log W$  is integrable on the complement  $K^c$  of a compact set in  $D$ , then  $H^p(\mu) \subseteq L_a^p(\mu)$ . In this section, we show that if  $\log W$  is locally integrable on  $K^c$ , then the same result is true. We give a necessary and sufficient condition for  $H^p(\mu) \subset L_a^p(\mu)$  using Riesz's function. Theorem 8 is a joint work with K. Takahashi. A subset  $E$  of  $D$  is a uniqueness set if  $E$  satisfies the following : If  $f$  in  $H$  is zero on  $E$ , then  $f \equiv 0$  on  $D$ .

Lemma 2. Suppose  $0 < p < \infty$  and  $\mu$  is a finite positive Borel measure on  $D$ . Then, the following (1)  $\sim$  (3) are equivalent.

- (1)  $\sup_{a \in K} R(\mu, p, a) < \infty$  for all compact set  $K$  in  $D$ .
- (2)  $\int_K R(\mu, p) dm < \infty$  for all compact set  $K$  in  $D$ .
- (3)  $\int_K \log R(\mu, p) dm < \infty$  for all compact set  $K$  in  $D$ .

Proof. Both (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are trivial. We will show (3)  $\Rightarrow$  (1). We may assume that  $\mu(D) = 1$ . For any  $f \in P$ ,

$$\log |f(0)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(0)} \log |f|^p dm.$$

If  $a \in D_r(0)$ , then for all  $f \in P$

$$\log |f(a)|^p \leq \frac{1}{m(D_r(0))} \int_{D_r(a)} \log |f|^p \frac{(1 - |a|^2)^2}{|1 - \bar{a}z|^4} dm.$$

Assuming  $\int |f|^p d\mu \leq 1$ , we get

$$\log R(\mu, p, a) \leq \frac{1}{m(D_r(0))} \frac{(1 + |a|)^2}{(1 - |a|)^2} \int_{D_r(a)} \log R(\mu, p) dm.$$

Since  $D_r(a) \subset D_{2r}(0)$  and  $R(\mu, p, a) \geq 1$ , for each  $a \in D_r(0)$  there exists a finite positive constant  $\gamma_r$  such that

$$\log R(\mu, p, a) \leq \gamma_r \int_{D_{2r}(0)} \log R(\mu, p) dm.$$

This implies (1).

Lemma 3. Let  $X$  be a Banach space which consists of analytic functions on  $D$  and contains 1. Suppose there exists a dense subspace  $Y$  of  $X$  such that if  $f$  in  $Y$ , then  $(f - f(a))/(z - a)$  belongs to  $Y$  for some  $a$  in  $D$ . If  $(z - a)X$  is not dense in  $X$ , then the functional  $f \rightarrow f(a)$  is bounded.

Proof. By the hypothesis, if  $f \in Y$ , then  $f = f(a) + (z - a)g$  for some  $g \in Y$ . Since  $(z - a)X$  is not dense in  $X$ , there exists  $\phi \in X^*$  such that  $\langle (z - a)h, \phi \rangle = 0$  for all  $h \in X$  but  $\langle 1, \phi \rangle \neq 0$ . Hence,  $\langle f, \phi \rangle = f(a) \langle 1, \phi \rangle$ . Thus  $|f(a)| \leq \gamma \|f\|$  for all  $f \in X$  where  $\gamma = |\langle 1, \phi \rangle|^{-1} \|\phi\|_*$ .

Theorem 8. Suppose  $1 \leq p < \infty$  and  $\mu$  is a finite positive Borel measure on  $D$  such that  $(\text{supp } \mu) \cap D$  is a uniqueness set for  $H$ .

(1)  $L_a^p(\mu)$  is closed if and only if for all compact set  $K$  in  $D$

$$\int_K \log r(\mu, p) dm < \infty \text{ or } \int_K \log s(\mu, p) dm > -\infty.$$

(2)  $H^p(\mu) \subset L_a^p(\mu)$  if and only if for all compact set  $K$  in  $D$

$$\int_K \log R(\mu, p) dm < \infty \text{ or } \int_K \log S(\mu, p) dm > -\infty.$$

Proof. (1) If  $f \in L_a^p(\mu)$ , then  $(f - f(0))/z$  belongs to  $H$ . Since  $(f - f(0))/z$  is bounded on  $|z| \leq t < 1$  and  $1/z$  is bounded on  $|z| \geq t$ ,  $(f - f(0))/z$  belongs to  $L_a^p(\mu)$ . This implies that  $\{f \in L_a^p(\mu) ; f(0) = 0\} = zL_a^p(\mu)$  and hence  $L_a^p(\mu) = \mathbb{C} \oplus zL_a^p(\mu)$ . If  $Af = zf$  for  $f \in L_a^p(\mu)$ , then  $A$  is a bounded operator on  $L_a^p(\mu)$  and the range of  $A$  is complemented in  $L_a^p(\mu)$  by what was just proved. By [4, Part III, Corollary 2.3], the range of  $A$  is closed and hence  $zL_a^p(\mu)$  is not dense in  $L_a^p(\mu)$ . Applying Lemma 3 with  $X = Y = L_a^p(\mu)$  and  $a = 0$ ,  $r(\mu, p, 0) < \infty$  follows. The same argument is true for all  $a \in D \setminus \{0\}$  and hence  $r(\mu, p, a) < \infty$  for all  $a \in D$ . By the boundedness of holomorphic functions on compact sets and the uniform boundedness principle,  $\sup_{a \in K} r(\mu, p, a) < \infty$  for all compact set  $K$  in  $D$ . As Lemma 2 is also for  $r(\mu, p, a)$ ,

$$\int_K \log r(\mu, p) dm < \infty \text{ or } \int_K \log s(\mu, p) dm > -\infty.$$

Conversely, suppose  $\int_K \log r(\mu, p) dm < \infty$  for any compact set  $K$ . Then by the above lemma,  $\sup_K r(\mu, p) < \infty$  for any compact set  $K$ . If  $f$  is in the  $L^p(\mu)$ -norm closure of  $L_a^p(\mu)$ , then there exists a sequence  $\{f_n\}$  in  $L_a^p(\mu)$  such that  $\int |f - f_n|^p d\mu \rightarrow 0$ . Then by hypothesis on  $r(\mu, p)$ ,  $\sup\{|f_n(z)| ; z \in D_r(0)\} < \infty$  for each  $r < \infty$ . Hence, for each  $r < \infty$  there exists a subsequence  $\{f_{n_j}\}$  in  $L_a^p(\mu)$  and an analytic function  $g_r$  on  $D_r(0)$  such that  $f_{n_j} \rightarrow g_r$  uniformly on  $D_r(0)$ . This implies that  $f = g_r \mu - a.e.$  on  $D_r(0)$  for all  $r < \infty$ . Thus  $g = \lim_r g_r$  is analytic on  $D$  and  $f = g \mu - a.e.$  on  $D$ .

(2) The 'if' part is same to (1) and hence we will show the 'only if' part. Put  $M = \{f \in L^p(\mu) ; zf \in H^p(\mu)\}$ , then  $M$  is a closed subspace of  $L^p(\mu)$  such that

$$M \supseteq H^p(\mu) \supseteq zM \supseteq H^p(\mu)_0$$

where  $H^p(\mu)_0 = \{f \in H^p(\mu); f(0) = 0\}$ .  $H^p(\mu)_0$  is well defined because  $H^p(\mu) \subset L_a^p(\mu)$ . Suppose  $H^p(\mu) \neq zM$ . Then  $H^p(\mu) = \mathbb{C} + H^p(\mu)_0 = \mathbb{C} + zM$  and  $\mathbb{C} \cap zM = \{0\}$ . As in the proof of (1), by [4, Part III, Corollary 2.3],  $zM$  is closed and hence  $zH^p(\mu)$  is not dense in  $H^p(\mu)$ . Applying Lemma 3 with  $X = H^p(\mu), Y = P$  and  $a = 0, R(\mu, p, 0) < \infty$  follows. Suppose  $H^p(\mu) = zM$ . Then  $z^{-1} \in L^p(\mu)$  and hence  $\mu(\{0\}) = 0$ . If  $Af = zf$  for  $f \in M$ , then  $A$  is a one-one bounded operator from  $M$  onto  $H^p(\mu)$ . Therefore  $A$  is invertible and hence  $A(zM) = zH^p(\mu)$  is closed. Since  $H^p(\mu) \subset L_a^p(\mu), zH^p(\mu) \neq H^p(\mu)$  and hence by Lemma 3,  $R(\mu, p, 0) < \infty$  follows. The same argument implies that  $R(\mu, p, a) < \infty$  for all  $a \in D$ . Now, as in the proof of (1), Lemma 2 implies the 'only if' part of (2).

Corollary 4. Suppose  $1 \leq p < \infty$  and  $d\mu/dm = W$ . If  $\log W$  is locally integrable on  $K_0^c$  for some compact set  $K_0$  in  $D$ , then  $L_a^p(\mu)$  is closed and  $H^p(\mu) \subseteq L_a^p(\mu)$ .

Proof. By (1) of Theorem 8, it is sufficient to prove that for any compact set  $K$  in  $D, \inf_K s(\mu, p) > -\infty$ . If  $\log W$  is integrable on  $K_0^c$ , then by the proof of Theorem 5  $\inf_K s(\mu, p) > -\infty$ . For a more general  $W$  in this corollary, we have to proceed as the following. Suppose  $a \in D$  and  $0 < \varepsilon < \delta < 1$ . As in the proof of Theorem 5,

$$\begin{aligned} s(\mu, p, a) &\geq (1 - |a|^2)^2 \int_\varepsilon^\delta \exp\left(\int_0^{2\pi} \log W \circ \phi_a d\theta / 2\pi\right) 2r dr \\ &\geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp\left(\frac{1}{\delta^2 - \varepsilon^2} \int_{\Delta_\delta(0) \setminus \Delta_\varepsilon(0)} \log W \circ \phi_a dm\right) \\ &\geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp\left(\frac{2^2}{(1 - |a|^2)^2 (\delta^2 - \varepsilon^2)} \int_{\Delta_\delta(a) \setminus \Delta_\varepsilon(a)} \log(W \wedge 1) dm\right). \end{aligned}$$

Suppose  $K$  is an arbitrary compact set in  $D$ . Put  $t = \max\{|z|; z \in K_0\}$  and  $k = \max\{|z|; z \in K\}$ . The Euclidean center and radius of  $\Delta_\gamma(k)$  ( $0 < \gamma < 1$ ) are

$$C(\gamma) = \frac{1 - \gamma^2}{1 - \gamma^2 k^2} k, \quad R(\gamma) = \frac{1 - k^2}{1 - \gamma^2 k^2} \gamma$$

respectively. Put  $\ell = R(\delta) + C(\delta)$  and  $s = R(\varepsilon) - C(\varepsilon)$ . There exist  $\delta$  and  $\varepsilon$  such that

$$\Delta_\ell(0) \setminus \Delta_s(0) \subset D \setminus \Delta_t(0).$$

Then for all  $a \in K$

$$\Delta_\delta(a) \setminus \Delta_\varepsilon(a) \subset \Delta_\ell(0) \setminus \Delta_s(0).$$

Hence for all  $a \in K$

$$\Delta_\delta(a) \setminus \Delta_\varepsilon(a) \subset K_0^c$$

and so for all  $a \in K$

$$s(\mu, p, a) \geq (1 - |a|^2)^2 (\delta^2 - \varepsilon^2) \exp \left( \frac{2^2}{(1 - |a|^2)^2 (\delta^2 - \varepsilon^2)} \int_{K_\delta} \log(W \wedge 1) dm \right).$$

This shows the corollary.

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