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# Strong Coupling Limit of the Zero-Energy-State Density of the Dirac-Weyl Operator with a Singular Vector Potential

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**Abstract.** For a self-adjoint extension of the Dirac-Weyl operator with a singular vector potential, the strong coupling limit of the zero-energy-state density (ZESD) is computed. The result shows that a limit theorem of L.Erdös ( Lett.Math.Phys.**29**:219–240,1993) concerning the ZESD of the Dirac-Weyl operator with a "regular" magnetic field does not hold in the present singular case.

## 1. Introduction

In a paper [3], Erdös considered the two-dimesional Pauli operator $H$ with a "regular" magnetic field $B$ and showed that the naturally rescaled ground-state density of $H$ with respect to the coupling constant $L$ converges to $B$ as $L \to \infty$.

The ground states of the Pauli operator considered in [3] are in fact zero-energy states. Hence they are also the zero-energy states of the corresponding Dirac-Weyl operator. Thus the Erdös' limit theorem mentioned above can also be regarded as one concerning the zero-energy-state density (ZESD) of the Dirac-Weyl operator.

It is interesting to examine if the Erdös' limit theorem holds for wider classes of magnetic fields. With this motivation, we consider in this paper the strong coupling limit of the ZESD of the Dirac-Weyl operator with a singular vector potential such that the magnetic field is a distribution concentrated on two points in $\mathbf{R}^2$ and show that the Erdös' limit theorem does not hold in this singular case.

Mathematical aspects of a quantum system with a magnetic field concentrated on isolated points in $\mathbf{R}^2$ were discussed in [1,2]. We begin with a brief review of some results obtained in these papers.

## 2. The Dirac-Weyl Operator with a Singular Vector Potential and the ZESD

We consider a quantum system of a charged particle with charge $q \in \mathbf{R}\setminus\{0\}$ moving in the plane $\mathbf{R}^2$ under the influence of a perpendicular magnetic field $B(\mathbf{r})$ ($\mathbf{r} = (x, y) \in \mathbf{R}^2$) concentrated on two points $\mathbf{a}_\nu = (a_{\nu 1}, a_{\nu 2}) \in \mathbf{R}^2, \nu = 1, 2$. A general form of such a magnetic field is given by

$$B(\mathbf{r}) = \sum_{\nu=1}^{2} \sum_{0 \le \alpha+\beta \le \ell} C_{\alpha,\beta}^{(\nu)} D_x^\alpha D_y^\beta \delta(\mathbf{r} - \mathbf{a}_\nu),$$

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where $\ell$ and $C^{(\nu)}_{\alpha,\beta}$ are a nonnegative integer and a real constant, respectively, $D_x$ and $D_y$ denote the distributional partial differential operators in $x$ and $y$, respectively, and $\delta(\mathbf{r})$ is the Dirac delta distribution on $\mathbf{R}^2$. A vector (gauge) potential of the magnetic field is a vector-valued distribution $\mathbf{A} = (A_1, A_2)$ satisfying $B = D_x A_2 - D_y A_1$. One of such distributions $\mathbf{A}$ is given by

$$A_1(\mathbf{r}) = -\sum_{\nu=1}^{2} \sum_{0\le\alpha+\beta\le\ell} \frac{C^{(\nu)}_{\alpha,\beta}}{2\pi} D_x^\alpha D_y^\beta \left( \frac{y - a_{\nu 2}}{|\mathbf{r} - \mathbf{a}_\nu|^2} \right),$$

$$A_2(\mathbf{r}) = \sum_{\nu=1}^{2} \sum_{0\le\alpha+\beta\le\ell} \frac{C^{(\nu)}_{\alpha,\beta}}{2\pi} D_x^\alpha D_y^\beta \left( \frac{x - a_{\nu 1}}{|\mathbf{r} - \mathbf{a}_\nu|^2} \right).$$

We use a system of units where the light speed $c$ and $\hbar$ (the Planck constant divided by $2\pi$) are equal to 1. The kinetic momentum operator $\mathbf{P} = (P_1, P_2)$ with the vector potential $\mathbf{A}$ is defined by

$$P_1 = -iD_x - qA_1, \quad P_2 = -iD_y - qA_2,$$

in $L^2(\mathbf{R}^2)$ with $D(P_1) = D(D_x) \cap D(A_1)$ and $D(P_2) = D(D_y) \cap D(A_2)$ (we denote by $D(\cdot)$ operator domain). Let

$$M = \mathbf{R}^2 \setminus \{\mathbf{a}_1, \mathbf{a}_2\}.$$

It was shown in [1] that each $P_j$ is essentially self-adjoint on $C_0^\infty(M)$.

The Dirac-Weyl operator with the vector potential $\mathbf{A}$ is given by

$$\not{D} := \sigma_1 P_1 + \sigma_2 P_2 = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

acting in $L^2(\mathbf{R}^2; \mathbf{C}^2)$, where $\sigma_j, j = 1, 2$, are the first two of the Pauli matrices and

$$D_\pm := P_1 \pm iP_2.$$

It is an interesting and important problem to discuss essential self-adjointness of $\not{D}$, which is non-trivial because of the singularity of the vector potential. But here we only define a self-adjoint extension of the "minimal" Dirac-Weyl operator $\not{D}_{\min} := \not{D} \restriction C_0^\infty(M)$ [2].

Let $D_{-,\min}$ be the closure of $D_- \restriction C_0^\infty(M)$. Then the operator

$$\widetilde{\not{D}} = \begin{pmatrix} 0 & D_{-,\min} \\ D^*_{-,\min} & 0 \end{pmatrix}$$

with $D(\widetilde{\not{D}}) = D(D^*_{-,\min}) \oplus D(D_{-,\min})$ is a self-adjoint extension of $\not{D}_{\min}$ (see Section IV-B in [2], where $\widetilde{\not{D}}$ is denoted $Q^{(2)}_{\min}$). The set of zero-energy states of $\widetilde{\not{D}}$ is given by $\ker \widetilde{\not{D}} := \{\psi \in D(\widetilde{\not{D}}) | \widetilde{\not{D}}\psi = 0\}$.

The constants

$$\gamma_\nu := C^{(\nu)}_{0,0}, \quad \nu = 1,2,$$

physically mean the magnetic fluxes passing through the points $\mathbf{a}_1, \mathbf{a}_2$, respectively. For simplicity, we assume that

$$q > 0, \quad \gamma_\nu > 0, \quad \nu = 1,2,$$

and set

$$\varepsilon_\nu(q) = \frac{q\gamma_\nu}{2\pi} - \left[\frac{q\gamma_\nu}{2\pi}\right], \quad \nu = 1,2,$$

where $[x]$ denotes the largest integer less than or equal to $x$. It follows that $0 \le \varepsilon_\nu(q) < 1$.

For each point $\mathbf{r} = (x,y) \in \mathbf{R}^2$, we denote by $z = x + iy$ the complex number corresponding to $\mathbf{r}$. Let

$$a_\nu := a_{\nu 1} + i a_{\nu 2}$$

and

$$F(z) = -\frac{1}{2\pi}\sum_{\nu=1}^{2}\sum_{k=1}^{\ell}\frac{C^{(\nu)}_k}{k(z-a_\nu)^k},$$

with $C^{(\nu)}_k = (-1)^k k! \sum_{\alpha=0}^{k} i^{k-\alpha} C^{(\nu)}_{\alpha,k-\alpha}$. The function $F$ is related to $\mathbf{A}$ [2]:

$$A_2(\mathbf{r}) + iA_1(\mathbf{r}) = \frac{\partial F(z)}{\partial z} + \frac{1}{2\pi}\sum_{\nu=1}^{2}\frac{\gamma_\nu}{z-a_\nu}.$$

Let

$$\Omega_q(\mathbf{r}) = \left(\prod_{\nu=1}^{2} |z-a_\nu|^{-q\gamma_\nu/2\pi}(z-a_\nu)^{[q\gamma_\nu/2\pi]}\right) e^{iq\mathrm{Im}\, F(z)}.$$

LEMMA 2.1. $\dim\ker\widetilde{\not{D}} = 1$ *if and only if*

$$\varepsilon_1(q) + \varepsilon_2(q) > 1. \tag{2.1}$$

*In that case, the zero-energy state of* $\widetilde{\not{D}}$ *is given by* $\begin{pmatrix}\Omega_q(\mathbf{r})\\ 0\end{pmatrix}$ *(up to constant multiples).*

*Proof.* This is a special case of Theorem 4.7 in [2] (the case $n = 2$). ∎

*Remark.* Under condition (2.1), $q\gamma_1/2\pi$ and $q\gamma_2/2\pi$ are not integers, i.e, the magnetic flux is not locally quantized (see [1,2]).

In what follows, we consider only the case where (2.1) is satisfied. Under this condition, the ZESD of $\widetilde{\not{D}}$ is given by

$$\varrho_q(\mathbf{r}) := \frac{|\Omega_q(\mathbf{r})|^2}{||\Omega_q||^2_{L^2(\mathbf{R}^2)}} = \frac{|\mathbf{r}-\mathbf{a}_1|^{-2\varepsilon_1(q)}|\mathbf{r}-\mathbf{a}_2|^{-2\varepsilon_2(q)}}{\int_{\mathbf{R}^2}|\mathbf{r}-\mathbf{a}_1|^{-2\varepsilon_1(q)}|\mathbf{r}-\mathbf{a}_2|^{-2\varepsilon_2(q)}d\mathbf{r}}. \tag{2.2}$$

## 3. Strong Coupling Limit of the ZESD

We now consider the strong coupling limit $q \to \infty$ of the ZESD $\varrho_q(\mathbf{r})$ given by (2.2). Since the magnetic field $B$ is not a function, but a distribution, it is natural to take the strong coupling limit in the distribution sense. For this purpose, for each $\mu, \lambda \in (0,1)$ satsifying $\mu + \lambda > 1$, we define a functional $\Phi_{\mu,\lambda}$ in $L^\infty(\mathbf{R}^2)^*$, the topological dual space of $L^\infty(\mathbf{R}^2)$, by

$$\Phi_{\mu,\lambda}(f) = \int_{\mathbf{R}^2} \frac{f(\mathbf{r})}{|\mathbf{r}-\mathbf{a}_1|^{2\mu}|\mathbf{r}-\mathbf{a}_2|^{2\lambda}} d\mathbf{r}, \quad f \in L^\infty(\mathbf{R}^2).$$

In terms of this functional, the zero-energy-state functional

$$\varrho_q(f) := \int_{\mathbf{R}^2} \varrho_q(\mathbf{r}) f(\mathbf{r}) d\mathbf{r}$$

is written

$$\varrho_q(f) = \frac{\Phi_{\varepsilon_1(q),\varepsilon_2(q)}(f)}{\Phi_{\varepsilon_1(q),\varepsilon_2(q)}(1)}. \tag{3.1}$$

As (2.2) ( or (3.1)) shows, the $q$-dependence of $\varrho_q$ comes only from the factors $\varepsilon_\nu(q), \nu = 1,2$. It is obvious that $\lim_{q\to\infty} \varepsilon_\nu(q)$ does not exist. But, for suitable monotone increasing sequences $\{q_n\}_{n=1}^\infty$ of positive numbers satisfying

$$q_n \to \infty \ (n \to \infty), \quad \varepsilon_1(q_n) + \varepsilon_2(q_n) > 1, \quad n \geq 1, \tag{3.2}$$

the limits

$$\lambda_\nu := \lim_{n\to\infty} \varepsilon_\nu(q_n), \quad \nu = 1,2, \tag{3.3}$$

may exist, depending on the choice of $\{q_n\}_{n=1}^\infty$. For this reason, we discuss the strong coupling limit of the ZESD according to the magnitude of $\lambda_\nu, \nu = 1,2$.

We denote by $\mathfrak{B}(\mathbf{R}^2)$ the set of bounded continuous functions on $\mathbf{R}^2$. Limiting behaviors of the functional $\Phi_{\mu,\lambda}$ in $\mu$ and $\lambda$ are given in the following lemma.

LEMMA 3.1.

(i) *Let $\mu_0, \lambda_0 \in (0,1)$ such that $\mu_0 + \lambda_0 > 1$. Then, for all $f \in L^\infty(\mathbf{R}^2)$,*

$$\lim_{\mu\to\mu_0, \lambda\to\lambda_0} \Phi_{\mu,\lambda}(f) = \Phi_{\mu_0,\lambda_0}(f). \tag{3.4}$$

(ii) *Let $\lambda_0 \in (0,1)$. Then, for all $f \in \mathfrak{B}(\mathbf{R}^2)$,*

$$\lim_{\mu\to 1, \lambda\to\lambda_0} (1-\mu)\Phi_{\mu,\lambda}(f) = \frac{\pi f(\mathbf{a}_1)}{|\mathbf{a}_1 - \mathbf{a}_2|^{2\lambda_0}}. \tag{3.5}$$

(iii) *Let $\tau > 0$. Then, for all $f \in \mathfrak{B}(\mathbf{R}^2)$,*

$$\lim_{\mu\to 1} (1-\mu)\Phi_{\mu,(\tau-1+\mu)/\tau}(f) = \frac{\pi}{|\mathbf{a}_1 - \mathbf{a}_2|^2}(f(\mathbf{a}_1) + \tau f(\mathbf{a}_2)). \tag{3.6}$$

*Proof.* (i) For $\delta > 0$, we define

$$B_\delta(\mathbf{a}_\nu) = \{\mathbf{r} \in \mathbf{R}^2 \,|\, 0 \leq |\mathbf{r} - \mathbf{a}_\nu| \leq \delta\}.$$

Without loss of generality, we can assume that $\mu_0 - \varepsilon < \mu < \mu_0 + \varepsilon < 1$ and

$$\lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon < 1, \tag{3.7}$$

where $\varepsilon > 0$ is a constant such that $\mu_0 + \lambda_0 - 2\varepsilon > 1$. Then we have for $\mathbf{r} \in M$

$$\frac{1}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}} \leq \frac{\chi_{B_1(\mathbf{a}_1)}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\mu_0+\varepsilon)}} + \frac{\chi_{B_1(\mathbf{a}_1)^c}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\mu_0-\varepsilon)}},$$
$$\frac{1}{|\mathbf{r} - \mathbf{a}_2|^{2\lambda}} \leq \frac{\chi_{B_1(\mathbf{a}_2)}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\lambda_0+\varepsilon)}} + \frac{\chi_{B_1(\mathbf{a}_2)^c}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\lambda_0-\varepsilon)}},$$

where $\chi_S$ denotes the characteristic function of the set $S$. Hence, for all $\mathbf{r} \in M$,

$$\frac{|f(\mathbf{r})|}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}|\mathbf{r} - \mathbf{a}_2|^{2\lambda}}$$
$$\leq ||f||_\infty \left(\frac{\chi_{B_1(\mathbf{a}_1)}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\mu_0+\varepsilon)}} + \frac{\chi_{B_1(\mathbf{a}_1)^c}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2(\mu_0-\varepsilon)}}\right)\left(\frac{\chi_{B_1(\mathbf{a}_2)}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_2|^{2(\lambda_0+\varepsilon)}} + \frac{\chi_{B_1(\mathbf{a}_2)^c}(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_2|^{2(\lambda_0-\varepsilon)}}\right).$$

The function on the right hand side (RHS) is independent of $\mu, \lambda$ and integrable on $\mathbf{R}^2$. Hence, by the dominated convergence theorem, we obtain (3.4).

(ii) Let $\mathbf{b} = \mathbf{a}_1 - \mathbf{a}_2 = (b_1, b_2)$ and $\delta > 0$ be a constant such that $0 < \delta < \min\{1, |\mathbf{b}|/2\}$. Then we can write

$$\Phi_{\mu,\lambda}(f) = I_{\mu,\lambda}(f) + J_{\mu,\lambda}(f) + K_{\mu,\lambda}(f) \tag{3.8}$$

with

$$I_{\mu,\lambda}(f) = \int_{B_\delta(\mathbf{a}_1)} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}|\mathbf{r} - \mathbf{a}_2|^{2\lambda}} d\mathbf{r},$$
$$J_{\mu,\lambda}(f) = \int_{B_\delta(\mathbf{a}_2)} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}|\mathbf{r} - \mathbf{a}_2|^{2\lambda}} d\mathbf{r},$$
$$K_{\mu,\lambda}(f) = \int_{B_\delta(\mathbf{a}_1)^c \cap B_\delta(\mathbf{a}_2)^c} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}|\mathbf{r} - \mathbf{a}_2|^{2\lambda}} d\mathbf{r}.$$

By the change of variables $x \to a_{11} + r\cos\theta, y \to a_{12} + r\sin\theta, 0 < r \leq \delta, 0 \leq \theta < 2\pi$, we have

$$I_{\mu,\lambda}(f) = \int_0^\delta dr \int_0^{2\pi} d\theta \frac{f(a_{11} + r\cos\theta, a_{12} + r\sin\theta)}{r^{2\mu-1}(r^2 + |\mathbf{b}|^2 + 2rb(\theta))^\lambda},$$

where $b(\theta) = b_1 \cos\theta + b_2 \sin\theta$. Moreover, the change of variable $s = r^{2(1-\mu)}$ gives

$$(1-\mu) I_{\mu,\lambda}(f) = \frac{1}{2} \int_0^1 ds \int_0^{2\pi} d\theta F_{\mu,\lambda}(s,\theta)$$

with

$$F_{\mu,\lambda}(s,\theta) = \frac{\chi_{[0,\delta(\mu)]}(s) f(a_{11} + s^{\frac{1}{2(1-\mu)}} \cos\theta, a_{12} + s^{\frac{1}{2(1-\mu)}} \sin\theta)}{(s^{\frac{1}{1-\mu}} + |\mathbf{b}|^2 + 2s^{\frac{1}{2(1-\mu)}} b(\theta))^\lambda},$$

where $\delta(\mu) = \delta^{2(1-\mu)} < 1$. We have

$$\lim_{\mu\to 1, \lambda\to\lambda_0} F_{\mu,\lambda}(s,\theta) = \begin{cases} \frac{f(\mathbf{a}_1)}{|\mathbf{b}|^{2\lambda_0}} & ; \text{for } 0 < s < 1 \\ 0 & ; \text{for } s > 1 \end{cases}.$$

Moreover, it is easy to see that

$$|F_{\mu,\lambda}(s,\theta)| \le \frac{||f||_\infty}{(|\mathbf{b}| - \delta)^\lambda} \le C_0 ||f||_\infty$$

with $C_0 = \sup_{0<x<1} (|\mathbf{b}| - \delta)^{-x} < \infty$. Hence, by the dominated convergence theorem, we obain

$$\lim_{\mu\to 1, \lambda\to\lambda_0} (1-\mu) I_{\mu,\lambda}(f) = \frac{\pi f(\mathbf{a}_1)}{|\mathbf{b}|^{2\lambda_0}}. \tag{3.9}$$

To estimate $J_{\mu,\lambda}(f)$, we note that there exist positive constants $c_1, c_2$ such that

$$c_1 \le |\mathbf{r} - \mathbf{a}_1| \le c_2, \quad \mathbf{r} \in B_\delta(\mathbf{a}_2),$$

so that, for all $\mathbf{r} \in B_\delta(\mathbf{a}_2)$,

$$\frac{1}{|\mathbf{r} - \mathbf{a}_1|^{2\mu}} \le \frac{1}{c_1^{2\mu}} \le C_1$$

with $C_1 = \sup_{0<x<1} c_1^{-2x} < \infty$. Let $\varepsilon > 0$ be such that (3.7) holds. Then we have

$$|\mathbf{r} - \mathbf{a}_2|^{2\lambda} \ge |\mathbf{r} - \mathbf{a}_2|^{2(\lambda_0 + \varepsilon)}, \quad \mathbf{r} \in B_\delta(\mathbf{a}_2),$$

which implies that

$$\frac{|f(\mathbf{r})|}{|\mathbf{r} - \mathbf{a}_1|^{2\mu} |\mathbf{r} - \mathbf{a}_2|^{2\lambda}} \le \frac{C_1 ||f||_\infty}{|\mathbf{r} - \mathbf{a}_2|^{2(\lambda_0+\varepsilon)}}, \quad \mathbf{r} \in B_\delta(\mathbf{a}_2).$$

The function on the RHS is integrable on $B_\delta(\mathbf{a}_2)$, since $\lambda_0 + \varepsilon < 1$. Thus we can apply the dominated convergence theorem to obtain

$$\lim_{\mu\to 1, \lambda\to\lambda_0} J_{\mu,\lambda}(f) = \int_{B_\delta(\mathbf{a}_2)} \frac{f(\mathbf{r})}{|\mathbf{r} - \mathbf{a}_1|^2 |\mathbf{r} - \mathbf{a}_2|^{2\lambda_0}} d\mathbf{r}. \tag{3.10}$$

By a method similar to that of the proof of part (i), one can easily show that

$$\lim_{\mu\to 1,\lambda\to\lambda_0} K_{\mu,\lambda}(f) = \int_{B_\delta(\mathbf{a}_1)^c\cap B_\delta(\mathbf{a}_2)^c} \frac{f(\mathbf{r})}{|\mathbf{r}-\mathbf{a}_1|^2|\mathbf{r}-\mathbf{a}_2|^{2\lambda_0}}d\mathbf{r}. \tag{3.11}$$

Formula (3.5) now follows from (3.8)–(3.11).

(iii) Let $\lambda = (\tau - 1 + \mu)/\tau$ and $\max\{1-\tau,(1+\tau)^{-1}\} < \mu < 1$. Then, $\lambda \in (0,1)$, $\lambda+\mu > 1$, and $\lambda \to 1$ as $\mu \to 1$. As is seen from the proof of (3.9), (3.9) holds also in the case $\lambda_0 = 1$. Hence

$$\lim_{\mu\to 1}(1-\mu)I_{\mu,\lambda}(f) = \frac{\pi f(\mathbf{a}_1)}{|\mathbf{b}|^2}.$$

Similarly we have

$$\lim_{\mu\to 1}(1-\lambda)J_{\mu,\lambda}(f) = \frac{\pi f(\mathbf{a}_2)}{|\mathbf{b}|^2}.$$

Formula (3.11) holds also in the case $\lambda_0 = 1$. Since $1-\mu = \tau(1-\lambda)$, (3.6) follows. ∎

A simple application of Lemma 3.1 to $\varrho_q(f)$ gives the following result.

THEOREM 3.2. *Let $\{q_n\}_{n=1}^\infty$ be a sequence satisfying (3.2) and (3.3).*

(i) *Suppose that $\lambda_1, \lambda_2 \in (0,1)$ such that $\lambda_1+\lambda_2 > 1$. Then, for all $f \in L^\infty(\mathbf{R}^2)$,*

$$\lim_{n\to\infty} \varrho_{q_n}(f) = \frac{\Phi_{\lambda_1,\lambda_2}(f)}{\Phi_{\lambda_1,\lambda_2}(1)}.$$

(ii) *Suppose that $\lambda_1 = 1, \lambda_2 \in (0,1)$. Then, for all $f \in \mathfrak{B}(\mathbf{R}^2)$,*

$$\lim_{n\to\infty} \varrho_{q_n}(f) = f(\mathbf{a}_1).$$

(iii) *Let $\tau > 0$ and suppose that, for all sufficiently large $n$, $1-\varepsilon_1(q_n) = \tau(1-\varepsilon_2(q_n))$ and $\lambda_1 = 1$ (hence $\lambda_2 = 1$). Then, for all $f \in \mathfrak{B}(\mathbf{R}^2)$,*

$$\lim_{n\to\infty} \varrho_{q_n}(f) = \frac{f(\mathbf{a}_1)+\tau f(\mathbf{a}_2)}{1+\tau}. \tag{3.12}$$

*Remark.* One can eaily find examples of $\{q_n\}_{n=1}^\infty$ for each case in Theorem 3.2.

As a corollary, we obtain the folloiwng:

COROLLARY 3.3. *Let $\{q_n\}_{n=1}^\infty$ be a sequence satisfying (3.2) and (3.3). Then, for all cases (i)–(iii) in Theorem 3.2,*

$$\lim_{n\to\infty} \frac{\varrho_{q_n}(f)}{q_n} = 0, \quad f \in \mathfrak{B}(\mathbf{R}^2).$$

This result shows that the Erdös' theorem established in [3] does not hold in the present case.

In a special case of part (iii) of Theorem 3.2, a strong coupling limit of the ZESD yields the magnetic field :

COROLLARY 3.4. *Let* $\{q_n\}_{n=1}^{\infty}$ *be a sequence satisfying (3.2) and (3.3) with* $\lambda_\nu = 1, \nu = 1, 2$. *Suppose that, for all sufficiently large* $n$, $\gamma_1(1 - \varepsilon_1(q_n)) = \gamma_2(1 - \varepsilon_2(q_n))$. *Then,*

$$\lim_{n\to\infty} (\gamma_1 + \gamma_2)\varrho_{q_n}(\mathbf{r}) = \gamma_1 \delta(\mathbf{r} - \mathbf{a}_1) + \gamma_2 \delta(\mathbf{r} - \mathbf{a}_2)$$

*in the distribution sense. In particular, if* $C_{\alpha,\beta}^{(\nu)} = 0$ *for* $\nu = 1, 2$ *and all* $\alpha, \beta$ *with* $\alpha + \beta \geq 1$, *then*

$$\lim_{n\to\infty} (\gamma_1 + \gamma_2)\varrho_{q_n}(\mathbf{r}) = B(\mathbf{r})$$

*in the distribution sense.*

*Proof.* We need only to apply part (iii) of Theorem 3.2 with $\tau = \gamma_2/\gamma_1$. ∎

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