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**Factorizations Of Outer Functions
And Extremal Problems**

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Factorizations Of Outer Functions

And

Extremal Problems

By

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Abstract. The author has proved that an outer function in the Hardy space H^1 can be factored into a product in which one factor is strongly outer and the other is the sum of two inner functions. In an endeavor to understand better the latter factor, we introduce a class of functions containing sums of inner functions as a special case. Using it, we describe the solutions of extremal problems in the Hardy spaces H^p for $1 \leq p < \infty$.

§1. Introduction

N, N_+ and H^p for $1 \leq p \leq \infty$ denote the Nevanlinna class, the Smirnov class and the Hardy space, respectively on the open unit disc U in the complex plane. A function h in N_+ is called outer if it is not divisible in N_+ by a non-constant inner function. A function g in H^1 is called strongly outer if the only functions f in H^1 such that f/g is non-negative are scalar multiples of g . If g is not outer and so $g = qh$ for some inner q , then $f = (1+q)^2h$ belongs to H^1 and $f/g = (1+q)^2/q$ is non-negative. A norm one function in H^1 is outer if and only if it is an extreme point of the unit ball of H^1 [2]. On the other hand, a norm one function in H^1 is strongly outer if and only if it is an exposed point of the unit ball of H^1 (cf. [2], [12]). Like outer functions, strongly outer functions appear in many important areas, for example, function theory, operator theory and prediction theory.

It is not difficult to give a characterization of a strongly outer function similar to the above definition of an outer function. If g is divisible in H^1 by a sum of two inner functions q_1, q_2 where $q_1 + q_2$ is not constant and $Im\bar{q}_1q_2 \leq 0$ almost everywhere, then $f = -i(q_1 - q_2)g/(q_1 + q_2)$ is not a scalar multiple of g and f/g is non-negative because $-i(q_1 - q_2)/(q_1 + q_2) \geq 0$ almost everywhere. Thus g is not strongly outer. The converse is also true by the following factorization theorem [12].

Theorem. If an outer function h in H^1 is not strongly outer, then $h = (q_1 + q_2)g$ where both q_1 and q_2 are inner, $Im\bar{q}_1q_2 \leq 0$ almost everywhere, $(q_1 - q_2)^{-1}$ is summable and g is strongly outer. If q_1 is a finite Blaschke product of degree n then so is q_2 .

The aim of this paper is to gain a better understanding of this theorem and of the sum of two inner functions. The sum of two inner functions appeared in H.Helson's papers [7] and [8]. D.Sarason [15] examined cases in which the sum of two non-constant inner functions is outer. In this paper, we introduce functions in H^2 which have the form $k = s + q\bar{s}$ where s is in L^2 and q is inner. If $s = 1$, then $k = 1 + q$. If $s = q_1$ and $q = q_1q_2$ where q_1 and q_2 are inner, then $k = q_1 + q_2$. If f is the square of H^2 function $s + q\bar{s}$, then put $q_1 =$ the inner part of $f + iq$ and $q_2 =$ the inner part of $f - iq$. Then $Im\bar{q}_1q_2 \leq 0$, $q_1 + q_2$ is non-constant and f is divisible in H^1 by $q_1 + q_2$. By the remark above the Theorem, f is not strongly outer. The following factorization theorem can be proved easily by a theorem of E.Hayashi ([5], [6]).

Theorem. If an outer function h in H^1 is not strongly outer, then $h = (s + q\bar{s})^2g$ where q is a non-constant inner function, $s + q\bar{s}$ is in H^2 and g is strongly outer.

Proof. Suppose $h = k^2$ and k is outer in H^2 . By a theorem of E.Hayashi ([4], [5]),

$$H^2 \cap (k/\bar{k})\bar{H}^2 = g_0(H^2 \ominus zqH^2)$$

and $k/\bar{k} = \bar{q}g_0/g_0$ where q is inner and g_0^2 is strongly outer. Hence $k = \ell g_0$ where $\ell \in H^2 \ominus qzH^2$ and $\bar{q}\ell^2 \geq 0$. Put $s = \ell/2$, then $\ell = s + q\bar{s}$ and $h = \ell^2 g_0^2$.

In this theorem, we should like to be able to choose $s + q\bar{s} = q_1 + q_2$ for some inner functions q_1 and q_2 . Unfortunately we could not do except in some special cases [12]. Note that by an example of J.Inoue [9], we cannot choose $s + q\bar{s} = 1 + q$.

§2. Bad parts of outer functions

In this section we study a function in H^2 which has the form $s + q\bar{s}$ where s is in L^2 and q is an inner function. If $\prod_{j=1}^n (q_j + q'_j)$ where q_j and q'_j are inner functions for $1 \leq j \leq n$, then $\prod_{j=1}^n (q_j + q'_j) = s + q\bar{s}$ for $q = \prod_{j=1}^n q_j q'_j$. Two natural questions are the following : (1) When is $s + q\bar{s}$ an outer function ? (2) When can $s + q\bar{s}$ be divisible in H^2 by $1 + q'$ where q' denotes some nonconstant inner function ? The question (1) is related with a paper of D.Sarason [15]. He studied it when $s + q\bar{s}$ is a sum of two inner functions. The question (2) is related with a paper of J.Inoue [9]. By the second theorem in the Introduction, Inoue's result is the following : There exists an outer function f in H^2 which is not divisible in H^2 by any nonconstant $1 + q'$ but is divisible in H^2 by some nonconstant $s + q\bar{s}$, where q and q' are inner functions. Because of the first theorem in the Introduction, we are also interested in nonconstant outer function $q_1 + q_2$ such that both q_1 and q_2 are inner functions, $Im\bar{q}_1 q_2 \leq 0$ almost everywhere and $(q_1 - q_2)^{-1}$ is summable.

Proposition 1. Suppose s is a nonnegative function in N_+ and s^{-1} is summable. If $i-s = q_1 \ell$ where q_1 is an inner function and ℓ is an outer function, then $q_2 = (i+s/i-s)q_1$ is an inner function, $q_1 + q_2$ is an outer function, $Im\bar{q}_1 q_2 \leq 0$ almost everywhere and $(q_1 - q_2)^{-1}$ is summable. If s is a rational function, then both q_1 and q_2 are finite Blaschke products of the same degree.

Proof. Since $|q_2| = 1$ a.e. on ∂U and $q_2 = (i+s)/\ell$, q_2 is inner. Since $q_1 + q_2 = 2i\ell$, $q_1 + q_2$ is outer. By a simple calculation,

$$\frac{-Im\bar{q}_1 q_2}{|q_1 + q_2|^2} = \frac{-i(q_1 - q_2)}{q_1 + q_2} = s \geq 0 \quad \text{a.e.}$$

and so $Im\bar{q}_1 q_2 \leq 0$ a.e. on ∂U . Since $(q_1 - q_2)^{-1} = (i-s)/(-2s)$ and s^{-1} is summable, $(q_1 - q_2)^{-1}$ is summable. If s is a rational function, by [7] the number of zeros of $s - i$ and that of $s + i$ are equal. Hence q_1 and q_2 are finite Blaschke products of the same degree.

In Proposition 1, if $s = -z/(1-z)^2$, then q_1 and q_2 have degree one. However even if q_1 and q_2 have degree one and $q_1 + q_2$ is outer, $Im\bar{q}_1q_2$ is not necessarily non-negative. In fact, suppose $|a| < 1$ and $|\rho| = 1$. Then, $\rho z + \bar{\rho} \left(\frac{z-a}{1-\bar{a}z} \right)$ is outer if and only if $|Re\rho| \leq |a|$, [15]. However $Im\bar{z} \left(\frac{z-a}{1-\bar{a}z} \right)$ is not non-negative on ∂U .

Proposition 2. Suppose $s + q\bar{s}$ is in H^2 , where q is an inner function and s is in L^2 . Then $s + q\bar{s}$ is an outer function if and only if there exists a function t in L^2 such that $s + (t - q\bar{t})$ is an outer function.

Proof. If $\ell = s + (t - q\bar{t})$ is outer, then $s + q\bar{s} = \ell + q\bar{\ell} \in H^2$ and $q\bar{\ell} \in H^2$. Hence $q\bar{\ell} = q_0\bar{\ell}$ for some inner function q_0 . Then $s + q\bar{s} = \ell(1 + q\frac{\bar{\ell}}{\ell}) = \ell(1 + q_0)$ and hence $s + q\bar{s}$ is outer. Conversely if $s + q\bar{s} = 2\ell$ is outer, then $q\bar{\ell} = \ell$ and hence $s + q\bar{s} = \ell + q\bar{\ell}$. Let $k = \ell - s$, then $k + q\bar{k} = 0$ and so $k = t - q\bar{t}$, where $t = k/2$. Thus $\ell = s + (t - q\bar{t})$ is outer.

Corollary 1. Suppose $s + q\bar{s}$ is in H^2 , where q is an inner function and s is in L^2 . If s and q satisfy one of the following (1) \sim (3), then $s + q\bar{s}$ is an outer function.

(1) s is an outer function.

(2) $q = q_1q_2$ and $s = q_1h$ where q_1 and q_2 are inner functions, h is an outer function and $q_2\bar{h} = \alpha h$ for some complex number α .

(3) $q = q_1q_2$ and $s = q_1h$ where $\{q_j\}_{j=1,2,3}$ are inner functions, h is an outer function, $q_2\bar{h} = q_3h$, and $q_1 + q_3$ is an outer function.

Proof. (1) is clear by Proposition 2 and (2) is a special case of (3). For (3), let $t = (q_3 - q_1)h/4$, then

$$q\bar{t} = \frac{1}{4}q(\bar{q}_2h - \bar{q}_1\bar{h}) = \frac{1}{4}(q_1h - q_2\bar{h}) = \frac{1}{4}(q_1 - q_3)h$$

because $q_2\bar{h} = q_3h$. Hence $t - q\bar{t} = (q_3 - q_1)h/4$ and so $s + (t - q\bar{t}) = (q_3 + q_1)h/2$. This implies (3) because $q_1 + q_3$ is outer.

Proposition 3. Suppose q_1 is an inner function and $s + q\bar{s}$ is a non-zero function in H^2 , where q is an inner function and s is in L^2 . Then $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$ if and only if there exists a function t in L^2 such that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$. In particular, if $q\bar{s} = q_1s$ then $s + q\bar{s}$ is divisible by $1 + q_1$.

Proof. If there exists a function t in L^2 such that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$, then $s + q\bar{s} = s + t - q\bar{t} + q(\bar{s} + \bar{t} - \bar{q}t) = s + t - q\bar{t} + q_1(s + t - q\bar{t}) = (s + t - q\bar{t})(1 + q_1)$ and hence $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$. Conversely if $\ell = (s + q\bar{s})/(1 + q_1)$ is in H^2 , then

$$\bar{q} = \frac{\bar{s} + \bar{q}s}{s + q\bar{s}} = \frac{\bar{\ell}1 + \bar{q}_1}{\ell1 + q_1} = \frac{\bar{\ell}}{\ell}\bar{q}_1$$

and hence $q\bar{\ell} = q_1\ell$. If $k = \ell - s$, then $k = t - q\bar{t}$ for some $t \in L^2$ and hence $\ell = s + (t - q\bar{t})$. This implies that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$.

Corollary 2. Suppose $s + q\bar{s}$ is in H^2 , where q is an inner function and s is in L^2 .

(1) If s is an outer function and $q\bar{s} \neq \alpha s$ for any α in C with $|\alpha| = 1$, then there exists a non-constant inner function q_1 such that $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$.

(2) If h is an outer function, $q\bar{h} = q_1q_2^2h$ and $s = q_2h$ where q_1 and q_2 are inner functions, then $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$.

(3) If q is a finite Blaschke product, then there exists a non-constant finite Blaschke product q_1 such that $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$, or $s + q\bar{s}$ is not an outer function.

Proof. (1) Since s is outer, $q\bar{s} = q_1s$ for some inner function q_1 . By the hypothesis, q_1 is non-constant and hence Proposition 3 implies (1). (2) $q\bar{s} = q_1q_2^2h = q_1s$ implies (2) by Proposition 3. (3) Since $\bar{q}(s + q\bar{s})^2 \geq 0$ a.e. on ∂U and q is a finite Blaschke product, $(s + q\bar{s})^2 = \prod_{j=1}^n (z - a_j)(1 - \bar{a}_jz)\ell^2$, where $|a_j| \leq 1$ ($1 \leq j \leq n$) and ℓ is outer in H^2 ([2],[11]). Therefore if $s + q\bar{s}$ is outer, then $s + q\bar{s} = \prod_{j=1}^n (-\bar{a}_j)^{1/2}(z - a_j)\ell$ and $|a_j| = 1$. Thus $s + q\bar{s}$ is divisible in H^2 by $z - a_j$.

When q_1 and q_2 are inner functions, we write $q_1 \prec q_2$ if there exists a nonzero function f in H^1 such that $\bar{q}_1q_2 = f/|f|$. If both q_1 and q_2 are finite Blaschke product, then $q_1 \prec q_2$ is equivalent to $(\text{degree of } q_1) \leq (\text{degree of } q_2)$. For each g in H^1 , $\text{sing } g$ denotes the set of the unit circle on which g cannot be analytically extended.

Proposition 4. If q_1 and q_2 are inner functions and the inner part of $q_1 + q_2$ is q , then $q \prec q_1$ and $q \prec q_2$.

Proof. Let $q_1 + q_2 = qh$, then $|\bar{q}q_1 - h| = |\bar{q}q_2 - h| = 1$. By a theorem of P.Koosis (cf.[4, Chapter 4, Lemma 5.4]), $q \prec q_1$ and $q \prec q_2$.

Corollary 3. Suppose q_1 and q_2 are inner functions and $q_1 + q_2 = qh$ where q is an inner function and h is an outer function.

(1) If q_1 is a finite Blaschke product, then q is also a finite Blaschke product and $(\text{degree of } q) \leq (\text{degree of } q_1)$.

(2) If $(\text{sing } q_1) \cap (\text{sing } q_2)$ is empty, then q is a finite Blaschke product.

(3) Suppose $q_1 = \exp\left(-\frac{a+z}{a-z}\right)$ and $q_2 = -\alpha \exp\left(-\frac{b+z}{b-z}\right)$, where $|a| = |b| = 1$, $b = -\bar{a}$ and $|\alpha| = 1$. If $\alpha = 1$, then $q = z$ or q is constant. If $\alpha \neq 1$, then q is always constant, that is, $q_1 + q_2$ is an outer function.

Proof. (1) By Proposition 4, $\bar{q}q_1 = f/|f|$ for some function $f \in H^1$ and hence $\bar{q}_1(qf) \geq 0$ a.e. on ∂U . If q_1 is a finite Blaschke product of degree m , $qf = \prod_{j=1}^n (z - a_j)(1 - \bar{a}_j z)\ell$ and $n \leq m$ where $|a_j| \leq 1$ ($1 \leq j \leq n$) and ℓ is strongly outer. Hence q is a finite Blaschke product of degree k and $k \leq n$. (2) By Proposition 4, $\bar{q}q_1 = f/|f|$ for some function $f \in H^1$ and hence $\bar{q}q_1 = g/\bar{g}$ for some outer function $g \in H^2$. Therefore $\bar{q}_1 qg = \bar{g}$ and so $qg \in H^2 \ominus q_1 z H^2$. Hence $\text{sing } q_1 \supseteq \text{sing } qg$ and by [10, Lemma 4], $\text{sing } q_1 \supseteq \text{sing } q$. Similarly $\text{sing } q_2 \supseteq \text{sing } q$ and by the hypothesis q is a finite Blaschke product. (3) By (2), q is a finite Blaschke product. If $q(x) = 0$ for some point $x \in U$, then $\exp\left(-\frac{a+x}{a-x}\right) = \alpha \exp\left(-\frac{b+x}{b-x}\right)$ and hence

$$-\frac{a+x}{a-x} = -\frac{b+x}{b-x} + i\rho \text{ and } \rho = t + 2n\pi$$

where n is some integer and $\alpha = e^{it}$. If $\rho = 0$ then $q = z$ because $a \neq b$. Suppose $\rho \neq 0$. Then

$$x^2 - \left\{ \left(1 - \frac{2i}{\rho}\right)b + \left(1 + \frac{2i}{\rho}\right)a \right\}x + ab = 0.$$

If A and B are the solutions of the above quadratic equation, then $AB = ab = -1$ and

$$A + B = \left(1 + \frac{2i}{\rho}\right)a - \overline{\left(1 + \frac{2i}{\rho}\right)a}.$$

This implies $|A| = |B| = 1$ and contradicts $|x| < 1$.

(1) of Corollary 3 was proved by D.Sarason [15, Proposition 3]. Our proof is different from his.

§3. Projection.

For each inner function q , we define two operators on L^2

$$L_q(s) = \frac{s + q\bar{s}}{2} \text{ and } L'_q(s) = \frac{s - q\bar{s}}{2}.$$

If $q = 1$, then $L_q(s)$ is the real part of s and $L'_q(s)$ is the imaginary part of s . In general, $|L_q(s)| \leq |s|$ and $|L'_q(s)| \leq |s|$. Hence L_q and L'_q are contractive. L_q and L'_q commute with multiplication operators by real valued functions in L^∞ . Moreover on L^2 , we have $L_q L_q = L_q$ and $L_q L'_q = 0$ and $L_q + L'_q$ is the identity operator. By results of the last section, we are interested in a function s such that $L_q(s)$ belongs to H^2 . Since $q = (1+q)^2/|1+q|^2$, we define $q^{1/2} = (1+q)/|1+q|$. Put

$$\mathcal{A}_q = \{g \in H^2 : \frac{g}{1+q} \text{ is a real valued function}\}.$$

Theorem 5. Let q be a non-constant inner function. Then

$$\{s \in L^2 : L_q(s) \in H^2\} = \mathcal{A}_q + iq^{1/2}L_R^2,$$

where $L_R^2 = \{s \in L^2 : s \text{ is a real valued function}\}$. In particular, if $s + q\bar{s}$ belongs to H^2 for some s in L^2 , then $s + q\bar{s} = t + q\bar{t}$ for some t in H^2 .

Proof. If $g \in \mathcal{A}_q$ then $u = g/(1+q)$ is real and $g = u(1+q)$. Hence $q\bar{g} = g$ and so $L_q(g) = g \in H^2$. If $s = iq^{1/2}u$ and $u \in L_R^2$ then $L_q(s) = 0$. This implies that $\{s \in L^2 : L_q(s) \in H^2\} \supseteq \mathcal{A}_q + iq^{1/2}L_R^2$. Conversely, suppose $g = L_q(s) \in H^2$. If $g = 0$, then $s = -q\bar{s}$ and $s^2 = -q|s|^2$. Hence $(iq^{1/2}s)^2 = -q\bar{s}^2 = |s|^2 \geq 0$ and so $iq^{1/2}s = -u$ is real. Thus $s = iq^{1/2}u$ and $u \in L_R^2$. If $g \neq 0$, $s + q\bar{s} = 2g$ and

$$\frac{s}{g} + \overline{\left(\frac{s}{g}\right)} = 2,$$

Put $t = s/g - 1$, then $t + \bar{t} = 0$ and so $t = iv$ for some $v \in L_R^2$. Hence $s = g + ivg$ and $vg = q^{1/2}u$, where $u = v\bar{q}^{1/2}g$ is in L_R^2 . Thus $s = g + iq^{1/2}u$. This completes the proof of the proposition.

Corollary 4. Let q be a non-constant inner function. Then

$$\{s \in H^2 : L_q(s) \in H^2\} = \mathcal{A}_q + i\mathcal{A}_q$$

and hence $H^2 \ominus qzH^2 = \mathcal{A}_q + i\mathcal{A}_q$. L_q is the projection from $H^2 \ominus qzH^2$ onto \mathcal{A}_q and has kernel $i\mathcal{A}_q$.

Proof. If $g \in \mathcal{A}_q$, then $g = v(1+q)$ for some real valued function v and so $g = q^{1/2}u$ where $u = v|1+q|$. Hence $\mathcal{A}_q \subset q^{1/2}L_R^2$ and $(q^{1/2}L_R^2) \cap H^2 = \mathcal{A}_q$. Now Theorem 5 implies the corollary.

The proof of Theorem 5 is related to that of [14, Theorem 3]. The equality in Corollary 4, that is, $H^2 \ominus qzH^2 = \mathcal{A}_q + i\mathcal{A}_q$ is known by [12, (1) of Theorem 3].

Corollary 5. Let q be an inner function.

(1) If $q = z^n$, then $\mathcal{A}_q = \{\sum_{j=0}^n b_j z^j : b_j = \bar{b}_{n-j}\}$.

(2) If $q = \prod_{\ell=1}^{\infty} \frac{-\bar{a}_\ell}{|a_\ell|} \frac{z - a_\ell}{1 - \bar{a}_\ell z}$ and $\sum_{\ell=1}^{\infty} (1 - |a_\ell|) < \infty$, then

$$\mathcal{A}_q = \left\{ \sum_{j=0}^{\infty} \frac{c_j B_j + \bar{c}_j z B'_j}{1 - \bar{a}_j z} : \sum_{j=0}^{\infty} \frac{|c_j|^2}{(1 - |a_j|)^2 (1 + |a_j|)} < \infty \right\}$$

where $B_j = \prod_{\ell=1}^{j-1} \frac{-\bar{a}_\ell}{|a_\ell|} \frac{z - a_\ell}{1 - \bar{a}_\ell z}$, $B'_j = \prod_{\ell=j}^{\infty} \frac{-\bar{a}_\ell}{|a_\ell|} \frac{z - a_\ell}{1 - \bar{a}_\ell z}$, $a_0 = 0$, $B_0 = 1$ and $B'_0 = q$.

Proof. (1) If $s \in H^2 \ominus qzH^2$, then $s = \sum_{j=0}^n a_j z^j$ and hence $s + q\bar{s} = \sum_{j=0}^n (a_j + \bar{a}_{n-j}) z^j$. Now Corollary 4 implies (1). (2) If $s \in H^2 \ominus qzH^2$, then by [1]

$$s = \sum_{j=0}^{\infty} c_j (1 + |a_j|)^{1/2} B_j (1 - \bar{a}_j z)^{-1} (1 - |a_j|)$$

and $\sum_{j=0}^{\infty} |c_j|^2 < \infty$. Hence

$$\begin{aligned} s + q\bar{s} &= \sum_{j=0}^{\infty} \left(c_j \frac{B_j}{1 - \bar{a}_j z} + \bar{c}_j \frac{q \bar{B}_j}{1 - \bar{a}_j z} \right) (1 + |a_j|)^{1/2} (1 - |a_j|) \\ &= \sum_{j=0}^{\infty} \left(\frac{c_j B_j + \bar{c}_j z B'_{j+1}}{1 - \bar{a}_j z} \right) (1 + |a_j|)^{1/2} (1 - |a_j|). \end{aligned}$$

Now Corollary 4 implies (2).

A theorem of P.R.Ahern and D.N.Clark [1, Theorem 3.1] lets one describe \mathcal{A}_q for arbitrary inner function q .

§4. Extremal problems.

Let $1 \leq p \leq \infty$ and $1/p + 1/\ell = 1$. If $\phi \in L^\ell$, we denote by T_ϕ^p the continuous functional defined on the Hardy space H^p by

$$T_\phi^p(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \phi(e^{i\theta}) d\theta / 2\pi.$$

A function f in H^p , which satisfies $T_\phi^p(f) = \|T_\phi^p\|$ and $\|f\|_p \leq 1$, is called an extremal function. A function ϕ in L^ℓ is called an extremal kernel when $\|\phi\|_\ell = \|T_\phi^p\|$. The existence and uniqueness of extremal functions and extremal kernels is known for $1 < p \leq \infty$ (cf.[3, Theorem 8.1]). For $p = 1$, the situation is very different. An extremal function may not exist, the dual extremal kernel always exists and is unique if an extremal function exists (cf.[3, Theorem 8.1]). For $p = 1$, the set S_ϕ of all extremal functions is defined by

$$S_\phi = \{f \in H^1 : T_\phi^1(f) = \|T_\phi^1\| \text{ and } \|f\|_1 = 1\}.$$

S_ϕ has been described in general by E.Hayashi [5],[6]. In this section, we describe S_ϕ completely in ways different from that of E.Hayashi. Moreover using the result we describe extremal kernels and extremal functions for $1 < p < \infty$.

Theorem 6. Suppose $p = 1$ and S_ϕ is nonempty. Then there exist an inner function q and a strong outer function g which satisfy the following (1) ~ (4).

- (1) The unique extremal kernel of T_ϕ^1 is $\bar{q}|g|/g$.
- (2) f is a member of S_ϕ if and only if

$$f = \gamma q_0 \left(\frac{s + q\bar{s}}{1 + q_0} \right)^2 g,$$

where γ is a positive constant, $\|f\|_1 = 1$, q_0 is an inner function, s is in $H^2 \ominus qzH^2$ and $(s + q\bar{s})/(1 + q_0)$ is an outer function in H^2 .

- (3) f is a member of S_ϕ if and only if

$$f = \gamma q_0 (t + \overline{q_0 t})^2 g,$$

where γ is a positive constant, $\|f\|_1 = 1$, q_0 is an inner function, t is in $H^2 \ominus qzH^2$ and $t + \overline{q_0 t}$ is an outer function in $H^2 \ominus qzH^2$.

- (4) f is a member of S_ϕ if and only if

$$f = \gamma \{(s + q\bar{s})^2 + (t + \overline{q_0 t})^2\} g,$$

where γ is a positive constant, $\|f\|_1 = 1$, and s and t are in $H^2 \ominus qzH^2$.

Proof. (1) is known from [5]. (2) If $f = \gamma q_0 (s + q\bar{s}/1 + q_0)^2 g$, then

$$\frac{|f|}{f} = q_0 \frac{|1 + q_0|^2 |s + q\bar{s}|^2 |g|}{(1 + q_0)^2 (s + q\bar{s})^2 g} = \bar{q} \frac{|g|}{g},$$

and hence $f \in S_\phi$. Conversely, if $f \in S_\phi$ and $f = q_0 h^2$, where q_0 is inner and h is outer, then $\gamma_1 (1 + q_0)^2 h^2 \in S_\phi$ for some positive constant γ_1 . Since $(1 + q_0)h$ is outer in H^2 , by a theorem of E.Hayashi ([5],[6]),

$$H^2 \cap q_0 (h/\bar{h}) \bar{H}^2 = q_0 (H^2 \ominus qzH^2)$$

and $q_0 (h/\bar{h}) = \bar{q} \bar{q}_0 / q_0$, where q is inner and $g = g_0^2$, is strongly outer. Hence $(1 + q_0)h = kg_0$ where $k \in H^2 \ominus qzH^2$ and $\bar{q} k^2 \geq 0$. Since $k \in \mathcal{A}_q$, by Corollary 4, $k = s + q\bar{s}$ for some function $\bar{s} \in H^2 \ominus qzH^2$. Now $q_0 h$ belongs to $g_0 (H^2 \ominus qzH^2)$ because $q_0 h = q_0 (k/\bar{h}) \bar{h}$. Therefore $q_0 h / g_0$ belongs to $H^2 \ominus qzH^2$ and hence $h/g_0 = (s + q\bar{s})/(1 + q_0)$ belongs to $N_+ \cap L^2 = H^2$. This implies (2).

- (3) Put $(s + q\bar{s})/(1 + q_0) = \ell$ in (2); then

$$\frac{\bar{\ell}}{\ell} = \frac{\bar{s} + \bar{q}s}{1 + \bar{q}_0} \frac{1 + q_0}{s + q\bar{s}} = q_0\bar{q}.$$

Hence $\ell = q\bar{q}_0\bar{\ell}$ and so $\ell = t + q\bar{q}_0\bar{t}$, where $t = \ell/2 \in H^2$. This implies (3). (4) By (2), the 'if' part is clear. Conversely if $f \in S_\phi$, then by (2) $f = q_0k^2g$, where $k = \gamma^{1/2}(s + q\bar{s})/(1 + q_0)$. Since $\bar{q}q_0k^2 = |k|^2$, $q\bar{k} = q_0k$ and hence $k \in H^2 \ominus qzH^2$. By Corollary 4, $k = \ell + im$ for some functions $\ell, m \in \mathcal{A}_q$ and hence $q_1k = \ell - im$ for some inner function q_1 . Thus $q_1k^2 = \ell^2 + m^2$ and hence $\bar{q}q_1k^2 = |k|^2$. Therefore $q_1 = q_0$. Corollary 4 implies (4) because $f = \gamma\{\ell^2 + m^2\}g$.

If $(s + q\bar{s})/(1 + q_0)$ belongs to H^2 , then $q_0 \prec q$ and $(s + q\bar{s})/(1 + q_0)$ belongs to $H^2 \ominus qzH^2$. In fact, if $\ell = (s + q\bar{s})/(1 + q_0)$, then by the proof of (3) of Theorem 6, $q\bar{\ell} = q_0\ell$. Hence ℓ belongs to $H^2 \ominus qzH^2$ and $q_0 \prec q$ because $\bar{q}_0q = \ell^2/|\ell|^2$. Theorem 7 and Theorem 1 in [13] describe extremal kernels and extremal functions in case $1 < p < \infty$.

Theorem 7. Suppose $1 < p < \infty$ and $1/p + 1/\ell = 1$. Then ϕ is the unique extremal kernel and f is the unique extremal function of T_ϕ^p if and only if there exist an inner function q and a strong outer function g which satisfy the following :

$$\phi = \|T_\phi^p\| \bar{q} \frac{|g|}{g} \left(\frac{s + q\bar{s}}{1 + q_0} \right)^{2/\ell} g^{1/\ell}$$

and

$$f = q_0 \left(\frac{s + q\bar{s}}{1 + q_0} \right)^{2/p} g^{1/p},$$

where q_0 is an inner function, $\|f\|_p = 1$, $\|\phi\|_\ell = \|T_\phi^p\|$, $s \in H^2 \ominus qzH^2$ and $(s + q\bar{s})/(1 + q_0)$ is an outer function in H^2 .

Proof. If ϕ is the unique extremal kernel and f is the unique extremal function of T_ϕ^p , then by [13, Theorem 1]

$$\phi = \phi_0 h, f = \|T_\phi^\ell\|^{-\ell/p} Q h^{\ell/p}$$

and

$$\|T_\phi^\ell\|^{-\ell} Q h^\ell \in S_{\phi_0}, \phi_0 = \bar{Q}|h|^\ell h^{-\ell},$$

where h is outer with $|\phi| = |h|$ and Q is the inner part of f . By Theorem 6,

$$\|T_\phi^\ell\|^{-\ell} Q h^\ell = q_0 \left(\frac{s + q\bar{s}}{1 + q_0} \right)^2 g,$$

where q and q_0 are inner, g is strongly outer, $\|q_0 \left(\frac{s + q\bar{s}}{1 + q_0} \right)^2 g\|_1 = 1$, $s \in H^2 \ominus qzH^2$ and $(s + q\bar{s})/(1 + q_0)$ is outer in H^2 . Hence $Q = q_0$, $h = \|T_\phi^\ell\| \left(\frac{s + q\bar{s}}{1 + q_0} \right)^{2/\ell} g^{1/\ell}$ and $\phi_0 = \bar{q}_0 \frac{|h|^\ell}{h^\ell} = \bar{q} \frac{|g|}{g}$.

Thus

$$\phi = \bar{q} \frac{|g|}{g} \|T_\phi^\ell\| \left(\frac{s + q\bar{s}}{1 + q_0} \right)^{2/\ell} g^{1/\ell}$$

and

$$f = \|T_\phi^\ell\|^{-\frac{1}{p}} q_0 h^{\frac{1}{p}} = q_0 \left(\frac{s + q\bar{s}}{1 + q_0} \right)^{2/p} g^{1/p}.$$

Theorem 6 is a generalization of [11, Theorem 2]. Theorem 7 is a generalization of [13, Theorem 2]. But the descriptions are different from the previous ones. In those descriptions, the bad part $q_0(s + q\bar{s}/1 + q_0)^{2/\ell}$ is important. If f is an inner function, then it is clear that $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$ for $1 \leq \ell \leq \infty$. If $f = q_0(s + q\bar{s}/1 + q_0)^{2/\ell}$, then, by Theorem 8, $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$ for $1 \leq \ell \leq \infty$. Theorem 8 also shows [13, Corollary 3]. To prove Theorem 8 we need the following lemma.

Lemma. Suppose $1 \leq \ell \leq \infty$ and $f = qh$ is in H^ℓ , where q is an inner function and h is an outer function. Then $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if $qh^{2-\ell}/|h|^{2-\ell}$ is an inner function.

Proof. For $\ell \neq 1$ the lemma is known [13, Corollary 2]. Suppose $\ell = 1$. By [3, p133], if $\|f + \bar{z}\bar{H}^1\| = \|f\|_1$, then there exists an extremal function $Q \in H^\infty$ and $|Q| = 1$ a.e. on $\{\theta; f(e^{i\theta}) \neq 0\}$ and $Q\bar{f} \geq 0$ a.e. on ∂U . Hence Q is inner and so $f/|f|$ is inner. The converse is clear.

Theorem 8. Suppose $1 \leq \ell \leq \infty$ and f is a nonzero function in H^ℓ .

- (1) $\|f + \bar{z}\bar{H}^2\| = \|f\|_2$ for an arbitrary function f in H^2 .
- (2) For $2 < \ell < \infty$, $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if

$$f = q \left(\frac{s + q\bar{s}}{1 + Q} \right)^{2/\ell-2},$$

where q and Q are inner functions with $Q \prec q$.

- (3) For $1 \leq \ell < 2$, $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if

$$f = q \left(\frac{s + Q\bar{s}}{1 + q} \right)^{2/2-\ell},$$

where q and Q are inner functions with $q \prec Q$.

(4) Suppose $\ell = \infty$ and $S_{\bar{f}}$ is nonempty. Then $\|f + \bar{z}\bar{H}^\infty\| = \|f\|_\infty$ if and only if f is an inner function.

Proof. (1) is clear because f is orthogonal to $\bar{z}\bar{H}^2$. Suppose $f = qh$ where q is inner and h is outer. (2) If $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$, then by Lemma $qh^{2-\ell}/|h|^{2-\ell} = Q$ is inner. Hence $\bar{q}Qh^{\ell-2} = |h|^{\ell-2}$. If $1 < t < \ell/\ell - 2$, then $h^{\ell-2} \in H^t$ and so $h^{\ell-2} \in H^1$. Now Theorem 6 implies that

$$Qh^{\ell-2} = Q \left(\frac{s + q\bar{s}}{1 + Q} \right)^2 \text{ and } Q \prec q.$$

Hence $h = (s + q\bar{s}/1 + Q)^{2/\ell-2}$ and so $f = q(s + q\bar{s}/1 + Q)^{2/\ell-2}$. Conversely if $f = q(s + q\bar{s}/1 + Q)^{2/\ell-2}$, then $h = (s + q\bar{s}/1 + Q)^{2/\ell-2}$ and hence

$$\bar{q} \frac{h^{\ell-2}}{|h|^{\ell-2}} = \bar{q} \frac{(s + q\bar{s})^2 |1 + Q|^2}{(1 + Q)^2 |s + q\bar{s}|^2} = Q.$$

The lemma implies $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$.

(3) If $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$, then by the lemma $qh^{2-\ell}/|h|^{2-\ell} = Q$ is inner. Hence $\bar{Q}qh^{2-\ell} = |h|^{2-\ell}$ and $h^{2-\ell} \in H^1$ because $h^\ell \in H^1$ and $\ell > 2 - \ell > 0$. Again by Theorem 6

$$qh^{2-\ell} = q \left(\frac{s + Q\bar{s}}{1 + q} \right)^2 \text{ and } q \prec Q.$$

Hence $h = (s + Q\bar{s}/1 + q)^{2/2-\ell}$ and so $f = q(s + Q\bar{s}/1 + q)^{2/2-\ell}$. Conversely if $f = q(s + Q\bar{s}/1 + q)^{2/2-\ell}$ and hence

$$\bar{q} \frac{h^{\ell-2}}{|h|^{\ell-2}} = \bar{q} \frac{(s + Q\bar{s})^2 |1 + q|^2}{(1 + q)^2 |s + Q\bar{s}|^2} = Q.$$

The lemma implies $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$. (4) If $S_{\bar{f}}$ is nonempty and $\|f + \bar{z}\bar{H}^\infty\| = \|f\|_\infty$, then f is inner by Theorem 6.

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