Factorizations Of Outer Functions And Extremal Problems

Takahiko Nakazi

Series ♯287. March 1995

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 263 Y. Giga, M. E. Gurtin, A comparison theorem for crystalline evolution in the plane, 14 pages. 1994.

♯ 264 J. Wierzbicki, On Commutativity of Diagrams of Type II_1 Factors, 26 pages. 1994.

♯ 265 N. Hayashi, T. Ozawa, Schrödinger Equations with nonlinearity of integral type, 12 pages. 1994.

♯ 266 T. Ozawa, On the resonance equations of long and short waves, 8 pages. 1994.

♯ 267 T. Mikami, A sufficient condition for the uniqueness of solutions to a class of integro-differential equations, 9 pages. 1994.

♯ 268 Y. Giga, Evolving curves with boundary conditions, 10 pages. 1994.

♯ 269 A. Arai, Operator-theoretical analysis of representation of a supersymmetry algebra in Hilbert space, 12 pages. 1994.

♯ 270 A. Arai, Gauge theory on a non-simply-connected domain and representations of canonical commutation relations, 18 pages. 1994.

♯ 271 S. Jimbo, Y. Morita and J. Zhai, Ginzburg landau equation and stable steady state solutions in a non-trivial domain, 17 pages. 1994.

♯ 272 S. Izumiya, A. Takiyama, A time-like surface in Minkowski 3-space which contains light-like lines, 7 pages. 1994.

♯ 273 K. Tsutaya, Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations, 11 pages. 1994.

♯ 274 H. Kubo, On the critical decay and power for semilinear wave equations in odd space dimensions, 22 pages. 1994.

♯ 275 N. Terai, T. Hibi, Alexander duality theorem and second Betti numbers of Stanley-Reisner rings, 2 pages. 1995.

♯ 276 N. Terai, T. Hibi, Stanley-Reisner rings whose Betti numbers are independent of the base field, 12 pages. 1995.

♯ 277 N. Terai, T. Hibi, Computation of Betti numbers of monomial ideals associated with cyclic polytopes, 11 pages. 1995.

♯ 278 N. Terai, T. Hibi, Computation of Betti numbers of monomial ideals associated with stacked polytopes, 8 pages. 1995.

♯ 279 N. Terai, T. Hibi, Finite free resolutions and 1-skeletons of simplicial$(d-1)$-spheres, 3 pages. 1995.

♯ 280 N. Terai, T. Hibi, Monomial ideals and minimal non-faces of Cohen-Macaulay complexes, 6 pages. 1995.

♯ 281 A. Arai, N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed d'Alembertians and the external field problem in quantum field theory, 44 pages. 1995.

♯ 282 T. Mikami, Asymptotic behavior of the first exit time of randomly perturbed dynamical systems with a repulsive equilibrium point, 29 pages. 1995.

♯ 283 K. Iwata, J. Schäfer, Markov property and cokernels of local operators, 17 pages. 1995.

♯ 284 T. Nakazi, M. Yamada, Riesz's Functions In Weighted Hardy And Bergman Spaces, 20 pages. 1995.

♯ 285 K. Hidano, K. Tsutaya, Scattering theory for nonlinear wave equations in the invariant Sobolev space, 32 pages. 1995.

♯ 286 A. Arai, Strong coupling limit of the zero-energy-state density of the Dirac-Weyl operator with a singular vector potential, 8 pages. 1995.

Factorizations Of Outer Functions And Extremal Problems

By

Takahiko Nakazi*

Department of Mathematics

Faculty of Science

Hokkaido University

Sapporo 060, Japan

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

1980 Mathematical subject classification (1985 Revision). Primary 30 D 55, 46 J 15 : Secondary 47 B 35.

Key words and phrases : inner function, outer function, exposed point, Hardy space, extremal problem.

Abstract. The author has proved that an outer function in the Hardy space H^1 can be factored into a product in which one factor is strongly outer and the other is the sum of two inner functions. In an endeavor to understand better the latter factor, we introduce a class of functions containing sums of inner functions as a special case . Using it, we describe the solutions of extremal problems in the Hardy spaces H^p for $1 \leq p < \infty$.

§1. Introduction

N, N_+ and H^p for $1 \leq p \leq \infty$ denote the Nevanlinna class, the Smirnov class and the Hardy space, respectively on the open unit disc U in the complex plane. A function h in N_+ is called outer if it is not divisible in N_+ by a non-constant inner function. A function g in H^1 is called strongly outer if the only functions f in H^1 such that f/g is non-negative are scalar multiples of g. If g is not outer and so $g = qh$ for some inner q, then $f = (1+q)^2h$ belongs to H^1 and $f/g = (1+q)^2/q$ is non-negative. A norm one function in H^1 is outer if and only if it is an extreme point of the unit ball of H^1 [2]. On the other hand, a norm one function in H^1 is strongly outer if and only if it is an exposed point of the unit ball of H^1 ($cf.$[2],[12]). Like outer functions, strongly outer functions appear in many important areas, for example, function theory, operator theory and prediction theory.

It is not difficult to give a characterization of a strongly outer function similar to the above definition of an outer function. If g is divisible in H^1 by a sum of two inner functions q_1, q_2 where $q_1 + q_2$ is not constant and $Im\bar{q}_1 q_2 \leq 0$ almost everywhere, then $f = -i(q_1 - q_2)g/(q_1 + q_2)$ is not a scalar multiple of g and f/g is non-negative because $-i(q_1 - q_2)/(q_1 + q_2) \geq 0$ almost everywhere. Thus g is not strongly outer. The converse is also true by the following factorization theorem [12].

Theorem. If an outer function h in H^1 is not strongly outer, then $h = (q_1+q_2)g$ where both q_1 and q_2 are inner, $Im\bar{q}_1 q_2 \leq 0$ almost everywhere, $(q_1 - q_2)^{-1}$ is summable and g is strongly outer. If q_1 is a finite Blaschke product of degree n then so is q_2.

The aim of this paper is to gain a better understanding of this theorem and of the sum of two inner functions. The sum of two inner functions appeared in H.Helson's papers [7] and [8]. D.Sarason [15] examined cases in which the sum of two non-constant inner functions is outer. In this paper, we introduce functions in H^2 which have the form ; $k = s + q\bar{s}$ where s is in L^2 and q is inner. If $s = 1$, then $k = 1 + q$. If $s = q_1$ and $q = q_1 q_2$ where q_1 and q_2 are inner, then $k = q_1 + q_2$. If f is the square of H^2 function $s + q\bar{s}$, then put q_1 = the inner part of $f + iq$ and q_2 = the inner part of $f - iq$. Then $Im\bar{q}_1 q_2 \leq 0$, $q_1 + q_2$ is non-constant and f is divisible in H^1 by $q_1 + q_2$. By the remark above the Theorem, f is not strongly outer. The following factorization theorem can be proved easily by a theorem of E.Hayashi ([5], [6]).

Theorem. If an outer function h in H^1 is not strongly outer, then $h = (s+q\bar{s})^2 g$ where q is a non-constant inner function, $s + q\bar{s}$ is in H^2 and g is strongly outer.

Proof. Suppose $h = k^2$ and k is outer in H^2. By a theorem of E.Hayashi ([4],[5]),

$$H^2 \cap (k/\bar{k})\bar{H}^2 = g_0(H^2 \ominus zqH^2)$$

and $k/\bar{k} = \bar{q}\bar{g}_0/g_0$ where q is inner and g_0^2 is strongly outer. Hence $k = \ell g_0$ where $\ell \in H^2 \ominus qzH^2$ and $\bar{q}\ell^2 \geq 0$. Put $s = \ell/2$, then $\ell = s + q\bar{s}$ and $h = \ell^2 g_0^2$.

In this theorem, we should like to be able to choose $s + q\bar{s} = q_1 + q_2$ for some inner functions q_1 and q_2. Unfortunately we could not do except in some special cases [12]. Note that by an example of J.Inoue [9]. we cannot choose $s + q\bar{s} = 1 + q$.

§2. Bad parts of outer functions

In this section we study a function in H^2 which has the form $s + q\bar{s}$ where s is in L^2 and q is an inner function. If $\prod_{j=1}^{n}(q_j + q_j')$ where q_j and q_j' are inner functions for $1 \leq j \leq n$, then $\prod_{j=1}^{n}(q_j + q_j') = s + q\bar{s}$ for $q = \prod_{j=1}^{n} q_j q_j'$. Two natural questions are the following : (1) When is $s + q\bar{s}$ an outer function ? (2) When can $s + q\bar{s}$ be divisible in H^2 by $1 + q'$ where q' denotes some nonconstant inner function ? The question (1) is related with a paper of D.Sarason [15]. He studied it when $s + q\bar{s}$ is a sum of two inner functions. The question (2) is related with a paper of J.Inoue [9]. By the second theorem in the Introduction, Inoue's result is the following : There exists an outer function f in H^2 which is not divisible in H^2 by any nonconstant $1 + q'$ but is divisible in H^2 by some nonconstant $s + q\bar{s}$, where q and q' are inner functions. Because of the first theorem in the Introduction, we are also interested in nonconstant outer function $q_1 + q_2$ such that both q_1 and q_2 are inner functions, $Im\bar{q}_1 q_2 \leq 0$ almost everywhere and $(q_1 - q_2)^{-1}$ is summable.

Proposition 1. Suppose s is a nonnegative function in N_+ and s^{-1} is summable. If $i - s = q_1\ell$ where q_1 is an inner function and ℓ is an outer function, then $q_2 = (i + s/i - s)q_1$ is an inner function, $q_1 + q_2$ is an outer function, $Im\bar{q}_1 q_2 \leq 0$ almost everywhere and $(q_1 - q_2)^{-1}$ is summable. If s is a rational function, then both q_1 and q_2 are finite Blaschke products of the same degree.

Proof. Since $|q_2| = 1$ a.e. on ∂U and $q_2 = (i + s)/\ell$, q_2 is inner. Since $q_1 + q_2 = 2i\ell$, $q_1 + q_2$ is outer. By a simple calculation,

$$\frac{-Im\bar{q}_1 q_2}{|q_1 + q_2|^2} = \frac{-i(q_1 - q_2)}{q_1 + q_2} = s \geq 0 \qquad \text{a.e.}$$

and so $Im\bar{q}_1 q_2 \leq 0$ a.e. on ∂U. Since $(q_1 - q_2)^{-1} = (i - s)/(-2s)$ and s^{-1} is summable, $(q_1 - q_2)^{-1}$ is summable. If s is a rational function, by [7] the number of zeros of $s - i$ and that of $s + i$ are equal. Hence q_1 and q_2 are finite Blaschke products of the same degree.

In Proposition 1, if $s = -z/(1-z)^2$, then q_1 and q_2 have degree one. However even if q_1 and q_2 have degree one and $q_1 + q_2$ is outer, $Im\bar{q}_1 q_2$ is not necessarily non-negative. In fact, suppose $|a| < 1$ and $|\rho| = 1$. Then, $\rho z + \bar{\rho}\left(\frac{z-a}{1-\bar{a}z}\right)$ is outer if and only if $|Re\rho| \le |a|$, [15]. However $Im\bar{z}\left(\frac{z-a}{1-\bar{a}z}\right)$ is not non-negative on ∂U.

Proposition 2. Suppose $s + q\bar{s}$ is in H^2, where q is an inner function and s is in L^2. Then $s + q\bar{s}$ is an outer function if and only if there exists a function t in L^2 such that $s + (t - q\bar{t})$ is an outer function.

Proof. If $\ell = s + (t - q\bar{t})$ is outer, then $s + q\bar{s} = \ell + q\bar{\ell} \in H^2$ and $q\bar{\ell} \in H^2$. Hence $q\bar{\ell} = q_0 \ell$ for some inner function q_0. Then $s + q\bar{s} = \ell(1 + q\frac{\bar{\ell}}{\ell}) = \ell(1 + q_0)$ and hence $s + q\bar{s}$ is outer. Conversely if $s + q\bar{s} = 2\ell$ is outer, then $q\bar{\ell} = \ell$ and hence $s + q\bar{s} = \ell + q\bar{\ell}$. Let $k = \ell - s$, then $k + q\bar{k} = 0$ and so $k = t - q\bar{t}$, where $t = k/2$. Thus $\ell = s + (t - q\bar{t})$ is outer.

Corollary 1. Suppose $s + q\bar{s}$ is in H^2, where q is an inner function and s is in L^2. If s and q satisfy one of the follwing (1) $\sim$ (3), then $s + q\bar{s}$ is an outer function.

(1) s is an outer function.

(2) $q = q_1 q_2$ and $s = q_1 h$ where q_1 and q_2 are inner functions, h is an outer function and $q_2\bar{h} = \alpha h$ for some complex number α.

(3) $q = q_1 q_2$ and $s = q_1 h$ where $\{q_j\}_{j=1,2,3}$ are inner functions, h is an outer function, $q_2\bar{h} = q_3 h$, and $q_1 + q_3$ is an outer function.

Proof. (1) is clear by Proposition 2 and (2) is a special case of (3). For (3), let $t = (q_3 - q_1)h/4$, then

$$q\bar{t} = \tfrac{1}{4}q(\bar{q}_2 h - \bar{q}_1\bar{h}) = \tfrac{1}{4}(q_1 h - q_2\bar{h}) = \tfrac{1}{4}(q_1 - q_3)h$$

because $q_2\bar{h} = q_3 h$. Hence $t - q\bar{t} = (q_3 - q_1)h/4$ and so $s + (t - q\bar{t}) = (q_3 + q_1)h/2$. This implies (3) because $q_1 + q_3$ is outer.

Proposition 3. Suppose q_1 is an inner function and $s + q\bar{s}$ is a non-zero function in H^2, where q is an inner function and s is in L^2. Then $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$ if and only if there exists a function t in L^2 such that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$. In particular, if $q\bar{s} = q_1 s$ then $s + q\bar{s}$ is divisible by $1 + q_1$.

Proof. If there exists a function t in L^2 such that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$, then $s + q\bar{s} = s + t - q\bar{t} + q(\bar{s} + \bar{t} - \bar{q}t) = s + t - q\bar{t} + q_1(s + t - q\bar{t}) = (s + t - q\bar{t})(1 + q_1)$ and hence $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$. Conversely if $\ell = (s + q\bar{s})/(1 + q_1)$ is in H^2, then

$$\bar{q} = \frac{\bar{s} + \bar{q}s}{s + q\bar{s}} = \frac{\bar{\ell}}{\ell}\frac{1 + \bar{q}_1}{1 + q_1} = \frac{\bar{\ell}}{\ell}\bar{q}_1$$

and hence $q\bar{\ell} = q_1\ell$. If $k = \ell - s$, then $k = t - q\bar{t}$ for some $t \in L^2$ and hence $\ell = s + (t - q\bar{t})$. This implies that $q\{\bar{s} + (\bar{t} - \bar{q}t)\} = q_1\{s + (t - q\bar{t})\}$.

Corollary 2. Suppose $s + q\bar{s}$ is in H^2, where q is an inner function and s is in L^2.

(1) If s is an outer function and $q\bar{s} \neq \alpha s$ for any α in C with $|\alpha| = 1$, then there exists a non-constant inner function q_1 such that $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$.

(2) If h is an outer function, $q\bar{h} = q_1 q_2^2 h$ and $s = q_2 h$ where q_1 and q_2 are inner functions, then $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$.

(3) If q is a finite Blaschke product, then there exists a non-constant finite Blaschke product q_1 such that $s + q\bar{s}$ is divisible in H^2 by $1 + q_1$, or $s + q\bar{s}$ is not an outer function.

Proof. (1) Since s is outer, $q\bar{s} = q_1 s$ for some inner function q_1. By the hypothesis, q_1 is non-constant and hence Proposition 3 implies (1). (2) $q\bar{s} = q_1 q_2 h = q_1 s$ implies (2) by Proposition 3. (3) Since $\bar{q}(s + q\bar{s})^2 \geq 0$ a.e. on ∂U and q is a finite Blaschke product, $(s + q\bar{s})^2 = \prod_{j=1}^{n}(z - a_j)(1 - \bar{a}_j z)\ell^2$, where $|a_j| \leq 1$ $(1 \leq j \leq n)$ and ℓ is outer in H^2 ([2],[11]). Therefore if $s + q\bar{s}$ is outer, then $s + q\bar{s} = \prod_{j=1}^{n}(-\bar{a}_j)^{1/2}(z - a_j)\ell$ and $|a_j| = 1$. Thus $s + q\bar{s}$ is divisible in H^2 by $z - a_j$.

When q_1 and q_2 are inner functions, we write $q_1 \prec q_2$ if there exists a nonzero function f in H^1 such that $\bar{q}_1 q_2 = f/|f|$. If both q_1 and q_2 are finite Blaschke product, then $q_1 \prec q_2$ is equivalent to (degree of q_1) $\leq$ (degree of q_2). For each g in H^1, *sing* g denotes the set of the unit circle on which g cannot be analytically extended.

Proposition 4. If q_1 and q_2 are inner functions and the inner part of $q_1 + q_2$ is q, then $q \prec q_1$ and $q \prec q_2$.

Proof. Let $q_1 + q_2 = qh$, then $|\bar{q}q_1 - h| = |\bar{q}q_2 - h| = 1$. By a theorem of P.Koosis (cf.[4, Chapter 4, Lemma 5.4]), $q \prec q_1$. and $q \prec q_2$.

Corollary 3. Suppose q_1 and q_2 are inner functions and $q_1 + q_2 = qh$ where q is an inner function and h is an outer function.

(1) If q_1 is a finite Blaschke product, then q is also a finite Blaschke product and (degree of q) $\leq$ (degree of q_1).

(2) If $(sing\ q_1) \cap (sing\ q_2)$ is empty, then q is a finite Blaschke product.

(3) Suppose $q_1 = exp\left(-\dfrac{a+z}{a-z}\right)$ and $q_2 = -\alpha\ exp\left(-\dfrac{b+z}{b-z}\right)$, where $|a| = |b| = 1, b = -\bar{a}$ and $|\alpha| = 1$. If $\alpha = 1$, then $q = z$ or q is constant. If $\alpha \neq 1$, then q is always constant, that is, $q_1 + q_2$ is an outer function.

Proof. (1) By Proposition 4, $\bar{q}q_1 = f/|f|$ for some function $f \in H^1$ and hence $\bar{q}_1(qf) \geq 0$ a.e. on ∂U. If q_1 is a finite Blaschke product of degree $m, qf = \prod_{j=1}^{n}(z-a_j)(1-\bar{a}_j z)\ell$ and $n \leq m$ where $|a_j| \leq 1$ $(1 \leq j \leq n)$ and ℓ is strongly outer. Hence q is a finite Blaschke product of degree k and $k \leq n$. (2) By Proposition 4, $\bar{q}q_1 = f/|f|$ for some function $f \in H^1$ and hence $\bar{q}q_1 = g/\bar{g}$ for some outer function $g \in H^2$. Therefore $\bar{q}_1 qg = \bar{g}$ and so $qg \in H^2 \ominus q_1 z H^2$. Hence *sing* $q_1 \supseteq$ *sing* qg and by [10, Lemma 4], *sing* $q_1 \supseteq$ *sing* q. Similarly *sing* $q_2 \supseteq$ *sing* q and by the hypothesis q is a finite Blaschke product. (3) By (2), q is a finite Blaschke product. If $q(x) = 0$ for some point $x \in U$, then $exp\left(-\dfrac{a+x}{a-x}\right) = \alpha\ exp\left(-\dfrac{b+x}{b-x}\right)$ and hence

$$-\frac{a+x}{a-x} = -\frac{b+x}{b-x} + i\rho \text{ and } \rho = t + 2n\pi$$

where n is some integer and $\alpha = e^{it}$. If $\rho = 0$ then $q = z$ because $a \neq b$. Suppose $\rho \neq 0$. Then

$$x^2 - \{(1 - \frac{2i}{\rho})b + (1 + \frac{2i}{\rho})a\}x + ab = 0.$$

If A and B are the solutions of the above quadratic equation, then $AB = ab = -1$ and

$$A + B = (1 + \frac{2i}{\rho})a - \overline{(1 + \frac{2i}{\rho})a}.$$

This implies $|A| = |B| = 1$ and contradicts $|x| < 1$.

(1) of Corollary 3 was proved by D.Sarason [15, Proposition 3]. Our proof is different from his.

§3. Projection.

For each inner function q, we define two operators on L^2

$$L_q(s) = \frac{s + q\bar{s}}{2} \text{ and } L'_q(s) = \frac{s - q\bar{s}}{2}.$$

If $q = 1$, then $L_q(s)$ is the real part of s and $L'_q(s)$ is the imaginary part of s. In general, $|L_q(s)| \leq |s|$ and $|L'_q(s)| \leq |s|$. Hence L_q and L'_q are contractive. L_q and L'_q commute with multiplication operators by real valued functions in L^∞. Moreover on L^2, we have $L_qL_q = L_q$ and $L_qL'_q = 0$ and $L_q + L'_q$ is the identity operator. By results of the last section, we are interested in a function s such that $L_q(s)$ belongs to H^2. Since $q = (1+q)^2/|1+q|^2$, we define $q^{1/2} = (1+q)/|1+q|$. Put

$$\mathcal{A}_q = \{g \in H^2 : \frac{g}{1+q} \text{ is a real valued function}\}.$$

Theorem 5. Let q be a non-contant inner function. Then

$$\{s \in L^2 : L_q(s) \in H^2\} = \mathcal{A}_q + iq^{1/2}L^2_R,$$

where $L^2_R = \{s \in L^2 : s \text{ is a real valued function}\}$. In particular, if $s + q\bar{s}$ belongs to H^2 for some s in L^2, then $s + q\bar{s} = t + q\bar{t}$ for some t in H^2.

Proof. If $g \in \mathcal{A}_q$ then $u = g/(1+q)$ is real and $g = u(1+q)$. Hence $q\bar{g} = g$ and so $L_q(g) = g \in H^2$. If $s = iq^{1/2}u$ and $u \in L^2_R$ then $L_q(s) = 0$. This implies that $\{s \in L^2 : L_q(s) \in H^2\} \supseteq \mathcal{A}_q + iq^{1/2}L^2_R$. Conversely, suppose $g = L_q(s) \in H^2$. If $g = 0$, then $s = -q\bar{s}$ and $s^2 = -q|s|^2$. Hence $(i\bar{q}^{1/2}s)^2 = -\bar{q}s^2 = |s|^2 \geq 0$ and so $i\bar{q}^{1/2}s = -u$ is real. Thus $s = iq^{1/2}u$ and $u \in L^2_R$. If $g \neq 0, s + q\bar{s} = 2g$ and

$$\frac{s}{g} + \overline{\left(\frac{s}{g}\right)} = 2,$$

Put $t = s/g - 1$, then $t + \bar{t} = 0$ and so $t = iv$ for some $v \in L^2_R$. Hence $s = g + ivg$ and $vg = q^{1/2}u$, where $u = v\bar{q}^{1/2}g$ is in L^2_R. Thus $s = g + iq^{1/2}u$. This completes the proof of the proposition.

Corollary 4. Let q be a non-constant inner function. Then

$$\{s \in H^2 : L_q(s) \in H^2\} = \mathcal{A}_q + i\mathcal{A}_q$$

and hence $H^2 \ominus qzH^2 = \mathcal{A}_q + i\mathcal{A}_q$. L_q is the projection from $H^2 \ominus qzH^2$ onto $\mathcal{A}_q$ and has kernel $i\mathcal{A}_q$.

Proof. If $g \in \mathcal{A}_q$, then $g = v(1+q)$ for some real valued function v and so $g = q^{1/2}u$ where $u = v|1+q|$. Hence $\mathcal{A}_q \subset q^{1/2}L^2_R$ and $(q^{1/2}L^2_R) \cap H^2 = \mathcal{A}_q$. Now Theorem 5 implies the corollary.

The proof of Theorem 5 is related to that of [14, Theorem 3]. The equality in Corollary 4, that is, $H^2 \ominus qzH^2 = \mathcal{A}_q + i\mathcal{A}_q$ is known by [12, (1) of Theorem 3].

Corollary 5. Let q be an inner function.

(1) If $q = z^n$, then $\mathcal{A}_q = \{\sum_{j=0}^{n} b_j z^j : b_j = \bar{b}_{n-j}\}$.

(2) If $q = \prod_{\ell=1}^{\infty} \frac{-\bar{a}_\ell}{|a_\ell|} \frac{z - a_\ell}{1 - \bar{a}_\ell z}$ and $\sum_{\ell=1}^{\infty}(1 - |a_\ell|) < \infty$, then

$$\mathcal{A}_q = \{\sum_{j=0}^{\infty} \frac{c_j B_j + \bar{c}_j z B'_j}{1 - \bar{a}_j z} \ : \ \sum_{j=0}^{\infty} \frac{|c_j|^2}{(1-|a_j|)^2(1+|a_j|)} < \infty\}$$

where $B_j = \prod_{\ell=1}^{j-1} \frac{-\bar{a}_j}{|a_j|} \frac{z - a_j}{1 - \bar{a}_j z}, B'_j = \prod_{\ell=j}^{\infty} \frac{-\bar{a}_j}{|a_j|} \frac{z - a_j}{1 - \bar{a}_j z}, a_0 = 0, B_0 = 1$ and $B'_0 = q$.

Proof. (1) If $s \in H^2 \ominus qzH^2$, then $s = \sum_{j=0}^{n} a_j z^j$ and hence $s + q\bar{s} = \sum_{j=0}^{n}(a_j + \bar{a}_{n-j})z^j$. Now Corollary 4 implies (1). (2) If $s \in H^2 \ominus qzH^2$, then by [1]

$$s = \sum_{j=0}^{\infty} c_j (1+|a_j|)^{1/2} B_j (1 - \bar{a}_j z)^{-1} (1 - |a_j|)$$

and $\sum_{j=0}^{\infty} |c_j|^2 < \infty$. Hence

$$\begin{aligned} s + q\bar{s} &= \sum_{j=0}^{\infty} \left(c_j \frac{B_j}{1 - \bar{a}_j z} + \bar{c}_j \frac{q\bar{B}_j}{1 - \bar{a}_j z} \right) (1+|a_j|)^{1/2}(1-|a_j|) \\ &= \sum_{j=0}^{\infty} \left(\frac{c_j B_j + \bar{c}_j z B'_{j+1}}{1 - \bar{a}_j z} \right) (1+|a_j|)^{1/2}(1-|a_j|). \end{aligned}$$

Now Corollary 4 implies (2).

A theorem of P.R.Ahern and D.N.Clark [1, Theorem 3.1] lets one describe $\mathcal{A}_q$ for arbitrary inner function q.

§4. Extremal problems.

Let $1 \le p \le \infty$ and $1/p + 1/\ell = 1$. If $\phi \in L^\ell$, we denote by T^p_ϕ the continuous functional defined on the Hardy space H^p by

$$T^p_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta}) d\theta / 2\pi.$$

A function f in H^p, which satisfies $T^p_\phi(f) = \|T^p_\phi\|$ and $\|f\|_p \le 1$, is called an extremal function. A function ϕ in L^ℓ is called an extremal kernel when $\|\phi\|_\ell = \|T^p_\phi\|$. The existence and uniqueness of extremal functions and extremal kernels is known for $1 < p \le \infty$ (cf.[3, Theorem 8.1]). For $p = 1$, the situation is very different. An extremal function may not exist, the dual extremal kernel always exists and is unique if an extremal function exists (cf.[3, Theorem 8.1]). For $p = 1$, the set S_ϕ of all extremal functions is defined by

$$S_\phi = \{f \in H^1 : T^1_\phi(f) = \|T^1_\phi\| \text{ and } \|f\|_1 = 1\}.$$

S_ϕ has been described in general by E.Hayashi [5],[6]. In this section, we describe S_ϕ completely in ways different from that of E.Hayashi. Moreover using the result we describe extremal kernels and extremal functions for $1 < p < \infty$.

Theorem 6. Suppose $p = 1$ and S_ϕ is nonempty. Then there exist an inner function q and a strong outer function g which satisfy the following (1) ~(4).

(1) The unique extremal kernel of T^1_ϕ is $\bar{q}|g|/g$.

(2) f is a member of S_ϕ if and only if

$$f = \gamma q_0 \left(\frac{s + q\bar{s}}{1 + q_0}\right)^2 g,$$

where γ is a positive constant, $\|f\|_1 = 1, q_0$ is an inner function, s is in $H^2 \ominus qzH^2$ and $(s + q\bar{s})/(1 + q_0)$ is an outer function in H^2.

(3) f is a member of S_ϕ if and only if

$$f = \gamma q_0 (t + q\overline{q_0 t})^2 g,$$

where γ is a positive constant, $\|f\|_1 = 1, q_0$ is an inner function, t is in $H^2 \ominus qzH^2$ and $t + q\overline{q_0 t}$ is an outer function in $H^2 \ominus qzH^2$.

(4) f is a member of S_ϕ if and only if

$$f = \gamma\{(s + q\bar{s})^2 + (t + q\bar{t})^2\}g,$$

where γ is a positive constant, $\|f\|_1 = 1$, and s and t are in $H^2 \ominus qzH^2$.

Proof. (1) is known from [5]. (2) If $f = \gamma q_0(s + q\bar{s}/1 + q_0)^2 g$, then

$$\frac{|f|}{f} = q_0 \frac{|1 + q_0|^2}{(1 + q_0)^2} \frac{|s + q\bar{s}|^2}{(s + q\bar{s})^2} \frac{|g|}{g} = \bar{q}\frac{|g|}{g},$$

and hence $f \in S_\phi$. Conversely, if $f \in S_\phi$ and $f = q_0 h^2$,where q_0 is inner and h is outer, then $\gamma_1(1 + q_0)^2 h^2 \in S_\phi$ for some positive constant γ_1. Since $(1 + q_0)h$ is outer in H^2, by a theorem of E.Hayashi ([5],[6]),

$$H^2 \cap q_0(h/\bar{h})\bar{H}^2 = q_0(H^2 \ominus qzH^2)$$

and $q_0(h/\bar{h}) = \bar{q}\bar{g}_0/g_0$, where q is inner and $g = g_0^2$, is strongly outer. Hence $(1+q_0)h = kg_0$ where $k \in H^2 \ominus qzH^2$ and $\bar{q}k^2 \geq 0$. Since $k \in \mathcal{A}_q$, by Corollary 4, $k = s + q\bar{s}$ for some function $s \in H^2 \ominus qzH^2$. Now $q_0 h$ belongs to $g_0(H^2 \ominus qzH^2)$ because $q_0 h = q_0(k/\bar{h})\bar{h}$. Thefore $q_0 h/g_0$ belongs to $H^2 \ominus qzH^2$ and hence $h/g_0 = (s + q\bar{s})/(1 + q_0)$ belongs to $N_+ \cap L^2 = H^2$. This implies (2).

(3) Put $(s + q\bar{s})/(1 + q_0) = \ell$ in (2) ; then

$$\frac{\bar{\ell}}{\ell} = \frac{\bar{s}+\bar{q}s}{1+\bar{q}_0}\frac{1+q_0}{s+q\bar{s}} = q_0\bar{q}.$$

Hence $\ell = q\overline{q_0\ell}$ and so $\ell = t + q\overline{q_0 t}$, where $t = \ell/2 \in H^2$. This implies (3). (4) By (2), the 'if' part is clear. Conversely if $f \in S_\phi$, then by (2) $f = q_0k^2g$, where $k = \gamma^{1/2}(s+q\bar{s})/(1+q_0)$. Since $\bar{q}q_0k^2 = |k|^2$, $q\bar{k} = q_0k$ and hence $k \in H^2 \ominus qzH^2$. By Corollary 4, $k = \ell + im$ for some functions $\ell, m \in \mathcal{A}_q$ and hence $q_1\bar{k} = \ell - im$ for some inner function q_1. Thus $q_1k^2 = \ell^2 + m^2$ and hence $\bar{q}q_1k^2 = |k|^2$. Therefore $q_1 = q_0$. Corollary 4 implies (4) because $f = \gamma\{\ell^2+m^2\}g$.

If $(s+q\bar{s})/(1+q_0)$ belongs to H^2, then $q_0 \prec q$ and $(s+q\bar{s})/(1+q_0)$ belongs to $H^2 \ominus qzH^2$. In fact, if $\ell = (s+q\bar{s})/(1+q_0)$, then by the proof of (3) of Theorem 6, $q\bar{\ell} = q_0\ell$. Hence ℓ belongs to $H^2 \ominus qzH^2$ and $q_0 \prec q$ because $\bar{q}_0q = \ell^2/|\ell|^2$. Theorem 7 and Theorem 1 in [13] describe extremal kernels and extremal functions in case $1 < p < \infty$.

Theorem 7. Suppose $1 < p < \infty$ and $1/p + 1/\ell = 1$. Then ϕ is the unique extremal kernel and f is the unique extremal function of T_ϕ^p if and only if there exist an inner function q and a strong outer function g which satisfy the following :

$$\phi = \|T_\phi^p\|\bar{q}\frac{|g|}{g}\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/\ell} g^{1/\ell}$$

and

$$f = q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/p} g^{1/p},$$

where q_0 is an inner function, $\|f\|_p = 1$, $\|\phi\|_\ell = \|T_\phi^p\|$, $s \in H^2 \ominus qzH^2$ and $(s+q\bar{s})/(1+q_0)$ is an outer function in H^2.

Proof. If ϕ is the unique extremal kernel and f is the unique extremal function of T_ϕ^p, then by [13, Theorem 1]

$$\phi = \phi_0 h, f = \|T_\phi^\ell\|^{-\ell/p}Qh^{\ell/p}$$

and

$$\|T_\phi^\ell\|^{-\ell}Qh^\ell \in S_{\phi_0},\ \phi_0 = \bar{Q}|h|^\ell h^{-\ell},$$

where h is outer with $|\phi| = |h|$ and Q is the inner part of f. By Theorem 6,

$$\|T_\phi^\ell\|^{-\ell}Qh^\ell = q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^2 g,$$

where q and q_0 are inner, g is strongly outer, $\|q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^2 g\|_1 = 1$, $s \in H^2 \ominus qzH^2$ and $(s+q\bar{s})/(1+q_0)$ is outer in H^2. Hence $Q = q_0, h = \|T_\phi^\ell\|\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/\ell} g^{1/\ell}$ and $\phi_0 = \bar{q}_0\frac{|h|^\ell}{h^\ell} = \bar{q}\frac{|g|}{g}$.

Thus

$$\phi = \bar{q}\frac{|g|}{g}\|T_\phi^\ell\|\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/\ell} g^{1/\ell}$$

and

$$f = \|T_\phi^\ell\|^{-\frac{\ell}{p}} q_0 h^{\frac{\ell}{p}} = q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/p} g^{1/p}.$$

Theorem 6 is a generalization of [11, Theorem 2]. Theorem 7 is a generalization of [13, Theorem 2]. But the descriptions are different from the previous ones. In those descriptions, the bad part $q_0(s+q\bar{s}/1+q_0)^{2/\ell}$ is important. If f is an inner function, then it is clear that $\|f+\bar{z}\bar{H}^\ell\| = \|f\|_\ell$ for $1 \le \ell \le \infty$. If $f = q_0(s+q\bar{s}/1+q_0)^{2/\ell}$, then, by Theorem 8, $\|f+\bar{z}\bar{H}^\ell\| = \|f\|_\ell$ for $1 \le \ell \le \infty$. Theorem 8 also shows [13, Corollary 3]. To prove Theorem 8 we need the following lemma.

Lemma. Suppose $1 \le \ell \le \infty$ and $f = qh$ is in H^ℓ, where q is an inner function and h is an outer function. Then $\|f+\bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if $qh^{2-\ell}/|h|^{2-\ell}$ is an inner function.

Proof. For $\ell \ne 1$ the lemma is known [13, Corollary 2]. Suppose $\ell = 1$. By [3, p133], if $\|f+\bar{z}\bar{H}^1\| = \|f\|_1$, then there exists an extremal function $Q \in H^\infty$ and $|Q| = 1$ a.e. on $\{\theta\ ;\ f(e^{i\theta}) \ne 0\}$ and $Q\bar{f} \ge 0$ a.e. on ∂U. Hence Q is inner and so $f/|f|$ is inner. The converse is clear.

Theorem 8. Suppose $1 \le \ell \le \infty$ and f is a nonzero function in H^ℓ.
(1) $\|f+\bar{z}\bar{H}^2\| = \|f\|_2$ for an arbitrary function f in H^2.
(2) For $2 < \ell < \infty$, $\|f+\bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if

$$f = q\left(\frac{s+q\bar{s}}{1+Q}\right)^{2/\ell-2},$$

where q and Q are inner functions with $Q \prec q$.
(3) For $1 \le \ell < 2$, $\|f+\bar{z}\bar{H}^\ell\| = \|f\|_\ell$ if and only if

$$f = q\left(\frac{s+Q\bar{s}}{1+q}\right)^{2/2-\ell},$$

where q and Q are inner functions with $q \prec Q$.

(4) Suppose $\ell = \infty$ and $S_{\bar{f}}$ is nonempty. Then $\|f + \bar{z}\bar{H}^\infty\| = \|f\|_\infty$ if and only if f is an inner function.

Proof. (1) is clear because f is orthogonal to $\bar{z}\bar{H}^2$. Suppose $f = qh$ where q is inner and h is outer. (2) If $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$, then by Lemma $qh^{2-\ell}/|h|^{2-\ell} = Q$ is inner. Hence $\bar{q}Qh^{\ell-2} = |h^{\ell-2}|$. If $1 < t < \ell/\ell - 2$, then $h^{\ell-2} \in H^t$ and so $h^{\ell-2} \in H^1$. Now Theorem 6 implies that

$$Qh^{\ell-2} = Q\left(\frac{s+q\bar{s}}{1+Q}\right)^2 \text{ and } Q \prec q.$$

Hence $h = (s + q\bar{s}/1 + Q)^{2/\ell-2}$ and so $f = q(s + q\bar{s}/1 + Q)^{2/\ell-2}$. Conversely if $f = q(s + q\bar{s}/1 + Q)^{2/\ell-2}$, then $h = (s + q\bar{s}/1 + Q)^{2/\ell-2}$ and hence

$$\bar{q}\frac{h^{\ell-2}}{|h|^{\ell-2}} = \bar{q}\frac{(s+q\bar{s})^2}{(1+Q)^2}\frac{|1+Q|^2}{|s+q\bar{s}|^2} = Q.$$

The lemma implies $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$.

(3) If $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$, then by the lemma $qh^{2-\ell}/|h|^{2-\ell} = Q$ is inner. Hence $\bar{Q}qh^{2-\ell} = |h|^{2-\ell}$ and $h^{2-\ell} \in H^1$ because $h^\ell \in H^1$ and $\ell > 2 - \ell > 0$. Again by Theorem 6

$$qh^{2-\ell} = q\left(\frac{s+Q\bar{s}}{1+q}\right)^2 \text{ and } q \prec Q.$$

Hence $h = (s + Q\bar{s}/1 + q)^{2/2-\ell}$ and so $f = q(s + Q\bar{s}/1 + q)^{2/2-\ell}$. Conversely if $f = q(s + Q\bar{s}/1 + q)^{2/2-\ell}$ and hence

$$\bar{q}\frac{h^{\ell-2}}{|h|^{\ell-2}} = \bar{q}\frac{(s+Q\bar{s})^2}{(1+q)^2}\frac{|1+q|^2}{|s+Q\bar{s}|^2} = Q.$$

The lemma implies $\|f + \bar{z}\bar{H}^\ell\| = \|f\|_\ell$. (4) If $S_{\bar{f}}$ is nonempty and $\|f + \bar{z}\bar{H}^\infty\| = \|f\|_\infty$, then f is inner by Theorem 6.

References

1. P.R.Ahern and D.N.Clark, On functions orthogonal to invariant subspaces, Acta Math. 124, 191-204.

2. K.deLeeuw and W.Rudin, Extreme points and extremum problems in H^1, Pacific J. Math. 8(1958), 467-485.

3. P.Duren, Theory of H^p Spaces (Academic Press, New York 1970).

4. J.B.Garnett, Bounded Analytic Functions, (Academic Press, 1981).

5. E.Hayashi, The solution sets of extremal problems in H^1, Proc. Amer. Math. Soc. 93(1985), 690-696.

6. E.Hayashi, The kernel of a Toeplitz operator, Integral E_q. Operator Th. 9(1986), 589-591.

7. H.Helson, Large analytic functions, Oper. Theory : Adv. Appl. 43(1989), 209-216.

8. H.Helson, Large analytic functions 2, in Analysis and Partial Differential Equations (C.Sadosky, ed.), Marcel Dekker, Basel, 1990.

9. J.Inoue, An example of a non-exposed extreme function on the unit ball of H^1, Proc. Edinburgh Math. Soc. 37(1993), 47-51.

10. M.Lee and D.Sarason, The spectra of some Toeplitz operators, J. Math. Anal. Appl. 33(1971), 529-543.

11. T.Nakazi, Exposed points and extremal problems in H^1, J. Funct. Anal. 53(1983), 224-230

12. T.Nakazi, Sum of two inner functions and exposed points in H^1, Proc. Edinburgh Math. Soc. 35(1992), 349-357.

13. T.Nakazi, Extremal problems in H^p, J. Austral. Math. Soc. (Series A) 52(1992), 103-110.

14. T.Nakazi and K.Takahashi, Hyponormal Toeplitz operators and extremal problems of Hardy spaces, Trans. Amer. Math. Soc. 338(1993), 753-767.

15. D.Sarason, Making an outer function from two inner functions, in preprint.