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The Rohlin property for automorphisms of UHF algebras

Akitaka Kishimoto

February 1995

Abstract

For an automorphism α of a UHF algebra it is shown that α has the Rohlin property if and only if α^m is uniformly outer for any $m \neq 0$. It is also shown that the automorphisms of a UHF algebra with the Rohlin property are outer conjugate with each other.

1 Introduction

Let α be an automorphism of a unital C^* -algebra A . First we recall some definitions [13, 6, 17, 15].

Definition 1.1 The automorphism α has the Rohlin property if for any $k \in \mathbb{N}$ there are positive integers $k_1, \dots, k_m \geq k$ satisfying the following condition: For any finite subset F of A and $\epsilon > 0$ there are projections $e_{i,j}$; $i = 1, \dots, m$, $j = 0, \dots, k_i - 1$ in A such that

$$\begin{aligned} \sum_{i=1}^m \sum_{j=0}^{k_i-1} e_{i,j} &= 1, \\ \|\alpha(e_{i,j}) - e_{i,j+1}\| &< \epsilon, \\ \|[x, e_{i,j}]\| &< \epsilon \end{aligned}$$

for $i = 1, \dots, m$, $j = 0, \dots, k_i - 1$, and $x \in F$ where $e_{i,k_i} = e_{i,0}$.

Definition 1.2 The automorphism α is uniformly outer if for any $a \in A$, any projection $p \in A$, and any $\epsilon > 0$, there are a finite number of projections p_1, \dots, p_n in A such that

$$\begin{aligned} p &= \sum_{i=1}^n p_i, \\ \|p_i a \alpha(p_i)\| &< \epsilon, \quad i = 1, \dots, n. \end{aligned}$$

If α has the Rohlin property then α^m is uniformly outer for any $m \neq 0$. The converse is shown for Cuntz algebras $A = O_n$ in [15]. In this note we shall show the converse for UHF algebras (and for certain unital simple AF algebras). This generalizes the results on shifts on UHF algebras M_{n^∞} [6, 15] and on quasi-free automorphisms of the CAR algebras [4]. Note that if A is UHF, the property that α^m is uniformly outer for any $m \neq 0$ is equivalent to that the crossed product $A \times_\alpha \mathbf{Z}$ has a unique tracial state [15]. To be more precise, the former condition implies, by [15], Lemma 4.3, that any tracial state of $A \times_\alpha \mathbf{Z}$ is invariant under the dual action $\hat{\alpha}$, and hence that $A \times_\alpha \mathbf{Z}$ has a unique tracial state; which implies in turn, by (the proof of) [15], Lemma 4.4, that $\bar{\alpha}^m$ is outer for any $m \neq 0$, where $\bar{\alpha}$ denotes the extension of α to an automorphism of the weak closure of A in the tracial representation; which implies, by [15], Theorem 4.5, the condition on α we started with.

If an automorphism α of a UHF algebra A has the Rohlin property, then for any $\epsilon > 0$ there is a unitary $u \in A$ such that $\|u - 1\| < \epsilon$ and $\text{Ad } u \circ \alpha$ is of infinite tensor product type [18, 4]. (This follows from the proof of [4], Theorem 1.7, which asserts the same conclusion for certain UHF algebras based on a special form of Rohlin property. Adopting the present form of Rohlin property as suggested by Rørdam [17], the conclusion holds for any UHF algebras.) Furthermore $A \times_\alpha \mathbf{Z}$ is approximately divisible [3] and is isomorphic to a unique simple AT algebra independent of α (as far as α has the Rohlin property) [4]. ($K_*(A \times_\alpha \mathbf{Z}) \cong K_0(A)$ by the Pimsner-Voiculescu exact sequence [1]; $A \times_\alpha \mathbf{Z}$ is an inductive limit of C^* -algebras of the form $C(\mathbf{T}) \otimes M_n$, called an AT algebra; $A \times_\alpha \mathbf{Z}$ has real rank zero [2]: These properties imply the above assertion by Elliott's theory [12].) Note that if we just assume that any non-zero power of α is outer (instead of uniformly outer), we cannot expect in general the closeness, in norm, of α to an automorphism of infinite tensor product type (cf. [8]). Summing up such results we obtain the following result; the corresponding result for quasi-free automorphisms of the CAR algebra is given in [4], Theorem 1.1:

Theorem 1.3 *Let A be a UHF algebra, i.e., the inductive limit of an increasing sequence $\{M_{n_1} \otimes \cdots \otimes M_{n_k}\}$ of full matrix algebras with natural embeddings. If α is an automorphism of A , the following conditions are equivalent:*

1. α has the Rohlin property.
2. α^m is uniformly outer for any $m \neq 0$.
3. $A \times_\alpha \mathbf{Z}$ has a unique tracial state.
4. $A \times_\alpha \mathbf{Z}$ has real rank zero.
5. There exists a unitary $u \in A$ such that $(A, \text{Ad } u \circ \alpha)$ is isomorphic to

$$\left(\bigotimes_{k=1}^{\infty} M_{m_k}, \bigotimes_{k=1}^{\infty} \text{Ad } u_k \right),$$

where u_k is a unitary of M_{m_k} and the eigenvalues of $\bigotimes_{k=1}^{\infty} u_k$ are uniformly distributed for any $l \in \mathbf{N}$.

6. $A \times_\alpha \mathbf{Z}$ is the inductive limit of an increasing sequence $\{B_k\}$ of C^* -algebras such that

$$B_k \equiv C(\mathbf{T}) \otimes M_{n_1} \otimes \cdots \otimes M_{n_k} \cong M_{n_k}(B_{k-1})$$

and the embedding of B_k into B_{k+1} is given by

$$f \mapsto f \oplus \beta_\theta(f) \oplus \beta_{2\theta}(f) \oplus \cdots \oplus \beta_{(n_{k+1}-1)\theta}(f),$$

where θ is irrational and $\beta_\theta(f)(t) = f(t - \theta)$ for $f \in B_k \cong C(\mathbf{T}, M_{n_1} \otimes \cdots \otimes M_{n_k})$.

When (1) holds, (6) is obtained by choosing a special α by using the uniqueness of $A \times_\alpha \mathbf{Z}$ mentioned above; (6) implies (3) (by an easy computation) and (4) (by [2]). See Section 5 for details of (5); (5) implies (1) by the condition of uniform distribution (Lemma 5.2). We have noted before the theorem that (5) \Leftrightarrow (1) \Rightarrow (2) \Leftrightarrow (3). We shall show that (4) implies (3). A C^* -algebra has real rank zero if and only if the self-adjoint elements with finite spectra are dense in the set of all self-adjoint elements [7]. Hence if (4) holds, the linear span of projections is dense in $A \times_\alpha \mathbf{Z}$ and so any tracial state on $A \times_\alpha \mathbf{Z}$ is invariant under the dual action of $\hat{\mathbf{Z}} \cong \mathbf{T}$ which is connected. Since A has only one tracial state, this shows that (4) implies (3). Sections 2-4 are devoted to the proof of the remaining implication (2) \Rightarrow (1).

In Section 2 we give a simple main technical lemma concerning the shift automorphism on the compact operators on $l^2(\mathbf{Z})$. This asserts that this automorphism has a kind of Rohlin property. In Section 3 using Connes' result [9] on the Rohlin property for automorphisms of the injective type II_1 factor, we show that we can *simulate* a system as in Section 2 in certain systems (A, α) if the automorphism α satisfies the property that α^m is uniformly outer for any $m \neq 0$. If the C^* -algebra A is UHF, this simulation can be done in the relative commutant of any finite-dimensional subalgebra. Combining these results we conclude, in Section 4, that α has the *approximate* Rohlin property [5]. Using the methods given in [15] we then conclude that α has the Rohlin property.

In Section 5 we shall show (cf. [14]):

Theorem 1.4 *Let α and β be automorphisms of a UHF algebra A . If α and β have the Rohlin property, then for any $\epsilon > 0$ there exists an automorphism γ of A such that $\|\alpha - \gamma \circ \beta \circ \gamma^{-1}\| < \epsilon$.*

The conclusion of this theorem is equivalent to saying that α and β are outer conjugate. Since an automorphism of a unital simple C^* -algebra is inner if it is close to the identity in norm, the outer conjugacy follows from the conclusion. If $\text{Ad } u \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}$ and $\|u - v^* \alpha(v)\| < \epsilon$ for some unitary $v \in A$, then $\|\alpha - \text{Ad } v \circ \gamma \circ \beta \circ \gamma^{-1} \circ \text{Ad } v^*\| < 2\epsilon$ follows; the existence of the v above follows from the Rohlin property by [13]. Hence the uniqueness of $A \times_\alpha \mathbf{Z}$ asserted by using Elliott's result [12] is also a consequence of this theorem.

In Section 6 we shall show by similar methods that an automorphism α of a simple non-commutative torus which comes from the gauge action has the Rohlin property if any non-zero power of α is outer. Since the crossed product by α depends on α in general [11], a result like the above theorem cannot hold in this case.

2 Shift on $l^2(\mathbf{Z})$

Let $\{E_{ij}; i, j \in \mathbf{Z}\}$ be matrix units:

$$(E_{ij})^* = E_{ji}, \quad E_{ij}E_{kl} = \delta_{jk}E_{il}.$$

Let K be the closed linear span of these E_{ij} with unique C^* -norm; K is the compact operators on $l^2(\mathbf{Z})$ by identifying E_{ii} with the one-dimensional projection onto the functions supported by $\{i\} \subset \mathbf{Z}$. Define an automorphism σ of K by $\sigma(E_{i,j}) = E_{i+1,j+1}$. For any $N \in \mathbf{N}$ let $P_N = \sum_{i=0}^{N-1} E_{ii}$.

Lemma 2.1 *Let K, σ, P_N , etc be as above. For any $\epsilon > 0$ and $n \in \mathbf{N}$ there exist $N \in \mathbf{N}$ and projections e_0, e_1, \dots, e_{n-1} in K such that*

$$\begin{aligned} \sum_{i=0}^{n-1} e_i &\leq P_N, \\ \|\sigma(e_i) - e_{i+1}\| &< \epsilon, \quad i = 0, \dots, n-1, \\ \frac{n \dim e_0}{N} &> 1 - \epsilon, \end{aligned}$$

where $e_n = e_0$.

Proof. We shall choose $k, l \in \mathbf{N}$ such that $1 \ll k \ll l$. Define

$$\begin{aligned} f = & \sum_{m=1}^{k-1} \left\{ \frac{m}{k} E_{nm, nm} + \frac{k-m}{k} E_{n(m+k+l), n(m+k+l)} \right. \\ & + \frac{\sqrt{m(k-m)}}{k} E_{nm, n(m+k+l)} + \frac{\sqrt{m(k-m)}}{k} E_{n(m+k+l), nm} \left. \right\} \\ & + \sum_{m=k}^{k+l} E_{nm, nm}. \end{aligned}$$

Then f is a projection such that

$$f \leq \sum_{m=1}^{2k+l-1} E_{nm, nm}$$

and the dimension of f is equal to $k+l$. For $m = 0, \dots, n-1$ let $e_m = \sigma^{m-n}(f)$. Let $N = n(2k+l-1)$. Then $\{e_m; m = 0, \dots, n-1\}$ are mutually orthogonal projections of K such that $e_0 + \dots + e_{n-1} \leq P_N$. Since $\sigma(e_{n-1}) - e_0 = f - e_0$ is equal to

$$\begin{aligned} & -\frac{1}{k} \sum_{m=0}^{k-1} E_{nm, nm} + \frac{1}{k} \sum_{m=0}^{k-1} E_{n(m+k+l), n(m+k+l)} \\ & + \sum_{m=0}^{k-1} \frac{1}{k} [\sqrt{m(k-m)} - \sqrt{(m+1)(k-m-1)}] \{ E_{nm, n(m+k+l)} + E_{n(m+k+l), nm} \} \end{aligned}$$

and for $m = 0, \dots, k-1$

$$|\sqrt{m(k-m)} - \sqrt{(m+1)(k-m-1)}| < \sqrt{k},$$

we have that

$$\|\sigma(e_{n-1}) - e_0\| < \frac{1}{k} + \frac{1}{\sqrt{k}}.$$

Hence if k is so large that $k^{-1} + k^{-1/2} < \epsilon$, then we have that $\|\sigma(e_i) - e_{i+1}\| < \epsilon$ for $i = 0, \dots, n-1$ with $e_n = e_0$. Since

$$\frac{n \dim e_0}{N} = \frac{k+l}{2k+l-1},$$

if l is so large that

$$\epsilon > \frac{k-1}{2k+l-1}$$

then we have that

$$\frac{n \dim e_0}{N} > 1 - \epsilon.$$

This concludes the proof.

3 Simulation

We consider a certain class of AF algebras which includes UHF algebras. Let A be a unital infinite-dimensional simple AF algebra. Note that $K_0(A)$ is of finite rank r and has no infinitesimal elements if and only if $K_0(A)$ is order isomorphic to a dense subgroup of \mathbf{R}^d for some $d \leq r$, provided with the relative strict order [10]. (The strict order on \mathbf{R}^d is defined as follows: $a \leq b$ if $a = b$ or $a_i < b_i$ for $i = 1, \dots, d$.) In this case A has d extreme tracial states.

Lemma 3.1 *Let A be a unital simple AF algebra such that $K_0(A)$ is of finite rank and has no infinitesimal elements. Let α be an automorphism of A such that α^m is uniformly outer for any $m \neq 0$. For any $m \in \mathbf{N}$, $\epsilon > 0$, and any finite-dimensional C^* -subalgebra B of A there exists an orthogonal family $\{e_i; i = 0, \dots, m\}$ of projections in $A \cap B'$ such that*

$$\begin{aligned} \|\alpha(e_i) - e_{i+1}\| &< \epsilon, \quad i = 0, \dots, m-1, \\ [e_0] &= [e_1] = \dots = [e_m], \\ m[1] &\leq (m+k)(m+1)[e_0] \end{aligned}$$

where $[\cdot]$ denotes the equivalence class in $K_0(A \cap B')$ and k is the period of α_* on $K_0(A)$.

Proof. Let $\{\tau_1, \dots, \tau_d\}$ be the set of extreme tracial states of A . For each $i = 1, \dots, d$ let k_i be the minimal positive integer such that $\tau_i \circ \alpha^{k_i} = \tau_i$ and let k be the least common multiple of k_1, \dots, k_d . Then k is the minimal positive integer such that $\tau \circ \alpha^k = \tau$ for any tracial state τ on A or equivalently $\alpha_*^k = id$ on $K_0(A)$.

Let $\rho = \bigoplus_{i=1}^d \pi_{\tau_i}$. Then there is a unique automorphism $\bar{\alpha}$ of $\rho(A)''$ such that $\bar{\alpha} \circ \rho = \rho \circ \alpha$. Since $\pi_{\tau_i}(A)''$ is an injective type II₁ factor and any non-zero power of $\bar{\alpha}^{k_i}$ on $\pi_{\tau_i}(A)''$ is outer [15], Theorem 4.5, we can apply [9] to $\bar{\alpha}^{k_i}|_{\pi_{\tau_i}(A)''}$ to conclude that $\bar{\alpha}^{k_i}|_{\pi_{\tau_i}(A)''}$ has the (von Neumann version of) Rohlin property. Thus we have, for any $l \in \mathbb{N}$, a central sequence $\{(E_0^{(j)}, \dots, E_{k_i l - 1}^{(j)})\}$ of orthogonal families of projections in $\bigoplus_{s=0}^{k_i l - 1} \pi_{\tau_i \circ \alpha^s}(A)''$ such that

$$\sum_{i=0}^{k_i l - 1} E_i^{(j)} \rightarrow 1,$$

$$\bar{\alpha}(E_i^{(j)}) - E_{i+1}^{(j)} \rightarrow 0, \quad i = 0, \dots, k_i l - 1,$$

strongly as $j \rightarrow \infty$, where $E_{k_i l}^{(j)} = E_0^{(j)}$. Since $\rho(A)''$ is the finite direct sum of such von Neumann algebras, we can conclude that for any $l \in \mathbb{N}$ there exists a central sequence $\{(E_0^{(j)}, \dots, E_{kl-1}^{(j)})\}$ of orthogonal families of projections in $\rho(A)''$ such that

$$\sum_{i=0}^{kl-1} E_i^{(j)} \rightarrow 1,$$

$$\bar{\alpha}(E_i^{(j)}) - E_{i+1}^{(j)} \rightarrow 0, \quad i = 0, \dots, kl - 1,$$

strongly as $j \rightarrow \infty$, where $E_{kl}^{(j)} = E_0^{(j)}$.

Then we can replace $\{E_i^{(j)}\}$ by a central sequence $\{e_i^{(j)}\}$ of orthogonal families of projections in A such that

$$\rho\left(\sum_{i=0}^{kl-1} e_i^{(j)}\right) \rightarrow 1,$$

$$\|\alpha(e_i^{(j)}) - e_{i+1}^{(j)}\| \rightarrow 0, \quad i = 0, \dots, kl - 2$$

as $j \rightarrow \infty$. (See [15], Section 4 for details.) Actually what we do is as follows: First find a central sequence $\{e_j\}$ of projections of A such that $\rho(e_j) - E_0^{(j)} \rightarrow 0$ strongly. Second, noting that $\rho(e_j(\sum_{s=1}^{kl-1} \alpha^s(e_j))e_j) \rightarrow 0$, find a subprojection p_j of e_j such that $\{p_j\}$ is a central sequence and

$$\|p_j(\sum_{s=1}^{kl-1} \alpha^s(e_j))p_j\| \rightarrow 0,$$

$$\rho(e_j - p_j) \rightarrow 0.$$

The condition of centrality can be satisfied by assuming that the algebra generated by $e_j, \alpha(e_j), \dots, \alpha^{kl-1}(e_j)$ lies in the relative commutant of a prescribed finite-dimensional

C^* -subalgebra and then by choosing p_j from this commutant. Then the projections $p_j, \alpha(p_j), \dots, \alpha^{kl-1}(p_j)$ will be nearly orthogonal, and hence we can construct, by a standard technique, an orthogonal family $\{f_0^{(j)}, \dots, f_{kl-1}^{(j)}\}$ of projections such that $f_s^{(j)} \approx \alpha^s(p_j)$. Furthermore we can assume that all $f_s^{(j)}$ are in $A \cap B'$.

Given $m \in \mathbb{N}$ let $m_0 \geq m$ be the smallest integer such that $m_0 \equiv 0 \pmod{k}$. Let $l = m_0 + 1$ and

$$e_i^{(j)} = \sum_{s=0}^{k-1} f_{i+s(m_0+1)}^{(j)}, \quad i = 0, \dots, m.$$

Then $\{e_i^{(j)}; i = 0, \dots, m\}$ is a central sequence in j and $[e_i^{(j)}]$ is independent of i in $K_0(A \cap B')$ for a sufficiently large j by Lemma 3.2 below. It also follows that

$$\begin{aligned} e^{(j)} &= \sum_{i=0}^m e_i^{(j)} \leq 1, \\ \|\alpha(e_i^{(j)}) - e_{i+1}^{(j)}\| &\rightarrow 0, \quad i = 0, \dots, m-1, \\ \tau_s(e^{(j)}) &\rightarrow \frac{m+1}{m_0+1}, \quad s = 1, \dots, d. \end{aligned}$$

Since τ_s is factorial, it follows that for any minimal central projection E of B ,

$$\tau_s(e^{(j)}E) \rightarrow \frac{m+1}{m_0+1} \tau_s(E).$$

Since the traces τ_1, \dots, τ_d on $(A \cap B')E$ determine the order of the dimension group of $(A \cap B')E$, which is order isomorphic to $K_0(A)$, it follows that

$$(m_0 + 1)[e^{(j)}] \geq m[1]$$

in $K_0(A \cap B')$ if j is sufficiently large. Since $m_0 \leq m + k - 1$, this concludes the proof.

Lemma 3.2 *In the situation of Lemma 3.1 let $\{p_j\}$ be a central sequence of projections in A and let $\beta = \alpha^{kl}$ for some $l \in \mathbb{N}$. Then there is a central sequence $\{x_j\}$ in A such that $x_j^*x_j = p_j$ and $x_jx_j^* = \beta(p_j)$.*

Proof. Let D be a finite-dimensional subalgebra of A and let v be a unitary of A such that $\beta|D = \text{Ad } v|D$. Then for a sufficiently large j , p_j almost commutes with D and $\beta(p_j)$ is close to $\text{Ad } v^* \circ \beta(p_j)$. Since $\text{Ad } v^* \circ \beta|D = \text{id}$, it follows that there is a unitary $u_j \in A \cap D'$ such that $\text{Ad } v^* \circ \beta(p_j) \approx \text{Ad } u_j(p_j)$. Thus we can find x_j with $x_j^*x_j = p_j$, $x_jx_j^* = \beta(p_j)$ in a small neighbourhood of $u_j p_j$, which implies that x_j almost commutes with D . Since D is arbitrary, we obtain the conclusion.

4 The Rohlin property

Theorem 4.1 *Let A be a unital simple AF algebra such that $K_0(A)$ is of finite rank and has no infinitesimal elements. Let α be an automorphism of A . If α satisfies the property that α^m is uniformly outer for any $m \neq 0$, then α has the Rohlin property.*

We note the following consequence obtained in [18]: In the above theorem further suppose that $\alpha_* = id$ on $K_0(A)$. Then for any $\epsilon > 0$ there are a unitary $u \in A$ and an increasing sequence $\{A_n\}$ of finite-dimensional subalgebras of A such that $\|u - 1\| < \epsilon$, $\cup A_n$ is dense in A , and $\text{Ad } u \circ \alpha(A_n) = A_n$ for all n .

Before giving the proof we introduce the following definition [5].

Definition 4.2 An automorphism α of a unital AF algebra A has the approximate Rohlin property if for any $m, n \in \mathbf{N}$, any $\epsilon > 0$, and any finite-dimensional C^* -subalgebra B of A there is an orthogonal family $\{e_i; i = 0, \dots, m-1\}$ of projections in $A \cap B'$ such that

$$\begin{aligned} \|\alpha(e_i) - e_{i+1}\| &< \epsilon, \quad i = 0, \dots, m-1, \\ [e_0] &= [e_1] = \dots = [e_{m-1}], \\ [e_0] &\geq n[1 - \sum_{i=0}^{m-1} e_i], \end{aligned}$$

where $[\cdot]$ denotes the equivalence class in $K_0(A \cap B')$ and $e_m = e_0$.

Theorem 4.1 follows from the following two lemmas.

Lemma 4.3 *Under the situation of Theorem 4.1 suppose that α^m is uniformly outer for any $m \neq 0$. Then α has the approximate Rohlin property.*

Proof. This follows from 2.1 and 3.1. Let k be the period of α_* on $K_0(A)$ as in the proof of 3.1. Let $m \in \mathbf{N}$, $\epsilon > 0$, and B a finite-dimensional C^* -subalgebra of A . By 2.1 one finds $N \in \mathbf{N}$ and projections e_0, \dots, e_{m-1} in K such that

$$\begin{aligned} e_0 + \dots + e_{m-1} &\leq P_N, \\ \|\sigma(e_i) - e_{i+1}\| &< \epsilon, \\ \frac{m \dim e_0}{N} &> 1 - \epsilon, \end{aligned}$$

where $e_m = e_0$. By 3.1 one finds $N+1$ projections p_0, \dots, p_N in $A \cap B'$ such that

$$\begin{aligned} p_0 + \dots + p_N &\leq 1, \\ \|\alpha(p_i) - p_{i+1}\| &< \epsilon/N, \quad i = 0, \dots, N-1, \end{aligned}$$

and $[p_0] = \dots = [p_N]$, $N[1] \leq (N+k)(N+1)[p_0]$ in $K_0(A \cap B')$. Then there is a unitary $u \in A$ such that $\|u - 1\| < 4\epsilon$ and $\text{Ad } u \circ \alpha(p_i) = p_{i+1}$ for $i = 0, \dots, N-1$. Let w be a partial isometry in $A \cap B'$ such that $w^*w = p_0$ and $ww^* = p_1$. Let $\alpha_1 = \text{Ad } u \circ \alpha$ and let C_1 (resp. C) be the C^* -algebra generated by $w, \alpha_1(w), \dots, \alpha_1^{N-2}(w)$ (resp. and $\alpha_1^{N-1}(w)$). Then $M_N \cong C_1 \subset C \cong M_{N+1}$. Define a homomorphism Φ of C into K (as in 2.1) by $\Phi(\alpha_1^i(w)) = E_{i+1,i}$. Then Φ satisfies that $\sigma \circ \Phi|_{C_1} = \Phi \circ \alpha_1|_{C_1}$ and $\Phi(C_1) = P_N K P_N$. Thus, denoting $\Phi^{-1}(e_i)$ by e_i , we obtain projections e_0, \dots, e_{m-1} in $C_1 \subset A \cap B'$ such that

$$\begin{aligned} e_0 + \dots + e_{m-1} &\leq 1, \\ \|\alpha(e_i) - e_{i+1}\| &< 9\epsilon, \quad i = 0, \dots, m-1, \\ m[e_0] &\geq \frac{N^2}{(N+k)(N+1)}(1-\epsilon)[1], \end{aligned}$$

where $e_m = e_0$ and the last inequality is in $K_0(A \cap B') \otimes \mathbf{R}$. For a sufficiently large N and a sufficiently small $\epsilon > 0$ we can conclude the proof.

Lemma 4.4 *Under the situation of Theorem 4.1 suppose that α has the approximate Rohlin property. Then α has the Rohlin property.*

Proof. This follows by the arguments in the proof of Theorem 2.1 of [15]. We just outline the proof. The approximate Rohlin property is not good enough only because

$$e = \sum_{i=0}^m e_i$$

is not 1, where e_i 's are as in Definition 4.2. But $1 - e$ is small compared with e_0 (if n is large), which is the fact we shall use below. Assuming $\epsilon > 0$ is sufficiently small, let us assume that $\alpha(e_i) = e_{i+1}$ and so $\alpha^m(e_0) = e_0$. We use the approximate Rohlin property for $(e_0 A e_0, \alpha^m)$ to obtain projections p_1, \dots, p_n in $A \cap B'$ such that

$$\begin{aligned} p_1 + \dots + p_n &\leq e_0, \\ \alpha^m(p_i) &\approx p_{i+1}, \quad i = 1, \dots, n-1, \end{aligned}$$

and $[p_1] = [1 - e]$ in $K_0(A \cap B')$. Let v be a partial isometry in $A \cap B'$ such that $v^*v = 1 - e$ and $vv^* = p_1$ and let

$$w' = n^{-1/2} \sum_{i=0}^{n-1} \alpha^{mi}(v).$$

Then w' is close to a partial isometry and we will obtain a partial isometry w in a small neighbourhood of w' such that $w^*w = 1 - e$, $ww^* \leq e_0$, w is almost invariant under α^m (if n is large), and w is almost central (if B is large). Then the C^* -algebra D generated by $w, \alpha(w), \dots, \alpha^{m-1}(w)$ is isomorphic to M_{m+1} , and $\alpha|_D$ is close to $\text{Ad } U|_D$, where U is a unitary in D whose eigenvalues are

$$1, e^{2\pi ij/m}; j = 0, 1, \dots, m-1.$$

Since the eigenvalues of U are almost uniformly distributed (if m is large), one can find projections $f_0, \dots, f_{l-1}; g_0, \dots, g_l$ in D (for a prescribed l) such that they add up to the identity of D and

$$\alpha(f_i) \approx f_{i+1}; \alpha(g_i) \approx g_{i+1}$$

where $f_l = f_0$ and $g_{l+1} = g_0$. Hence the projections

$$f_0, \dots, f_{l-1}; g_0, \dots, g_l; e_0 - ww^*, \dots, e_{m-1} - \alpha^{m-1}(ww^*)$$

satisfy the required properties.

5 Outer conjugacy

Definition 5.1 Let S_k be a finite sequence $\{s_{k1}, \dots, s_{ka_k}\}$ in \mathbf{T} for each $k \in \mathbf{N}$. The sequence $\{S_k\}$ is uniformly distributed on \mathbf{T} if for any continuous function f on \mathbf{T}

$$\lim_k a_k^{-1} \sum_{i=1}^{a_k} f(s_{ki}) = \int_{\mathbf{T}} f(\lambda) d\lambda$$

where $d\lambda$ is normalized Haar measure on \mathbf{T} .

See [4] for other equivalent conditions.

Lemma 5.2 Let α be an automorphism of a UHF algebra A such that $\alpha = \bigotimes_{k=1}^{\infty} \text{Ad } u_k$ on $A = \bigotimes_{k=1}^{\infty} M_{m_k}$, where u_k is a unitary in the k -th factor M_{m_k} . For $k, l \in \mathbf{N}$ with $l \leq k$ let $S_{l,k}$ be a sequence consisting of the eigenvalues of the unitary $\bigotimes_{i=l}^k u_i$ repeated as often as multiplicity indicates. Then α has the Rohlin property if and only if $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed for any $l \in \mathbf{N}$.

Proof. This is almost contained in [14, 4]. Suppose that α has the Rohlin property, and that a sufficiently large $N \in \mathbf{N}$ is given for k in Definition 1.1. Then there exist $N_1, \dots, N_t \geq N$ such that for any $\epsilon > 0$ there are projections $e_{ij}; i = 1, \dots, t, j = 0, \dots, N_i - 1$ which are adding up to 1 and satisfy that $\|\alpha(e_{i,j}) - e_{i,j+1}\| < \epsilon$, where $e_{i,k_i} = e_{i,0}$. This shows that for a sufficiently large $k \in \mathbf{N}$, the unitary $\bigotimes_{i=1}^k u_i$ is almost unitarily equivalent to

$$\bigoplus_{i=1}^t U_i \otimes V_i$$

where U_i is a unitary of M_{N_i} with eigenvalues

$$e^{2\pi ij/N_i}; j = 0, \dots, N_i - 1$$

and V_i is a unitary of $M_{m_1 \dots m_k \tau_i}$ with $\tau_i = \tau(e_{i,0})$. This shows that the eigenvalues of $\bigotimes_{i=1}^k u_i$ is almost uniformly distributed whatever V_i 's are, i.e., $\{S_{1,k}\}$ is uniformly distributed. This argument applies to the restriction of α to the subalgebra $\bigotimes_{k=l}^{\infty} M_{m_k}$ of A for any $l \in \mathbf{N}$, concluding that $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed.

Suppose that $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed for any $l \in \mathbf{N}$. Then it is easy to construct the required projections as in Definition 1.1 with $F = \emptyset$ in $\otimes_{i=l}^k M_{m_i}$ for a sufficiently large k . (This is a linear algebra problem.) Hence the Rohlin property follows (see [4, 16] for details).

When $\{S_{l,k}\}_{k=l}^{\infty}$ is uniformly distributed in Lemma 5.2, we say that the eigenvalues of $\otimes_{i=l}^{\infty} u_i$ are uniformly distributed.

Proof of Theorem 1.4. As noted in Section 1, if α and β have the Rohlin property, for any $\epsilon > 0$ there exist unitaries $u_1, u_2 \in A$ such that $\|u_1 - 1\| < \epsilon$, $\|u_2 - 1\| < \epsilon$ and $\text{Ad } u_1 \circ \alpha$ and $\text{Ad } u_2 \circ \beta$ are of infinite tensor product type. Hence we may assume that there is a sequence $\{m_k\}$ of positive integers such that (A, α) is isomorphic to

$$(M_{m_1} \otimes \bigotimes_{i=1}^{\infty} (M_{m_{2i}} \otimes M_{m_{2i+1}}), \bigotimes_{i=0}^{\infty} \text{Ad } u_{2i})$$

where $u_{2i} \in M_{m_{2i}} \otimes M_{m_{2i+1}}$, with $M_{m_0} = \mathbf{C}$ and (A, β) is isomorphic to

$$(\bigotimes_{i=1}^{\infty} (M_{m_{2i-1}} \otimes M_{m_{2i}}), \bigotimes_{i=0}^{\infty} \text{Ad } u_{2i-1})$$

where $u_{2i-1} \in M_{m_{2i-1}} \otimes M_{m_{2i}}$. Since the eigenvalues of $\otimes_{i=1}^{\infty} u_{2i-1}$ are uniformly distributed, it follows that for a sufficiently large k , $\otimes_{i=1}^k u_{2i-1}$ is almost equal to $u_0 \otimes V$ in norm (up to conjugacy) where V is a unitary in $M_{m_2 \dots m_{2k}}$. Then replacing M_{m_2} by $M_{m_2 \dots m_{2k}}$ we may assume that there is a unitary $w_1 \in M_{m_1} \otimes M_{m_2}$ such that $\|w_1 - 1\| < \epsilon/2$, $w_1 u_1 = u_0 \otimes v_2$ where v_2 is a unitary of M_{m_2} . Note that (A, α) is still of the same form. In the same way we may assume that there is a unitary $w_2 \in M_{m_2} \otimes M_{m_3}$ such that $\|w_2 - 1\| < \epsilon/2^2$, $w_2 u_2 = v_2 \otimes v_3$ where v_3 is a unitary of M_{m_3} . Repeating this procedure we obtain unitaries $w_i \in M_{m_i} \otimes M_{m_{i+1}}$ and $v_i \in M_{m_i}$ such that

$$\begin{aligned} \|w_{2i} - 1\| &< 2^{-i}\epsilon, \\ w_i u_i &= v_i \otimes v_{i+1}, \end{aligned}$$

for $i = 1, 2, \dots$ where $v_1 = u_0$. Let

$$w_e = \otimes_{i=0}^{\infty} w_{2i}, \quad w_o = \otimes_{i=0}^{\infty} w_{2i-1}.$$

Then w_e, w_o are unitaries of A with $\|w_e - 1\| < \epsilon$, $\|w_o - 1\| < \epsilon$ and

$$\begin{aligned} \text{Ad } w_e \circ \alpha &= \otimes_{i=0}^{\infty} \text{Ad } w_{2i} u_{2i} = \otimes_{i=0}^{\infty} \text{Ad } v_i, \\ \text{Ad } w_o \circ \beta &= \otimes_{i=0}^{\infty} \text{Ad } w_{2i-1} u_{2i-1} = \otimes_{i=0}^{\infty} \text{Ad } v_i. \end{aligned}$$

This completes the proof.

6 Non-commutative tori

Let $\Theta = (\theta_{ij}) \in M_n(\mathbf{R})$ be such that $\theta_{ij} = -\theta_{ji}$ and let A_Θ be the universal C^* -algebra generated by n unitaries u_1, \dots, u_n with relations $u_i u_j u_i^* u_j^* = e^{2\pi i \theta_{ij}} 1$; A_Θ is called a non-commutative torus. We call Θ to be completely irrational if for any $x \in \mathbf{Z}^n \setminus \{0\}$ there is a $y \in \mathbf{Z}^n$ with $(\Theta x, y) \notin \mathbf{Z}$, i.e., $u^x = u_1^{x_1} \cdots u_n^{x_n}$ is not in the center of A_Θ . It follows that A_Θ is simple if and only if Θ is completely irrational. In this case A_Θ has a unique tracial state. Define an action α of \mathbf{T}^n on A_Θ by $\alpha_\lambda(u_i) = \lambda_i u_i$ with $\lambda = (\lambda_1, \dots, \lambda_n)$. The action α is ergodic and the above tracial state is defined as the average over this action. For a $\lambda \in \mathbf{T}^n$ α_λ is inner if and only if $\lambda_j = e^{2\pi i (\Theta x)_j}$ for some $x \in \mathbf{Z}^n$, i.e., α_λ is implemented by u^x for some $x \in \mathbf{Z}^n$.

Theorem 6.1 *Let A_Θ be as above and suppose that A_Θ is a simple C^* -algebra. Let $\lambda \in \mathbf{T}^n$ be such that α_{λ^m} is outer for any $m \neq 0$. Then α_λ has the Rohlin property.*

Proof. We may assume that θ_{12} is irrational and $\lambda_1^m \notin \{e^{2\pi i (\Theta x)_1}; x \in \mathbf{Z}^n\}$ for all $m \neq 0$. Let $\rho = e^{2\pi i \theta_{12}}$. Then, by using the proof of Lemma 4.6 of [3], for any $\epsilon > 0$ there are $p, q \in \mathbf{N}$ such that

$$\begin{aligned} & |\rho^p - 1|, |\lambda_1^p - \lambda_1|, |e^{2\pi i p \theta_{k1}} - 1| \quad (k = 3, \dots, n), \\ & |\rho^q - 1|, |\lambda_2^q - 1|, |e^{2\pi i q \theta_{k1}} - 1| \quad (k = 3, \dots, n), \\ & |\rho^{pq} - \rho| \end{aligned}$$

are all smaller than ϵ . (First choose a sufficiently large p such that the values in the first line above are all smaller than ϵ , and then choose q such that $|\rho^q - \rho^{1/p}|$, $|\lambda_2^q - 1|$, $|e^{2\pi i q \theta_{k1}} - 1|$ are sufficiently small, where $\rho^{1/p} = e^{2\pi i \theta_{12}/p}$. Then the estimate on $|\rho^q - \rho^{1/p}|$ will yield both $|\rho^q - 1| < \epsilon$ and $|\rho^{pq} - \rho| < \epsilon$.) For $\epsilon = 1/m$ let $v_m = u_1^p$ and $w_m = u_2^q$. Then $\{v_m, w_m\}$ forms a central sequence of pairs of unitaries in A_Θ such that

$$\begin{aligned} v_m w_m v_m^* w_m^* &= \rho_m 1, \quad \rho_m \rightarrow \rho, \\ \alpha_\lambda(v_m) &= \mu_m v_m, \quad \mu_m \rightarrow \lambda_1, \\ \alpha_\lambda(w_m) &= \nu_m w_m, \quad \nu_m \rightarrow 1. \end{aligned}$$

By applying Lemma 6.3 below we can conclude the proof.

Lemma 6.2 *Let θ be an irrational number and let A_θ be the non-commutative torus corresponding to $\Theta \in M_2(\mathbf{R})$ with $\theta_{12} = \theta$. Let $\lambda \in \mathbf{T}^2$ be such that any non-zero power of α_λ is outer. Then α_λ has the approximate Rohlin property in the sense that for any $m \in \mathbf{N}$ there is a central sequence of orthogonal families $\{e_i^{(j)}; i = 0, \dots, m-1\}$ of projections in A_θ such that*

$$\begin{aligned} \|\alpha_\lambda(e_i^{(j)}) - e_{i+1}^{(j)}\| &\rightarrow 0, \\ \tau(e_0^{(j)}) &\rightarrow 1/m, \end{aligned}$$

where $e_m^{(j)} = e_0^{(j)}$ and τ is the unique tracial state of A_θ .

Proof. Note that A_θ is a nuclear C^* -algebra of real rank zero [3] and that $K_0(A_\theta) \cong \mathbf{Z} + \theta\mathbf{Z}$ and A_θ has cancellation [1]. We can use the proof of Lemma 4.3 to obtain such a sequence if we drop the requirement of centralness. We needed the AF property there to make the sequence central; otherwise the above mentioned properties would suffice. To obtain a central sequence of such families, we use the argument in the proof of Theorem 6.1; assuming that $(2\pi)^{-1} \log \lambda_1$ is rationally independent of θ , we obtain a central sequence $\{v_k, w_k\}$ of pairs of unitaries in A_θ such that

$$\begin{aligned} v_k w_k v_k^* w_k^* &= \rho_k 1, \quad \rho_k \rightarrow e^{2\pi i \theta}, \\ \alpha_\lambda(v_k) &= \mu_k v_k, \quad \mu_k \rightarrow \lambda_1, \\ \alpha_\lambda(w_k) &= \nu_k w_k, \quad \nu_k \rightarrow 1. \end{aligned}$$

On the other hand, for any $\epsilon > 0$ and $m \in \mathbf{N}$ we have an orthogonal family $\{e_i; i = 0, \dots, m-1\}$ of projections in A_θ such that

$$\begin{aligned} \|\alpha_{(\lambda_1, 1)}(e_i) - e_{i+1}\| &< \epsilon, \\ m\tau(e_0) &> 1 - \epsilon, \end{aligned}$$

where $e_m = e_0$. Then we obtain a continuous field of such families $\{e_i(t); i = 0, \dots, m-1\}$ in A_t with t in a small neighbourhood of θ [11]. Since the action α of \mathbf{T}^2 acts continuously on this field, we have that for t in a small neighbourhood of θ and for μ in a small neighbourhood of $(\lambda_1, 1)$,

$$\begin{aligned} \|\alpha_\mu(e_i(t)) - e_{i+1}(t)\| &< \epsilon, \\ m\tau(e_i(t)) &> 1 - \epsilon, \end{aligned}$$

where $e_m(t) = e_0(t)$. By identifying the C^* -algebra B_k generated by v_k, w_k with A_{θ_k} with $\rho_k = e^{2\pi i \theta_k}$, we obtain the family $\{e_i(\theta_k)\}$ of projections in B_k . Since they form a central sequence, we can now conclude the proof.

Lemma 6.3 *Let A_θ, α_λ be as in Lemma 6.2. Then for any $k \in \mathbf{N}$ and any $\epsilon > 0$ there are $k_1, \dots, k_m \geq k$ and projections $e_{i,j}; i = 1, \dots, m, j = 0, \dots, k_i - 1$ in A_θ such that*

$$\begin{aligned} \sum_{i=1}^m \sum_{j=0}^{k_i-1} e_{i,j} &= 1, \\ \|\alpha_\lambda(e_{i,j}) - e_{i,j+1}\| &< \epsilon, \end{aligned}$$

where $e_{i,k_i} = e_{i,0}$.

Proof. This can be proved as Lemma 4.4, by using the fact that if two projections in A_θ have the same trace value then they are equivalent. Furthermore, in Definition 1.1 of the Rohlin property, we can choose $m = 2, k_1 = k, k_2 = k + 1$ as in the case of UHF algebras (cf. [17]). This follows by noting that we can choose, in Lemma 6.2, matrix units $\{e_{i,k}^{(j)}\}$ such that $\|\alpha_\lambda(e_{i,k}^{(j)}) - e_{i+1,k+1}^{(j)}\| < \epsilon, \tau(e_{0,0}) \rightarrow 1/m$ instead of just families of projections (see the proof of Theorem 2.1 in [15] for details).

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