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INTERIOR DERIVATIVE BLOW-UP FOR QUASILINEAR PARABOLIC EQUATIONS

Dedicated to Professor Kôji Kubota on
the occasion of his sixtieth birthday

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Abstract. We give examples of bounded solutions whose gradient blow up in a finite time but it stays bounded on the boundary for a class of quasilinear parabolic equations with zero boundary data. The method reflects a geometric argument for curve evolution equations.

1. Introduction. We consider a quasilinear parabolic equation

$$\begin{aligned} u_t - a(u_x)u_{xx} - f(u)g(u_x) = 0 \quad &\text{in} \quad Q = \Omega \times (0,T) \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0 \quad &\text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

where Ω is an open interval. Here functions a, f, g are assumed to be C^1 on $\mathbf{R}$ and $a > 0$. It is well known [LSU] that the initial-boundary value problem has a unique classical solution for small $T > 0$ if $u_0 \in C^1(\overline{\Omega})$ and u_0 is compatible with the boundary condition $u = 0$. We are interested in constructing derivative blow-up solutions when the equation is not uniformly parabolic in the sense that

$$a(p) \to 0 \quad \text{as} \quad |p| \to \infty.$$

For simplicity of explanation we set

$$a(p) = 1/(1+p^2)^\alpha, \quad g(p) = (1+p^2)^\theta, \quad \alpha \geq 0,\ \theta \geq 0 \tag{1.2}$$

in the introduction. It is well known [LSU], [Li] that (1.1) with (1.2) admits a unique global classical solution for every $T > 0$ provided that

(i) $\theta + \alpha < 1$

(ii) f is globally Lipschitz and nonnegative.

A more general sufficient condition is given in Section 3.6 for the reader's convenience (Note that (ii) implies (M) and (D3) in Section 3.6 so that the conditions (i), (ii) are sufficient for global existence.) For some θ, α with $\theta + \alpha > 1$ there are examples of the existence of the (first) derivative blow-up solutions for $(1.1)_{1,3}$ with (1.2) and some smooth inhomogeneous boundary data (depending on t mostly) even if f is constant (e.g. [D], [Ku1], [Ku2], [Ku3], [Li].) However, if f is constant the gradient blow-up occurs only at the boundary. This is easily verified by the maximum principle for the equation of u_x obtained from $(1.1)_1$ by differentiation in x. For homogeneous boundary data there is an interesting result [FL] on derivative blow-up. In [FL] it is shown that derivatives of all nontrivial solutions blow up on the left boundary of Ω for the initial-boundary value problem

$$\begin{aligned} u_t = u_{xx} + e^{u_x} \quad &\text{in} \quad Q \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \end{aligned}$$

provided that the length of Ω is sufficiently large. In fact a more general equation is discussed in [FL]; however the equation (1.1) with (1.2) ($\alpha = 0$) is not included. The profile of solution at the derivative blow-up time is studied and the solution is extended beyond the blow up time in [FL].

Our goal in this paper is to construct a derivative blow-up solution for (1.1) with (1.2) when $\theta + \alpha > 1$ with non constant f(satisfying (ii).) Our solution is also arranged so that the derivative on the boundary is bounded up to the blow-up time of derivative. In other words the derivative blow-up occurs in Ω not on $\partial\Omega$. Because of our method we are forced to restrict $\theta = 1/2$.

In [B] the Neumann boundary value problem of a quasilinear parabolic equation of the divergence type is studied. The typical form of the equation is

$$u_t = (G(u, u_x))_x, \quad G(u,p) = u^{5/2}p/(1+u|p|).$$

Under the Neumann boundary condition $u_x = 0$ on $\partial\Omega$ it is shown [B] that the derivative of solutions may blow up in a finite time for some initial data. Since the Neumann condition is imposed, the blow up of derivatives occurs in Ω in this problem.

Recently blow up of derivatives of solutions is studied for a semilinear parabolic equation in [AF]. A typical example of their equations is

$$u_t = u_{xx} + u|u_x|^{m-1}u_x, \quad m > 2.$$

For a suitable choice of constant bounary data it is shown [AF] that the derivative of all solutions blow up in a finite time in Ω (not on the boundary.)

As far as we know so far there were no examples of solutions (for quasilinear equations) whose derivative actually blow up in Ω not on $\partial\Omega$ for the Dirichlet problem. Our results actually provide such examples.

Our method of the proof is geometric. To describe it we consider a typical example of $(1.1)_1$ with (1.2) and $f(u) = u$ where $\alpha = 1$, i.e.,

$$u_t - \frac{u_{xx}}{1+u_x^2} - u(1+u_x^2)^{1/2} = 0. \tag{1.3}$$

This equation is interpreted as an equation of the motion of graph of u:

$$\Gamma_t = \{(x,y);\ y = u(x,t),\ x \in \Omega\}.$$

In fact, if V denotes the upward normal velocity of Γ_t and k denotes upward curvature of Γ_t, (1.3) is expressed as

$$V = k + y \tag{1.4}$$

since $V = u_t/(1+u_x^2)^{1/2}$, $k = u_{xx}/(1+u_x^2)^{3/2}$. The equation (1.4) is a curve shortening equation with driving force term y. Such a type of equations has been studied extensively. A key property is a comparison principle which is also fundamental to construct generalized solution by a level set method (cf. [CGG] see also [GGI], [ES]). We consider a circle C_0 with radius R moved by (1.4). For fixed R the solution C_t of (1.4) with initial data C_0 grows if the vertical component of the center of C_0 is very large since the driving force term dominates the curvature; here the normal of C_0 is taken outward. This can be proved by a comparison principle. On the other hand if the circle C_0' (with radius R) is located near x axis, the solution C_t' of (1.4) with initial data C_0' does not disappear instanteneously even if C_0' is oriented in an opposite way. To construct desired blow-up solutions we may assume that Ω is symmetric with respect to the origin, i.e. $\Omega = (-s, s)$ and that u_0 is an even function. We take R such that $3R$ is slightly less than s. Let C_0 be a circle of radius R centered at $(0, y_0)$ whose unit normal vector field is taken outward and C_0' be a circle of the same radius centered at $(s - R, R)$ whose unit normal vector field is taken inward. We take y_0 so large that the projection $\pi(C_t)$ (to the x axis) of the solution C_t of (1.4) with initial data C_0 overlaps $\pi(C_t')$ before C_t' disappears. If u_0 is taken so that Γ_0 is located over C_0 and under C_0' in $\pi(C_0) \times \mathbf{R}$ and $\pi(C_0') \times \mathbf{R}$ respectively, then the solution u ceases to be classical before $\pi(C_t')$ and $\pi(C_t)$ overlaps since Γ_t always located above C_t and below C_t' by comparison. Estimating evolution of C_t and C_t' carefully we are able to estimate the maximal existence time T_0 (of classical solutions) from above, say $T_0 \leq c$. It is possible to construct a boundary barrier until $t \leq c$ to prevent the blow up of u_x on $\partial\Omega$. This is a qualitative picture of the construction of blowing up solution whose derivative stays bounded on the boundary. (Note that the derivative should blow up at T_0; otherwise solution can be extended beyond T_0.)

To carry out this idea more rigorously we estimate evolution of C_t, C_t' from inside by a selfsimilar sub and supersolutions of (1.4). It turns out only evolution of upper hemicircle of C_0 and lower hemicircle is necessary to study. The method can be generalized to (1.1) with (1.2) for $\alpha > 1/2$, $\theta = 1/2$. This is because we are forced to assume $\theta = 1/2$ so that $\alpha > 1/2$ to construct sub and supersolutions. Unfortunately this restriction $\alpha > 1/2$ should not be removed if we use the above mentioned geometric argument. In fact if $\alpha = 0$ it seems natural to guess that C_t instanteneously becomes empty as suggested in the paper of [E]. Heuristically, if $\alpha \leq 1/2$, the horizontal diffusion is so fast that the closed curve cannot stay as a closed curve.

In section 2 we construct necessary sub and supersolutions of (1.1). In section 3 we construct desired initial data. The presentation of the proofs uses no geometric language like (1.4) although the underlying idea is based on the above mentioned geometric idea. In section 3 we give a fairly general class of equations so that our argument works.

As we mentioned before, if f is constant, derivative blow-up occurs only on the boundary. It turns out that even if f is close to a positive constant in $C^2(\mathbf{R})$ topology there is a blow-up solution whose derivative stays bounded on the boundary at least for

$$u_t - \frac{u_{xx}}{1 + u_x^2} - f(u)(1 + u_x^2)^{1/2} = 0$$

with $(1.1)_{2,3}$. This follows from Section 3.5 and 3.9.

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2. Construction of sub- and supersolutions. Let α be a positive continuous function defined on $(0, \infty)$ and let K be a nonzero real number. We consider an ordinary differential equation of the form

$$\alpha(u_x)u_{xx} = K. \tag{2.1}$$

It is convenient to set

$$\begin{aligned} d_0 &= \int_0^\infty \alpha(p)dp, & h_0 &= \int_0^\infty \alpha(p)p\, dp \\ d &= d_0/|K|, & h &= h_0/|K|. \end{aligned} \tag{2.2}$$

2.1. Existence Lemma. *Assume that*

$$\int_0^1 \alpha(p)dp < \infty \quad \text{and} \quad \int_1^\infty p\alpha(p)dp < \infty. \tag{2.3}$$

Then there is a unique C^2 solution u of (2.1) on $I = (0, d)$ such that

$$\begin{aligned} &\lim_{x\to 0} u(x) = -h \text{ sgn } K, & &\lim_{x\to 0} u_x(x) = 0, \\ &\lim_{x\to d} u(x) = 0, & &\lim_{x\to d} u_x(x) \text{sgn } K = \infty, \end{aligned} \tag{2.4}$$

where sgn K denotes the sign of K. If α is continuous up to zero, then u is of course C^2 up to zero.

Proof. It suffices to prove the first statement. We may assume $K = 1$ by considering the equation for rescaled function $Ku(x/|K|)$. We set

$$A(p) = \int_0^p \alpha(q)dq$$

so that A is C^1 on $(0, \infty)$ and continuous up to zero by $(2.3)_1$. Since $\alpha > 0$ and $d_0 < \infty$ by (2.3), A has its inverse A^{-1} continuous on $[0, d_0)$ with

$$\lim_{x\to d_0} A^{-1}(x) = \infty. \tag{2.5}$$

Since a solution of (2.1) is convex if $K > 0$, we may assume u_x is nonnegative if we consider $(2.4)_2$. If $(2.4)_2$ is imposed, integrating (2.1) (with $K = 1$) on $(x_0, x)(0 < x_0 < x < d_0)$ and letting x_0 tend to zero yields

$$A(u_x(x)) = x, \quad \text{in} \quad I.$$

Thus, u solves (2.1) on $(0, d_0)$ with $(2.4)_2$ if and only if

$$u(x) = \int_0^x A^{-1}(z)dz + C \quad (C : \text{constant}). \tag{2.6}$$

The property $(2.4)_4$ follows from (2.5).

It remains to show that v in (2.6) satisfies $(2.4)_1$ and $(2.4)_3$ if and only if $C = -h_0$. Here $(2.3)_2$ is invoked in a substantial way. Multiplying u_x with (2.1) and integrating from zero to $x(< d_0)$ yields

$$u(x) - u(0) = \int_0^{u_x(x)} a(p)pdp.$$

We send x to d_0 so that $u_x(x) \to \infty$ by $(2.4)_4$ and observe that

$$\lim_{x \to d_0} u(x) = u(0) + h_0.$$

The proof is now complete.

Remark. As easily observed, our assumption (2.3) is a necessary condition to guarantee the existence of u satisfying (2.1) and (2.4).

2.2. An irregularly parabolic equation. We consider a quasilinear parabolic equation of form

$$u_t - a(u_x)u_{xx} - Mg(u_x) = 0, \tag{2.7}$$

where M is a positive constant. We list assumptions on a and g which we use in the sequel.

(A0) a and g are continuous functions on $\mathbf{R}$ and are even, i.e., $a(p) = a(-p)$, $g(p) = g(-p)$ for $p \in \mathbf{R}$.

(A1) a and g are positive.

(A2) $\int_1^\infty pa(p)dp/g(p) < \infty$.

(A3) $\lim_{p\to\infty} \sup g(p)/|p| < \infty$.

(A4) $\lim_{p\to\infty} \inf g(p)/|p| > 0$.

The symmetry assumption in (A0) is just for simplicity. Our method (for constructing sub and supersolutions) works without assuming the symmetry if we also assume (A2)-(A4) with p replaced by $-p$ in a and g. The positivity assmption on a simply implies that (2.7) is parabolic. The positivity of g is essentially used in our method. If we recall irregularly parabolicity for (2.3) in the sense of Serrin [EP, (20),(21)], i.e.,

$$\lim_{|p|\to\infty} a(p)/g(p) = 0 \text{ and } \int_1^\infty \frac{a(p)p^3}{g(p)|p|^2}dp < \infty, \tag{2.8}$$

we find the second condition is the same as (A2). A typical example satisfying (A0)-(A4) is

$$a(p) = 1/(1+p^2)^\alpha, \quad g(p) = (1+p^2)^{1/2} \quad \text{with} \quad \alpha > 1/2 \tag{2.9}$$

which also satisfies (2.8) so that (2.7) is irregularly parabolic. The condition (A3) together with (A2) excludes $0 \le \alpha \le 1/2$, so in particular semilinear case for (2.7) is excluded.

We assume (A0)-(A3) so that $\alpha = a/g$ satisfies (2.3). Lemma 2.1 implies that there is a unique solution v of (2.1) with $K = -1$ satisfying (2.4). Extending v for $x < 0$ by $v(x) = v(-x)$ we see the extended function (denoted U) solves

$$\frac{a(U_x)}{g(U_x)}U_{xx} \equiv \alpha(|U_x|)U_{xx} = -1 \quad \text{in} \quad (-d_0, d_0) \tag{2.10}$$

with d_0 in (2.2). We introduce two numbers.

$$\beta_0 = \inf_{|x|<d_0} B(x), \quad \gamma_0 = \sup_{|x|<d_0} B(x) \quad \text{with} \tag{2.11}$$

$$B(x) := \frac{U(x) - xU_x(x)}{g(U_x(x))}.$$

Since $B(x) > 0$ for $|x| < d_0$ and

$$\lim_{x \to d_0} U_x(x) = -\infty \tag{2.12}$$

(A4) is equivalent to $\beta_0 > 0$. Since B is bounded and (2.12) holds, (A3) is equinvalent to $0 < \gamma_0 < \infty$. In general $\gamma_0 \geq \beta_0$, however it may happen that $\gamma_0 = \beta_0$. If a and g are as in (2.9) with $\alpha = 1$, then $U(x) = (1-x^2)^{1/2}$ with $d_0 = h_0 = 1$. A direct calculation shows that B identically equals one so that $\gamma_0 = \beta_0 = 1$.

We seek selfsimilar sub- and supersolutions of (2.7) of the form

$$v_\lambda^K(x,t) = -\lambda(t)K^{-1}U(|K|x/\lambda(t)), \quad \lambda(0) = 1 \tag{2.13}$$

with a nonnegative function λ defined on $[0,\sigma)$. The function v_λ^K is defined in

$$Q_\sigma = \{(x,t); \quad |x| < \lambda(t)d, \quad 0 \leq t < \sigma\},$$

where $d = d_0/|K|$ as in (2.2) (with $\alpha = a/g$, of course.)

2.3. Lemma. *Assume that a and g satisfy (A0)-(A3) and $M > 0$ in (2.7).*

(i) (Subsolution) Assume that $K < 0$ with $M + K > 0$ so that there is a unique global solution λ of a differential equation

$$\frac{\lambda' \gamma_0}{|K|} - \frac{K}{\lambda} - M = 0 \quad \text{for} \quad t > 0, \ \lambda(0) = 1, \tag{2.14}$$

which satisfies $\lambda'(t) > 0$ for all $t \geq 0$. Then v_λ^K is a subsolution of (2.7) in Q_∞.

(ii) (Supersolution) Assume (A4) so that β_0 in (2.11) is positive. Assume that $K > 0$ with $K > M$ so that a unique local solution of

$$\frac{\beta_0\lambda'}{K} + \frac{K}{\lambda} - M = 0 \quad \text{on} \quad (0,\sigma) \quad \text{with} \quad \lambda(0) = 1 \tag{2.15}$$

satisfies $\lambda' < 0$ on the maximal existence interval $[0, \sigma)$. Then v_λ^K is a supersolution (2.7) in Q_σ.

Proof. (i) Differentiating $v = v_\lambda^K$ in t yields

$$v_t(x,t) = -K^{-1}\lambda'(t)[U(|K|x/\lambda(t)) - |K|x(\lambda(t))^{-1}U_x(|K|x/\lambda(t))]. \tag{2.16}$$

Since $K < 0$ and $\lambda' > 0$, estimating (2.16) yields

$$\frac{v_t}{g(v_x)} \le \frac{\gamma_0}{|K|}\lambda'.$$

Since U solves (2.10), we see, by the definition of v in (2.13), that

$$\frac{a(v_x)v_{xx}}{g(v_x)} = \frac{K}{\lambda}.$$

Since λ solves (2.14), the above two formulas yield

$$\frac{v_t}{g(v_x)} - \frac{a(v_x)v_{xx}}{g(v_x)} - M \le \frac{\gamma_0}{|K|}\lambda' - \frac{K}{\lambda} - M = 0 \text{ in } Q_\infty.$$

This shows that v is a subsolution of (2.7) in Q_∞.
(ii) Since $K > 0$ and $\lambda' < 0$, estimating (2.16) yields

$$\frac{v_t}{g(v_x)} \ge -\frac{\beta_0\lambda'}{K}.$$

As in (i) we now observe that

$$\frac{v_t}{g(v_x)} - \frac{a(v_x)v_{xx}}{g(v_x)} - M \ge -\frac{\beta_0\lambda'}{K} - \frac{K}{\lambda} - M.$$

Since λ solves (2.15), v is a supersolution in Q_∞.

3. Break down phenomena. We consider a quasilinear parabolic equation of form

$$u_t - a(u_x)u_{xx} - f(u)g(u_x) = 0. \tag{3.1}$$

We shall always assume that a and g satisfy structure assumptions (A0)-(A4). Let d_0 and h_0 be numbers defined by (2.2) and let γ_0 and β_0 be numbers defined by (2.11). The function f we consider here is not nonincreasing. Actually we assume more:
(F) There is $M > 0$ and $\rho > 0$ such that
(i) $f \ge M$ on some interval J of the length 2ℓ with $\ell = 2h_0/\rho M$.
(ii) $\theta_0 = (1-\rho) - 2\gamma_0\rho/\beta_0 > 0$.

(iii) $f \leq N$ on some interval J' of the length ℓ such that $N > 0$ is some number strictly less than $\beta_0\theta_0 M/\gamma_0$ and that $\sup J' < \inf J$, i.e., the interval J' is located in the left of J. (Of course $\theta_0 < 1$ and $\beta_0/\gamma_0 \leq 1$ imply $N < M$.)

Although this assumption looks artifical, it is invariant under a scaling transformation:

$$u^\lambda = \lambda^{-1}u(\lambda x, \lambda^2 t), \quad (\lambda > 0, \text{number})$$

which solves

$$u_t^\lambda - a(u_x^\lambda)u_{xx}^\lambda - f^\lambda(u^\lambda)g(u_x^\lambda) = 0 \quad \text{with} \quad f^\lambda(r) = \lambda^{-1}f(r\lambda)$$

if u solves (3.1). For an open interval Ω where we consider (3.1) we assume that it is not too short compared with J, J'.

(Ω) The length of Ω is larger than $2\ell'$ with $\ell' = 2d_0/\rho M$, where ρ and M be as in (F).

We shall consider a class of functions on Ω having a 'sharply decreasing part'. To formulate this we set $t_0 = 2\gamma_0/M$ so that λ in (2.14) satisfies $\lambda(t) \leq 2$ for $0 \leq t \leq t_0$. Let $\sigma_0 = \sigma_0(M, K) > 0$ be a number so that for λ in (2.15) $\lambda(t) \geq 1/2$ for $0 \leq t \leq \sigma_0$. We then set $t_1 = \min(t_0, \sigma_0)$. Let r and r' be the center of J and J' in (F), respectively.

3.1. Definition. We say a continuous function u_0 on Ω has a *sharply decreasing part* with respect to f if there are intervals I_1, I_2 in Ω such that

(i) $u_0(x) > U(\rho M(x - x_1))/\rho M + r$ on $I_1 = (x_1 - \ell'/2, x_1 + \ell'/2)$

(ii) $u_0(x) < -U(\rho M(x - x_2))/\rho M + r'$ on $I_2 = (x_2 - \ell'/2, x_2 + \ell'/2)$

(iii) $x_2 - x_1 = \ell' + \delta$ with $\delta > 0$ so that I_1 and I_2 are disjoint.

(iv) $\delta < t_1 L$ with $L = d_0 M(\theta_0/\gamma_0 - N/M\beta_0)$ so that I_1 is not far from I_2.

(v) $x_1 - \ell' \in \Omega$ so that $2(I_1 - -x_1) + x_1$ is still contained in Ω.

The existence of such a u_0 is guaranteed by (F) and (Ω).

3.2. Locally well-posed problem. We consider the initial boundary value problem

$$\text{(IBVP)} \qquad \begin{aligned} &u_t - a(u_x)u_{xx} - f(u)g(u_x) = 0 \quad \text{in} \quad Q = \Omega \times (0,T) \\ &b(x, u, u_x) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ &u(x, 0) = u_0(x) \quad x \in \Omega, \end{aligned}$$

where Ω is an open interval in $\mathbf{R}$. We say that u_0 is *compatible* with the boundary condition $b = 0$ if

$$\begin{aligned} &b(x, u_0, u_{0x}) = 0 \quad \text{on} \quad \partial\Omega \\ &b_u(x, u_0, u_{0x})\{u_0\}_t - b_p(x, u_0, u_{0x})\{u_{0x}\}_t = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\{v\}_t$ denotes

$$\{v\}_t = a(v_x)v_{xx} - f(v)g(v_x)$$

for a function v on Ω and $b = b(x, u, p)$. We say (IBVP) is *locally well posed* if for each $u_0 \in C^{2+\nu}(\overline{\Omega})$ compatible with $b = 0$ there is a unique $u \in C^{2+\nu, 1+\nu/2}(\overline{Q})$ solving (IBVP) for sufficiently small $T > 0$, where $0 < \nu < 1$ is the Hölder exponent. The space $C^{2+\nu,1+\nu/2}$ is denoted $H^{2+\nu,1+\nu/2}$ in [LSU]. There are many locally well-posed (IBVP). For example if $a > 0$ and a, f, g is sufficiently smooth, then (IBVP) is locally well-posed for the Dirichlet problem ($b(x,u,p) = u$) and the Neumann problem ($b(x,u,p) = p$). We postpone to study solvability in the last part of Section 3.6 and 3.7. We say that T_0 is the *maximal existence time* if T_0 is the supremum of T such that a unique $C^{2+\nu,1+\nu/2}$ local solution of (IBVP) exists on $\Omega \times (0,T)$. Of course, T_0 depends on u_0. If $T_0 = \infty$ for every u_0, then (IBVP) is called *globally well-posed.*

3.3. Existence Theorem of break down. *Assume that $(3.2)_1$ satisfies structure conditions (A0)-(A4) and (F). Assume that (IBVP) is locally well posed. If (Ω) holds and the initial data u_0 has a sharply decreasing part with respect to f, then the maximal existence time T_0 fulfils $T_0 < \delta/L$. Here δ and L are defined in Definition 3.1.*

Proof. We shall compare u with

$$w_1(x,t) = v_{\lambda_1}^{-\rho M}(x - x_1, t) + r$$
$$w_2(x,t) = v_{\lambda_2}^{\rho M}(x - x_2, t) + r',$$

where $v_{\lambda_1}^{K}$ and $v_{\lambda_2}^{-K}$ are given by (2.13) and defined in Lemma 2.3 (i) with M and (ii) with $M = N$, respectively; subscripts of λ is dropped in Lemma 2.3. Numbers x_1, x_2, r, r' are as in Definition 3.1. By Lemma 2.3 w_1 and w_2 are a sub- and supersolution of (2.7) and (2.7) with $M = N$ in W_∞^1, W_σ^2 respectively, where

$$W_s^i = \{(x,t); |x - x_i| < \lambda_i(t)d_0/\rho M, \quad 0 < t < s\}, \quad i = 1,2.$$

By (F) and Definition 3.1 (v) u is a supersolution of (2.7) in $W_{t_0'}^1$ with $t_0' = \min(t_0, T_0), t_0 = 2\gamma_0/M$. By (F) u is a subsolution of (2.7) with $M = N$ in $W_{\sigma_0'}^2$, with $\sigma_0' = \min(\sigma_0, T_0)$, where σ_0 is defined right after (Ω).

We claim that $u > w_1$ in $W_{t_0'}^1$ and $u < w_2$ in $W_{\sigma_0'}^2$. Since the proof for $u < w_2$ parallels that of $u < w_1$, we only present the proof of $u > w_1$. Suppose that $u > w_1$ in $W_{t_0'}^1$ were false then there would exist t_* so that

$$0 < t_* = \sup\{t \le t_0'; u > w_1 \quad \text{in} \quad W_t^1\} < t_0'.$$

At $t = t_*, u = w_1$ at some point of the interval $R^1(t_*)$, where

$$R^i(t) = \{x; |x - x_i| \le \lambda_i(t)d_0/\rho M\}.$$

Since u is C^1 and $w_{1x} = \pm\infty$ on $\partial R^1(t)$, $u > w_1$ on $\partial R^1(t_*)$. By continuity of u and w_1 on the parabolic boundary of $W_{t_*}^1$ there is small $\epsilon > 0$ such that $u > w_1 + \epsilon$ since u_0 fulfills Definition 3.1(i),(ii). By comparison for (2.7) we have $u > w_1 + \epsilon$ in $W_{t_*}^1$. (Note that the classical weak maximum principle applies to get $u > w + \epsilon$ if a and g are C^1. If not, we apply comparison for viscosity solutions [CIL].) The property $u > w_1 + \epsilon$ in $W^1_{t_*}$ contradicts the definition if t_* so $u > w_1$ in $W^1_{t_0'}$ follows.

Since $\lambda_1(t)$ is increasing, from (2.14) it follows that $\lambda_1'(t) \geq \rho M \cdot M(1-\rho)/\gamma_0$. Since $\lambda_2(t)$ is decreasing, from (2.15) it follows that

$$-\lambda_2'(t) \leq \rho M(2\rho M + N)/\beta_0, \quad \text{for} \quad 0 \leq t \leq \sigma_0,$$

where σ_0 has been the time such that $\lambda_2 \geq 1/2$ on $(0, \sigma_0]$.
Let $\delta(t)$ denote the distance of $R^1(t)$ and $R^2(t)$. By definition $\delta(0) = \delta$ and

$$-\delta'(t) \geq (\lambda_1'(t) - (-\lambda_2'(t)))d_0/\rho M$$

until $R^1(t)$ and $R^2(t)$ touch each other, say $t \leq t_2$. Using estimates of λ_1 and λ_2 we see

$$-\delta'(t) \geq L = d_0 M(\theta_0/\gamma_0 - N/M\beta_0), \quad t \leq t_2$$

at least for $t \leq \sigma_0$. From Definition 3.1 (iv) it follows that $\delta/L < t_1 \leq \sigma_0$. The estimate of $-\delta'(t)$ and $\delta(0) = \delta$ now implies that $t_2 \leq \delta/L$. We recall that $u > w_1$ on $R^1(t)$ and $u < w_2$ on $R^2(t)$ for $0 \leq t \leq t_1' = \min(t_1, T_0)$. Since $\sup J' < \inf J$, this implies that $R^1(t)$ cannot touch $R^2(t)$ for $0 \leq t \leq t_1'$, i.e. $t_1' < t_2$. Since $t_1 > \delta/L \geq t_2$, this implies $T_0 < t_2 \leq \delta/L$.

3.4. Remark. We did not use C^2 regularity in the proof. Actually our proof shows that even if we extend a local solution in a weak sense such as a Lipschitz solution (in viscosity sense [CIL]), we still obtain the same estimate of existence time T_0. Moreover, the same estimate still holds even if we extend a solution as a continuous (viscosity) solution. This needs analysis of singular equations (see [OS]) for inverse functions so we shall discuss this property elsewhere.

3.5. Examples. (i) If f is a nondecreasing function such that $\lim_{r\to\infty} f(r) = \infty$, then f fulfills (F) for some M and ρ.

Indeed, we take $N = f(r_0)$ for some r_0 and fix ρ such that $\theta_0 > 0$ in (F)(ii). Then take M large enough to get (F)(i)(iii) with infinite length intervals J and J'. So f satisfies (F). Note that a function f satisfying (F) is not necessarily monotone because we do not have any restriction for $f(w)$ if $w > \sup J$ or $\sup J' < w < \inf J$. Also unboundedness of f is not necessary for (F).
(ii) If $\gamma_0 = \beta_0$, then for each $c > 0$ there is a function f very close to a constant c in C^k norm for each k satisfying (F).

Indeed for each $0 < \epsilon < 1$ we take $N = c(1-\epsilon), M = c$ and ρ sufficiently small, say $\epsilon = 3\rho$ so that $N < \theta_0 M$. Let f be a smooth function such that

$$\begin{aligned} &f = c(1-\epsilon) \quad \text{on} \quad J' = [0,\ 2h_0/\rho] \\ &f = c \quad \text{on} \quad J = [4h_0/\rho,\ 6h_0/\rho] \\ &\|f - c\|_{C^k} \leq 2c\epsilon; \end{aligned}$$

such a function f exists at least for sufficiently small $\epsilon > 0$.

3.6. Solvability for the Dirichlet problem. We consider the Dirichlet problem for a quasilinear parabolic equation

$$\begin{aligned} u_t - a(u_x)u_{xx} - f(u)g(u_x) &= 0 \qquad \text{in} \quad \Omega \times (0,T), \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x), \qquad x \in \Omega, \end{aligned} \tag{3.2}$$

where Ω is a bounded open interval. There are many sufficient conditions for *global well posedness* (e.g. [LSU], [Li], [Ku 1-3]). We present a sufficient condition so that it is comparable to our break down phenomena. For this purpose we always assume (A0), (A1) for a and g. We list further conditions for a, g and f.

(R) a, g, f are C^1 on $\mathbf{R}$.

(M) (growth assumption on f)
f is positive for large $|r|$, say $|r| > R$. Moreover, $\int_R^\infty dr/f(r) = \int_{-\infty}^{-R} dr/f(r) = \infty$.

(D1) (relation of a and g)
$\liminf_{p\to\infty} a(p)p^2/g(p) = \infty, \quad \liminf_{p\to\infty} a(p)p/|g'(p)| > 0.$

(D2) $\limsup_{p\to\infty} |a'(p)p|/a(p) < \infty.$

(D3) f' is bounded form above on $\mathbf{R}$.

The condition (M) gives a priori bound on $\max_{\overline{Q}}|u|$ of solution u of (3.2) with $Q = \Omega \times (0,T)$. We denote a bound on $\max_{\overline{Q}}|u|$ by R_0.

The condition $(D1)_1$ gives a priori bound (depending R_0) in $|u_x|$ on $\partial\Omega \times (0,T)$. A priori bound on $\max_{\overline{Q}}|u_x|$ is derived by (D1)-(D3) together with a priori bound on u_x on $\partial\Omega \times (0,T)$ and R_0. These a priori estimates are obtained in the same way as in [LSU]. Actually the priori bound R_0 for u is trivial in our setting by deriving inequality for $\mu(t) = \max_{\overline{\Omega}} u(x,t)$ from $(3.2)_1$:

$$\mu_t - f(\mu)g(0) \leq 0$$

and similarly for $\min_{\overline{\Omega}} u(x,t)$. Other bounds can be obtained essentially in the same way as in [LSU, VI Lemma 3.1, Theorem 3.1, 3.2, p.535-548]. In [LSU] it is always assumed that $a(p)$ is comparable with $(1+p^2)^m, \quad m > 0$ for large p but the proof works under our assumptions. Note that $(\mathrm{D1})_1$ can be weaken so that

$$\liminf_{p\to\infty} a(p)p^2/g(p) = c_0 > 0. \tag{3.3}$$

For this case we need to assume $f' \leq \epsilon$ in (D3) for sufficiently small ϵ depending only on R_0.

If we get a priori bound for gradient of solutions, we get a global solvability as in [LSU].

Lemma. *Assume that (A0), (A1), (R), (M), (D1)-(D3). Then (3.2) is globally well-posed.*

Remark. If we assume (A3), (A4), then (D1) (or even (3.3)) excludes (A2) so this Lemma is consistent with Theorem 3.3.

This Lemma gives local well-posedness for (3.2) without assuming structure conditions (M), (D1)-(D3).

Lemma. *Assume that (A0), (A1) and (R). Then (3.2) is locally well-posed.*

Proof. For initial data u_0 we take a number K_0 such that

$$\max_{\overline{\Omega}} |u_0|, \quad \max_{\overline{\Omega}} |u_{0x}| \leq K_0.$$

We modify $a(p), g(p), f(r)$ only for $|r| > 2K_0, |p| > 2K_0$ so that new a, g, f satisfies (M), (D1)-(D3). This is of course possible and the new Dirichlet problem is globally well-posed. However, solution u should be a solution of the original problem as far as $|u| < 2K_0$, $|u_x| < 2K_0$. Since u and u_x is continuous, this u solves the original problem at least locally in time. ∎

3.7. Lemma on derivative blow up. *Assume that (A0), (A1), (M) and (R). Let T be the maximal existence time for (3.2) with initial data $u_0 \in C^{2+\nu}(\overline{\Omega})$, $0 < \nu < 1$. Then*

$$\limsup_{t \to T} \max_{\overline{\Omega}} |u|(t) < \infty, \quad \limsup_{t \to T} \max_{\overline{\Omega}} |u_x|(t) = \infty.$$

Proof. By (M) we have a bound R_0 on $\max_{\overline{\Omega}} |u|$. If $\max_{\overline{\Omega}} |u_x|(t)$ has a bound K_1 independent of time, then we modify a, g, f only for $|p| > 2K_1, |r| > 2R_0$ so that the new problem is globally well posed. Its solution agrees with u for $t < T$ and it also solves the original equation a little beyond T. This contadicts the maximality of T so $\max_{\overline{\Omega}} |u_x|(t)$ blows up at $t = T$. ∎

3.8. Theorem on interior derivative blow up. *Assume the hypotheses of Theorem 3.3 with $\inf J' = 0$ in (F) and that the length of Ω is larger than $4\ell'$ in (Ω). Then there is an initial data $u_0 \in C^{2+\nu}(\overline{\Omega})$, $0 < \nu < 1$ compatible with $u = 0$ such that*

$$\limsup_{t \to T_0} \max_{\partial\Omega} |u_x|(t) < \infty \quad \textit{though} \quad \limsup_{t \to T_0} \max_{\overline{\Omega}} |u_x|(t) = \infty.$$

Here T_0 is the maximal existence time for (3.2) of solution u with initial data u_0.

Proof. Since for $\Omega = (a_1, a_2)$ the difference $a_2 - a_1 > 4\ell'$, we observe that there is an data $u_0 \geq 0 (u_0 \in C^{2+\nu}(\overline{\Omega}))$ such that

$$u_0(x) = 0 \quad \text{on} \quad (a_1, a_1 + \ell'), (a_2 - \ell', a_2)$$

and that u_0 has a sharply decreasing part.

Since $\inf J' = 0$, we observe that

$$\varphi(x) = U(Nx)/N + r'$$

is a supersolution of $(3.2)_1$, where U is defined in (2.10). For each $\epsilon > 0$ there is $\nu = \nu(\epsilon)$, $a_1 < \nu < a_1 + \ell'/2$ such that

$$\psi_\epsilon(x) = \varphi(x - \nu) - \epsilon$$

satisfies

$$\psi_\epsilon(a_1) = 0, \quad 0 \leq \psi_{\epsilon x}(a_1) < \infty \quad \text{and} \quad \nu(\epsilon) \to a_1 + \ell'/2$$

as $\epsilon \to 0$.

We shall use ψ_ϵ as a barrier of u near $x = a_1$. For this purpose we shall first compare u with a supersolution

$$w(x,t) = v_\lambda^{\rho M}(x - a_1 - \ell'/2,\ t) + r'$$

where v_λ^K is given by (2.13) and defined in Lemma 2.3(ii) with $M = N$. Since T_0 is the maximal existence time so that $|u_x|(x,t)$ is bounded on Ω for each $t < T_0$, we see by comparison that

$$u(x,t) \leq w(x,t), \quad a_1 \leq x \leq a_1 + \ell', \quad 0 \leq t < T_0$$

in the domain where w is defined. Since $T_0 < \delta/L$ by Theorem 3.3, this yields

$$\sup\{u(x,t);\ a_1 \leq x \leq a_1 + \ell', \quad 0 \leq t < T_0\} \leq r' - \epsilon < r'.$$

In particular,

$$u(\nu,t) \leq \psi_\epsilon(\nu)$$

for sufficiently small $\epsilon > 0$. Since $u(a_1,t) = \psi_\epsilon(a_1) = 0$, and $u(x,0) \leq \psi_\epsilon(x)$, we see by comparison that

$$u(x,t) \leq \psi_\epsilon(x)$$

for $a \leq x \leq \nu$ and $0 \leq t < T_0$. This yields an upper bound

$$u_x(a_1,t) \leq c_1, \quad 0 \leq t < T_0.$$

Since $v \equiv 0$ is a subsolution if (3.2) so that $u \geq 0$, this gives a bound for $|u_x(a_1,t)|$, $0 \leq t < T_0$. A bound for $|u_x(a_2,t)|$ is obtained in the same way so is omitted. The derivative blow-up in $\overline{\Omega}$ is proved in Lemma 3.7 so the interior derivative blow-up is proved. ∎

3.9. Corollary. *For $\alpha > 1/2$ consider the initial-boundary value problem*

$$u_t - \frac{u_{xx}}{(1+u_x^2)^\alpha} - u(1+\ u_x^2)^{1/2} = 0 \quad in \quad \Omega \times (0, T_0)$$
$$u = 0 \quad on \quad \partial\Omega \times (0, T_0)$$
$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where Ω is a bounded open interval and T_0 is the maximal existence time. Then there exist an initial data $u_0 \in C^{2+\nu}(\overline{\Omega}), 0 < \nu < 1$ compatible with $u = 0$ such that

$$\limsup_{t\to T_0} \max_{\partial\Omega} |u_x|(t) < \infty \quad though \quad \limsup_{t\to T_0} \max_{\overline{\Omega}} |u_x|(t) = \infty.$$

Proof. In the condition (F) we may take M large for $f(w) = w$ so that ℓ and ℓ' are sufficiently small. We now apply Theorem 3.8 to get the desired result of interior derivative blow-up. ∎

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