



Title	A Generalization of the Morita-Mumford Classes to Extended Mapping Class Groups for Surfaces
Author(s)	Kawazumi, N.
Citation	Hokkaido University Preprint Series in Mathematics, 292, 1-11
Issue Date	1995-4-1
DOI	10.14943/83439
Doc URL	<a href="http://hdl.handle.net/2115/69043">http://hdl.handle.net/2115/69043</a>
Type	bulletin (article)
File Information	pre292.pdf



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**A GENERALIZATION OF THE  
MORITA-MUMFORD CLASSES  
TO EXTENDED MAPPING  
CLASS GROUPS FOR  
SURFACES**

**Nariya Kawazumi**

**Series #292. April 1995**

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# A GENERALIZATION OF THE MORITA-MUMFORD CLASSES TO EXTENDED MAPPING CLASS GROUPS FOR SURFACES

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ABSTRACT. Let  $\Sigma_{g,1}$  be an orientable compact surface of genus  $g$  with 1 boundary component, and  $\Gamma_{g,1}$  the mapping class group of  $\Sigma_{g,1}$ . We define a bigraded series of cohomology classes  $m_{i,j} \in H^{2i+j-2}(\Gamma_{g,1}; \bigwedge^j H_1(\Sigma_{g,1}; \mathbb{Z}))$ ,  $2i+j-2 \geq 1$ ,  $i, j \geq 0$ . When  $j = 0$ , the class  $m_{i+1,0}$  is the  $i$ -th Morita-Mumford class [Mo][Mu]. It is proved that  $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$  is generated by  $m_{i,j}$ 's for the case  $r+s = 2$  and the case  $g \geq 5$  and  $(r, s) = (1, 3)$ . Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes  $m_{0,3}$  and  $m_{0,2}m_{1,1}$  over  $\mathbb{Q}$ .

## INTRODUCTION

Let  $g \geq 2$ ,  $r, n \geq 0$  be integers. Let  $\Sigma_{g,r}^n$  denote a 2-dimensional compact oriented  $C^\infty$  manifold (i.e., compact oriented surface) of genus  $g$  with  $r$  boundary components and (ordered)  $n$  punctures. The group of path-components  $\pi_0(\text{Diff}^+(\Sigma_{g,r}^n))$  is denoted by  $\Gamma_{g,r}^n$  (or  $\mathcal{M}_{g,r}^n$ ) and called the mapping class group of genus  $g$  with  $r$  boundary components and (ordered)  $n$  punctures. Here  $\text{Diff}^+(\Sigma_{g,r}^n)$  denotes the topological group (endowed with  $C^\infty$  topology) consisting of all orientation preserving diffeomorphisms of  $\Sigma_{g,r}^n$  which fix the boundary components and the punctures pointwise. When  $n = 0$ , we drop the indices:  $\Sigma_{g,r} = \Sigma_{g,r}^0$ ,  $\Gamma_{g,r} = \Gamma_{g,r}^0$  and similarly  $\Sigma_g = \Sigma_{g,0}^0$ ,  $\Gamma_g = \Gamma_{g,0}^0$ . Throughout this paper we denote by  $H_1(\Sigma_{g,r}^n)$  the first integral singular homology of the space  $\Sigma_{g,r}^n$ , on which the group  $\Gamma_{g,r}^m$  act in an obvious way provided that  $s \geq r$  and  $m \geq n$ .

By the *extended mapping class group* we mean the semi-direct product

$$\widetilde{\Gamma}_{g,r}^n := H_1(\Sigma_{g,1}) \rtimes \Gamma_{g,r}^n.$$

The purpose of the present paper is to define a bigraded series  $\widetilde{m}_{i,j}$  of cohomology classes of the extended group  $\widetilde{\Gamma}_{g,1}$ , which is a generalization of the Morita-Mumford cohomology classes of the group  $\Gamma_g$ , and to investigate the ones of lower degree.

In §1 we prepare a theory of cohomology of pairs of groups, which is essential to the construction of the classes in the succeeding two sections. The  $E_2$ -term of the

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1991 Mathematical Subject Classification. Primary 57R20. Secondary 14H15, 20J05, 57R32, 20F36.

Lyndon-Hochschild-Serre spectral sequence of the group  $\widetilde{\Gamma}_{g,1}$  with respect to the normal subgroup  $H_1(\Sigma_{g,1})$  is given by

$$E_2^{p,q} = H^p(\Gamma_{g,1}; \bigwedge^q H^1(\Sigma_{g,1})).$$

So the classes  $\widetilde{m}_{i,j}$  induce cohomology classes  $m_{i,j}$  of the group  $\Gamma_{g,1}$  with values in  $\bigwedge^* H^1(\Sigma_{g,1})$ . When  $j = 0$ , the class  $m_{i+1,0}$  is the  $i$ -th Morita-Mumford class [Mo][Mu]. In §4, in order to see the non-triviality, we evaluate the classes  $m_{0,2}$ ,  $m_{1,1}$  and  $m_{0,3}$  and prove that  $H^r(\Gamma_{g,1}; \bigwedge^s H^1(\Sigma_{g,1}; \mathbb{Q}))$  is generated by  $m_{i,j}$ 's for the case  $r + s = 2$  (Proposition 4.1, Theorem 4.2, Corollary 4.6) and the case  $g \geq 5$  and  $(r, s) = (1, 3)$  (Theorem 4.5). Especially the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3] has an implicit representation by the classes  $m_{0,3}$  and  $m_{0,2}m_{1,1}$  over  $\mathbb{Q}$ .

The author would like to express his gratitude to Prof. S. Morita and Prof. A. Kohno for helpful discussions.

#### Contents.

- §1. Cohomology of Pairs of Groups.
- §2. Mapping Class Groups.
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### 1. Cohomology of Pairs of Groups.

In this section we define cohomology groups  $H^*(G, H : M)$  of a pair of groups  $(G, H)$  in the most naive sense. Denote by  $C^*(G; M)$  the normalized cochain complex of a group  $G$  with values in a  $G$ -module  $M$ .

Let  $G$  be a group,  $H$  a subgroup of  $G$ , and  $M$  a  $G$ -module. We denote by  $H^*(G, H; M)$  the cohomology group of the kernel of the restriction map

$$\text{res} : C^*(G; M) \rightarrow C^*(H; M)$$

and call it *the cohomology group of the pair of groups  $(G, H)$  with values in the  $G$ -module  $M$* . Since the restriction map  $\text{res}$  is surjective in the cochain level, we have a cohomology exact sequence

$$(1.1) \quad \dots \rightarrow H^{q-1}(H; M) \rightarrow H^q(G, H; M) \rightarrow H^q(G; M) \rightarrow H^q(H; M) \rightarrow \dots,$$

In a natural way the cup product

$$\cup : H^*(G; M') \otimes H^*(G, H; M'') \rightarrow H^*(G, H; M' \otimes M'')$$

is defined.

Let  $K \triangleleft G$  be a normal subgroup satisfying the condition

$$(1.2) \quad HK = G.$$

Then we have the following Lyndon-Hochschild-Serre (LHS) spectral sequence [HS].

**Proposition 1.3.** *There is a spectral sequence converging to  $H^*(G, H; M)$  whose  $E_2$  term is given by*

$$E_2^{p,q} = H^p(G/K; H^q(K, K \cap H; M)).$$

It should be remarked how the quotient group  $G/K$  acts on the cohomology group  $H^*(K, K \cap H; M)$ . Since  $K$  is a normal subgroup of  $G$ , the group  $H$  acts on the normalized complex  $C^*(K, K \cap H; M)$  by

$$(h \cdot c)(x_1, \dots, x_n) := h(c(h^{-1}x_1h, \dots, h^{-1}x_nh)),$$

where  $h \in H$ ,  $c \in C^n(K, K \cap H; M)$  and  $x_1, \dots, x_n \in K$ . For any element  $h \in K \cap H$  consider a homotopy map

$$\Phi = \Phi_h : C^n(K, K \cap H; M) \rightarrow C^{n-1}(K, K \cap H; M)$$

given by

$$(\Phi_h c)(x_1, \dots, x_{n-1}) := \sum_{j=0}^{n-1} (-1)^j c(x_1, \dots, x_j, h, h^{-1}x_{j+1}h, \dots, h^{-1}x_{n-1}h),$$

This map is well-defined and satisfies a homotopy equation

$$(d\Phi_h + \Phi_h d)c = h \cdot c - c \quad (\forall c \in C^*(K, K \cap H; M)).$$

Hence the subgroup  $K \cap H$  acts on the cohomology group  $H^*(K, K \cap H; M)$  trivially. From the condition (1.2) and the Second Isomorphism Theorem we have a natural isomorphism

$$G/K = H/K \cap H.$$

Thus the quotient group  $G/K$  acts on the cohomology group  $H^*(K, K \cap H; M)$ .

Let  $M$ ,  $M_1$  and  $M_2$  be  $G/K$ -modules. Suppose

$$(1.4) \quad H^q(K, K \cap H; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } q = n, \\ 0, & \text{if } q > n. \end{cases}$$

Then the spectral sequence (1.3) induces a homomorphism

$$(1.5) \quad \pi_! : H^p(G, H; M) \rightarrow H^{p-n}(G/K; M),$$

which is called *the Gysin map* or *the fiber integral*. As usual we have

$$(1.6) \quad \pi_!(u \cup \pi^*v) = (\pi_!u) \cup v \in H^{p+q-n}(G/K; M_1 \otimes M_2),$$

for  $u \in H^p(G, H; M_1)$  and  $v \in H^q(G/K; M_2)$ .

## 2. Mapping Class Groups.

From now on we consider mainly the mapping class groups  $\Gamma_{g,1}$  and  $\Gamma_{g,1}^1$ . First we remark that the surface  $\Sigma_{g,1}^1$  is obtained by glueing the surfaces  $\Sigma_{g,1}$  and  $\Sigma_{0,2}^1$  along the boundaries. So the diffeomorphism of  $\Sigma_{g,1}$  is naturally extended to that of  $\Sigma_{g,1}^1$ . The infinite cyclic group  $\mathbb{Z}$  acts on the surface  $\Sigma_{0,2}^1$  by rotating the puncture and fixing the boundaries pointwise. Similarly this action is extended to that on  $\Sigma_{g,1}^1$  in a natural way. Thus we obtain a natural homomorphism  $\Gamma_{g,1} \times \mathbb{Z} \rightarrow \Gamma_{g,1}^1$ , which is injective (see [I] §5). In the sequel we regard the group  $\Gamma_{g,1} \times \mathbb{Z}$  as a subgroup of  $\Gamma_{g,1}^1$  through the injection. Especially we may consider the cohomology group  $H^*(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M)$  for an arbitrary  $\Gamma_{g,1}^1$ -module  $M$ . By forgetting the puncture we obtain an extension

$$(2.1) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \Gamma_{g,1}^1 \xrightarrow{\pi} \Gamma_{g,1} \rightarrow 1.$$

Next we prepare a cycle induced by the "fiber"  $\pi_1(\Sigma_{g,1})$ . Choose a usual symplectic generator system of the fundamental group  $\pi_1(\Sigma_{g,1})$ :

$$a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g.$$

The loop on the boundary induces an element of  $\pi_1(\Sigma_{g,1})$

$$w := \prod_{i=1}^g [a_i b_i], \quad [a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}.$$

We identify the group  $\mathbb{Z}$  with the subgroup generated by  $w$  in  $\pi_1(\Sigma_{g,1})$ , and consider the cohomology group of the pair  $H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z})$ .

Following Meyer [Me], we construct a normalized bar 2-chain  $[\Sigma_{g,1}, \partial]$  as follows. For  $1 \leq j \leq 4g$  let  $w_j = a_i^{\pm 1}, b_i^{\pm 1}$  be the  $j$ -th generator in the element  $w$ , and  $\widetilde{w}_j := w_1 w_2 \cdots w_j = a_1 b_1 \cdots w_j$ . Let  $\widetilde{w}_0 = 1$ . We define

$$(2.2) \quad [\Sigma_{g,1}, \partial] := \sum_{j=1}^{4g} [\widetilde{w}_{j-1} | w_j] - \sum_{i=1}^g ([a_i | a_i^{-1}] + [b_i | b_i^{-1}]) \in C_2(\pi_1(\Sigma_{g,1})).$$

**Lemma 2.3.** *For any trivial  $\pi_1(\Sigma_{g,1})$ -module  $M$ , we have*

$$H^*(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) = \begin{cases} H \otimes M, & \text{if } * = 1, \\ M, & \text{if } * = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where  $H = H_1(\Sigma_{g,1}; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ . The evaluation

$$\langle \cdot, [\Sigma_{g,1}, \partial] \rangle: H^2(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M) \rightarrow M$$

is a well-defined isomorphism.

The first half of the lemma follows from the exact sequence (1.1), and the second from straightforward calculations.

Now let  $M$  be a  $\Gamma_{g,1}$ -module. The condition (1.2) is satisfied for our case  $G = \Gamma_{g,1}^1$ ,  $H = \Gamma_{g,1} \times \mathbb{Z}$  and  $K = \pi_1(\Sigma_{g,1})$ . It follows from Proposition 1.3 there exists a spectral sequence converging to

$$H^*(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; M),$$

whose  $E_2$  term is given by

$$H^p(\Gamma_{g,1}; H^q(\pi_1(\Sigma_{g,1}), \mathbb{Z}; M)) = \begin{cases} H^p(\Gamma_{g,1}; H \otimes M), & \text{if } * = 1, \\ H^p(\Gamma_{g,1}; M), & \text{if } * = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence it induces a Gysin exact sequence

$$\begin{aligned} \dots \rightarrow H^{q-1}(\Gamma_{g,1}; M) \rightarrow H^{q+1}(\Gamma_{g,1}; H \otimes M) \\ \rightarrow H^{q+2}(\Gamma_{g,1}^1, \partial\Gamma_{g,1}^1; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \rightarrow \dots \end{aligned}$$

Here the homomorphism  $\pi_!$  is *the fiber integral* introduced in (1.5).

The Gysin sequence splits as follows. The identity map  $1_{\mathbb{Z}} : \mathbb{Z} \rightarrow \mathbb{Z}$  generates the cohomology group  $H^1(\mathbb{Z}) \cong \mathbb{Z}$ . Regard  $1_{\mathbb{Z}}$  as an element of  $H^1(\Gamma_{g,1} \times \mathbb{Z})$  through the natural projection  $\Gamma_{g,1} \times \mathbb{Z} \rightarrow \mathbb{Z}$  and denote by  $\theta$  the image of  $1_{\mathbb{Z}}$  under the connecting homomorphism  $\delta^*$ :

$$\theta := \delta^*(1_{\mathbb{Z}}) \in H^2(\Gamma_{g,1}^1, \partial\Gamma_{g,1}^1; \mathbb{Z}).$$

Since  $\langle \theta, [\Sigma_{g,1}, \partial] \rangle = -1$ , we have

$$(2.4) \quad \pi_! \theta = -1 \in H^0(\Gamma_{g,1}; \mathbb{Z}).$$

Thus, from the property (1.6) of the fiber integral  $\pi_!$ , the sequence splits. Consequently we have

**Proposition 2.5.** *For any  $\Gamma_{g,1}$ -module  $M$ , we have an exact sequence*

$$0 \rightarrow H^{q+1}(\Gamma_{g,1}; H \otimes M) \rightarrow H^{q+2}(\Gamma_{g,1}^1, \partial\Gamma_{g,1}^1; M) \xrightarrow{\pi_!} H^q(\Gamma_{g,1}; M) \rightarrow 0,$$

which splits as follows:

$$H^{q+2}(\Gamma_{g,1}^1, \partial\Gamma_{g,1}^1; M) = H^{q+1}(\Gamma_{g,1}; H \otimes M) \oplus \theta \cup H^q(\Gamma_{g,1}; M).$$

On the other hand, taking the semi-direct product of the extension (2.1) and the  $\Gamma_{g,1}$ -module  $H_1(\Sigma_{g,1}; \mathbb{Z})$ , we have an extension of groups

$$(2.6) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \widetilde{\Gamma}_{g,1}^1 \xrightarrow{\widetilde{\pi}} \widetilde{\Gamma}_{g,1}^1 \rightarrow 1.$$

In a similar way to the fiber integral  $\pi_!$  we obtain *the fiber integral*

$$\widetilde{\pi}_! : H^q(\widetilde{\Gamma}_{g,1}^1, \widetilde{\Gamma}_{g,1}^1 \times \mathbb{Z}; \mathbb{Z}) \rightarrow H^{q-2}(\widetilde{\Gamma}_{g,1}^1; \mathbb{Z}).$$



### 3. Construction of Cohomology Classes.

For the rest we often abbreviate

$$H := H_1(\Sigma_{g,1}; \mathbb{Z}) = H^1(\Sigma_{g,1}; \mathbb{Z}).$$

The isomorphism on the right-hand side is the Poincaré duality, which is  $\Gamma_{g,1}$ -equivariant. We remark this  $H$  plays a different role in the sequel from the subgroup  $H$  in the preceding sections.

The surface  $\Sigma_{g,1}^1$  may be regarded as a subspace of  $\Sigma_{g,1}$  in an obvious way, so one obtains a natural homology exact sequence

$$(3.1) \quad 0 \rightarrow H_2(\Sigma_{g,1}, \Sigma_{g,1}^1) = \mathbb{Z} \xrightarrow{\iota} H_1(\Sigma_{g,1}^1) \rightarrow H_1(\Sigma_{g,1}) = H \rightarrow 0,$$

which induces a central extension

$$(3.2) \quad 0 \rightarrow \mathbb{Z} \rightarrow H_1(\Sigma_{g,1}^1) \times \Gamma_{g,1}^1 \rightarrow \widetilde{\Gamma_{g,1}^1} \rightarrow 0.$$

Let

$$\tilde{\omega} \in H^2(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z})$$

denote the 2-dimensional cohomology class defined by the central extension (3.2). In order to investigate the class  $\tilde{\omega}$  geometrically, we choose a simple curve  $l$  on  $\Sigma_{g,1}^1$  connecting the puncture to a point on the boundary. Define a 2-chain  $\tilde{\omega}_l \in C^2(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z})$  by

$$(3.3) \quad \tilde{\omega}_l(u_1\gamma_1, u_2\gamma_2) := \gamma_1(\gamma_2 l - l) \cdot u_1, \quad u_1, u_2 \in H, \gamma_1, \gamma_2 \in \Gamma_{g,1}^1,$$

where we remark  $\gamma_2 l - l$  can be regarded as a closed curve on  $\Sigma_{g,1}$ .

**Lemma 3.4.** *The 2-cochain  $\tilde{\omega}_l$  is a cocycle and its cohomology class is equal to  $\tilde{\omega}$ :*

$$\tilde{\omega} = [\tilde{\omega}_l] \in H^2(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z}).$$

*Proof.* Since the intersection number  $l \cdot \iota(1)$  is equal to 1, the curve  $l$  induces a splitting homomorphism  $\sigma = \sigma_l : H_1(\Sigma_{g,1}) \rightarrow H_1(\Sigma_{g,1}^1)$  of the exact sequence (3.1). By definition the 2-cocycle induced by the splitting homomorphism  $\sigma_l$  is given by

$$\begin{aligned} & \iota^{-1}(\sigma(u_1)\gamma_1\sigma(u_2)\gamma_2(\sigma(u_1\gamma_1(u_2))\gamma_1\gamma_2)^{-1}) = \iota^{-1}(\gamma_1(\sigma(u_2)) - \sigma(\gamma_1(u_2))) \\ & = l \cdot \gamma_1(\sigma(u_2)) - l \cdot \sigma(\gamma_1(u_2)) = l \cdot \gamma_1(\sigma(u_2)) = (\gamma_1^{-1}(l) - l) \cdot \sigma(u_2) \\ & = (\gamma_1^{-1}(l) - l) \cdot u_2. \end{aligned}$$

Here we use the fact  $l \cdot \sigma(u_2) = 0$ .

On the other hand give  $c \in C^1(\widetilde{\Gamma_{g,1}^1}; \mathbb{Z})$  by

$$c(u\gamma) := l \cdot \gamma\sigma(\gamma^{-1}(u)), \quad u \in H, \gamma \in \Gamma_{g,1}^1,$$

then we have

$$dc(u_1\gamma_1, u_2\gamma_2) = \tilde{\omega}_l(u_1\gamma_1, u_2\gamma_2) - (\gamma_1^{-1}(l) - l) \cdot u_2.$$

Thus  $d\tilde{\omega}_l = 0$  and the cohomology class of  $\tilde{\omega}_l$  is equal to  $\tilde{\omega}$ .  $\square$

Next consider the 1-chain  $\omega_l \in C^1(\Gamma_{g,1}^1; H)$  defined by

$$(3.5) \quad \omega_l(\gamma) = \gamma l - l \in H, \quad \gamma \in \Gamma_{g,1}^1.$$

Recall  $H^1(\Gamma_{g,1}^1; H)$  is the  $E_2^{1,1}$ -term of the LHS spectral sequence of  $\widetilde{\Gamma}_{g,1}$  with respect to the normal subgroup  $H$ . Since we have

$$\tilde{\omega}_l(u_1, u_2\gamma_2) = \omega_l(\gamma_2) \cdot u_1$$

for  $\forall u_1, u_2 \in H$  and  $\gamma_2 \in \Gamma_{g,1}^1$ , the cochain  $\omega_l$  is a 1-cocycle and its cohomology class

$$(3.6) \quad \omega := [\omega_l] \in H^1(\Gamma_{g,1}^1; H)$$

is equal to that induced by  $\tilde{\omega} \in H^2(\widetilde{\Gamma}_{g,1}^1; \mathbb{Z})$ , and so the class is independent of the choice of  $l$ . We remark that

$$(3.7) \quad \omega_l(\gamma) = -\gamma \in H, \quad \forall \gamma \in \pi_1(\Sigma_{g,1}).$$

Now we can define the cohomology classes  $\widetilde{m}_{i,j}$  and  $m_{i,j}$ . Consider two extensions of groups

$$(2.1) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \Gamma_{g,1}^1 \xrightarrow{\pi} \Gamma_{g,1}^1 \rightarrow 1$$

$$(2.6) \quad 1 \rightarrow \pi_1(\Sigma_{g,1}) \rightarrow \widetilde{\Gamma}_{g,1}^1 \xrightarrow{\tilde{\pi}} \Gamma_{g,1}^1 \rightarrow 1.$$

Let  $e \in H^2(\Gamma_g^1; \mathbb{Z})$  be the Euler class of the central extension

$$1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1} \rightarrow \Gamma_g^1 \rightarrow 1.$$

The class  $e$  may be regarded as a cohomology class in  $H^2(\Gamma_{g,1}^1, \Gamma_{g,1} \times \mathbb{Z}; \mathbb{Z})$  in an obvious way. We define

$$(3.8) \quad \begin{aligned} m_{i,j} &:= \pi_!(e^i \omega^j) \in H^{2i+j-2}(\Gamma_{g,1}; \bigwedge^j H) \\ \widetilde{m}_{i,j} &:= \tilde{\pi}_!(e^i \omega^j) \in H^{2i+2j-2}(\widetilde{\Gamma}_{g,1}; \mathbb{Z}) \end{aligned}$$

for  $i, j \in \mathbb{N}$ . Here  $\pi_!$  and  $\tilde{\pi}_!$  are the fiber integrals introduced in the previous section. Clearly  $m_{i+1,0}$  and  $\widetilde{m}_{i+1,0}$  are equal to (the image of) the  $i$ -th Morita-Mumford (tautological) class  $e_i (= \kappa_i) \in H^{2i}(\Gamma_g; \mathbb{Z})$  [Mo][Mu]:

$$(3.9) \quad m_{i+1,0} = \widetilde{m}_{i+1,0} = e_i \in H^{2i}(\Gamma_{g,1}; \mathbb{Z}).$$

*Remark 3.10.* Let  $\mathcal{F}_{g-1}$  be the dressed moduli of pairs of compact Riemann surfaces of genus  $g$  and holomorphic line bundles of degree  $g-1$  on the surfaces. The space  $\mathcal{F}_{g-1}$  is aspherical and its  $\pi_1$  is equal to  $\widetilde{\Gamma}_{g,1}$ . As is known, the Lie algebra of holomorphic differential operators "near  $S^1$ " has an infinitesimal and transitive action on the dressed moduli  $\mathcal{F}_{g-1}$  [ADKP]. The  $\widetilde{m}_{i,j}$ 's have their origins in the equivariant cohomology of  $\mathcal{F}_{g-1}$  under this action [Ka1].

#### 4. Evaluations.

The purpose of this section is to evaluate the classes  $m_{2,0}$ ,  $m_{1,1}$  and  $m_{0,3}$  and to prove that  $H^r(\Gamma_{g,1}; \bigwedge^s H_1(\Sigma_{g,1}; \mathbb{Q}))$  is generated by  $m_{i,j}$ 's for the case  $r + s = 2$  and the case  $g \geq 5$  and  $(r, s) = (1, 3)$ .

Denote by  $\Omega$  the symplectic form on  $H$  induced by the cup product:

$$\Omega := \sum_{i=1}^g a_i \otimes b_i - b_i \otimes a_i \in \bigwedge^2 H,$$

where  $\{a_i, b_i; 1 \leq i \leq g\}$  is (the homology classes induced by) a symplectic generator system of the fundamental group  $\pi_1(\Sigma_{g,1})$  as in §2.

#### Proposition 4.1.

$$m_{0,2} = \pi_1(\omega^2) = 2\Omega \in H^0(\Gamma_{g,1}; \bigwedge^2 H).$$

*Proof.* It suffices to show that

$$\langle \omega^2, [\Sigma_{g,1}, \partial] \rangle = 2\Omega.$$

Here  $[\Sigma_{g,1}, \partial]$  is a 2-chain introduced in (2.2). Since  $\omega(\widetilde{w_{4i}}) = 0$ , we have

$$\begin{aligned} \langle \omega^2, [\Sigma_{g,1}, \partial] \rangle &= \sum_{j=1}^{4g} \omega^2(\widetilde{w_{j-1}}, w_j) - \sum_{i=1}^g (\omega^2(a_i, a_i^{-1}) + \omega^2(b_i, b_i^{-1})) \\ &= \sum_{i=1}^g a_i \wedge b_i - (a_i + b_i) \wedge a_i - (a_i + b_i - a_i) \wedge b_i + a_i \wedge a_i + b_i \wedge b_i \\ &= \sum_{i=1}^g a_i \wedge b_i - b_i \wedge a_i = 2\Omega, \end{aligned}$$

as was to be shown.  $\square$

Next we study the classes  $m_{1,1}$  and  $m_{0,3}$ . In [Mo1] and [Mo2] Morita proved

$$(4.2) \quad H^1(\Gamma_{g,1}; H) = \mathbb{Z}, \quad \text{and} \quad H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z}^2,$$

where we denote  $H = H_1(\Sigma_{g,1}; \mathbb{Z})$  as before. Our results are

**Theorem 4.3.** *The class  $m_{1,1}$  generates the group  $H^1(\Gamma_{g,1}; H)$ .*

**Theorem 4.4.** *If  $g \geq 5$ , the classes  $m_{0,2}m_{1,1}$  and  $m_{0,3}$  generate the group  $H^1(\Gamma_{g,1}; \bigwedge^3 H \otimes \mathbb{Q})$ .*

The rest of this section is devoted to the proof of the theorems. As was shown by Harer [H], if  $g \geq 3$ , we have  $H^2(\Gamma_{g,1}; \mathbb{Q}) = \mathbb{Q}$  and the class  $m_{2,0} = e_1$  generates it. Hence in the case  $r + s = 2$  the groups  $H^r(\Gamma_{g,1}; \bigwedge^s H \otimes \mathbb{Q})$  are generated by the classes  $m_{i,j}$ 's. Consequently

**Corollary 4.5.** *If  $g \geq 3$ , the group  $H^2(\widetilde{\Gamma}_{g,1}; \mathbb{Q})$  is isomorphic to  $\mathbb{Q}^3$  and the classes  $\widetilde{m}_{0,2}$ ,  $\widetilde{m}_{1,1}$  and  $\widetilde{m}_{2,0}$  form its free basis.*

The first half of the corollary has been already shown by Arbarello et. al. ([ADKP] §5).

To prove the theorems we endow the surface  $\Sigma_g$  with a Riemannian metric. Fix a sufficiently small positive real  $\epsilon$ . Let  $\varpi : ST\Sigma_g \rightarrow \Sigma_g$  be the unit tangent bundle of the surface  $\Sigma_g$ . Denote by  $D^2$  the unit disk in  $\mathbb{C}$ :  $D^2 := \{z \in \mathbb{C}; |z| \leq 1\}$ . We define a disk bundle  $D_g$  over  $ST\Sigma_g$  by

$$D_g := \{(v_1, x_2) \in ST\Sigma_g \times \Sigma_g; \text{dist}(\varpi(v_1), x_2) \leq \epsilon\},$$

The first projection induces its projection  $p_1 : D_g \rightarrow ST\Sigma_g$ . The disk bundle is trivial through the projection

$$ST\Sigma_g \times D^2 \rightarrow D_g, \quad (v, z) \mapsto (v, \text{Exp}_{\varpi(v)}(\epsilon z v)).$$

Here we use the (almost) complex structure induced by the given Riemannian metric.

Consider a  $\Sigma_{g,1}$ -bundle

$$p_1 : Y_g := ST\Sigma_g \times \Sigma_g - \text{int } D_g \rightarrow ST\Sigma_g$$

induced by the first projection. The fundamental group  $\pi_1(ST\Sigma_g)$  is embedded into the group  $\Gamma_{g,1}$  through the classifying map  $\iota$  of the bundle  $p_1 : Y_g \rightarrow ST\Sigma_g$ , and is identified with the kernel of the forgetting map  $\Gamma_{g,1} \rightarrow \Gamma_g$ :

$$1 \rightarrow \pi_1(ST\Sigma_g) \xrightarrow{\iota} \Gamma_{g,1} \rightarrow \Gamma_g \rightarrow 1.$$

Since the spaces  $\Sigma_g$ ,  $ST\Sigma_g$ ,  $D_g$  and  $Y_g$  are all aspherical, we drop the notations  $\pi_1(\cdot)$  in the cohomology groups.

The identity map  $1_H \in \text{Hom}(H, H)$  induces a cohomology class

$$1_H \in H^1(\Sigma_g; H) \cong \text{Hom}(H, H).$$

By abuse of notation we denote also by  $1_H$  the pull-back  $\varpi^*(1_H)$  through the projection  $\varpi : ST\Sigma_g \rightarrow \Sigma_g$ :

$$1_H = \varpi^*(1_H) \in H^1(ST\Sigma_g; H) \cong \text{Hom}(H, H).$$

In [Mo1] Morita proved the following theorem (see also [Mo2] p.81 1.4 ff).

**Theorem 4.6 (Morita).**

$$H^1(\Gamma_{g,1}; H) = \mathbb{Z}.$$

Furthermore a crossed homomorphism  $k : \Gamma_{g,1} \rightarrow H$  represents a generator of the group  $H^1(ST\Sigma_g; H)$  if and only if the restriction of  $k$  to  $\pi_1(ST\Sigma_g)$  is equal to  $\pm(2 - 2g)1_H$ :

$$\iota^*(k) = \pm(2 - 2g)1_H \in H^1(ST\Sigma_g; H).$$

As for  $\bigwedge^3 H = \bigwedge^3 H_1(\Sigma_{g,1}; H)$  he proved the following ([Mo3] Theorem 5.1, see also the proof of Corollary 5.7). Let  $k_0$  be a generator of the group  $H^1(\Gamma_{g,1}; H)$ .

**Theorem 4.7 (Morita).** *If  $g \geq 3$ ,*

$$H^1(\Gamma_{g,1}; \bigwedge^3 H) = \mathbb{Z} \oplus \mathbb{Z}.$$

*The class  $\Omega \wedge k_0$  and a class he named  $2\tilde{k}$  form its free basis. Furthermore their restriction to  $\pi_1(ST\Sigma_g)$  are given by*

$$\begin{aligned} \iota^*(\Omega \wedge k_0) &= \pm(2 - 2g)\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H), \\ \iota^*(2\tilde{k}) &= 2\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H). \end{aligned}$$

Therefore our theorems are reduced to

**Assertion 4.8.**

- (1)  $\iota^*(m_{1,1}) = -(2 - 2g)1_H \in H^1(ST\Sigma_g; H)$
- (2)  $\iota^*(m_{0,3}) = -6\Omega \wedge 1_H \in H^1(ST\Sigma_g; \bigwedge^3 H)$

In fact, (1) implies Theorem 4.3 by Theorem 4.6. So we have  $m_{2,0}m_{1,1} = \pm 2\Omega \wedge k_0$ . From Theorem 4.7 the class  $m_{0,3}$  has a representation  $m_{0,3} = a\Omega \wedge k_0 + b(2\tilde{k})$  for some integers  $a$  and  $b$ . Since  $H^1(ST\Sigma_g; \bigwedge^3 H) = H \otimes \bigwedge^3 H$  is  $\mathbb{Z}$ -free, we have

$$-6 = \pm a(2 - 2g) + 2b,$$

and so  $b \equiv -3 \pmod{g-1}$ , while  $g-1 \geq 4$ . Thus we have  $b \neq 0$ .

This completes the proof of Theorems 4.3 and 4.4 modulo Assertion 4.8.

Let  $M$  be a  $\pi_1(ST\Sigma_g)$ -module. By excision we may consider the map

$$j^* : H^*(Y_g, \partial Y_g; M) \underset{\text{exc.}}{\cong} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \rightarrow H^*(ST\Sigma_g \times \Sigma_g; M).$$

The fiber integral  $p_{1!} : H^*(Y_g, \partial Y_g; M) \rightarrow H^{*-2}(ST\Sigma_g; M)$  decomposes itself into

$$H^*(Y_g, \partial Y_g; M) \xrightarrow{j^*} H^*(ST\Sigma_g \times \Sigma_g, D_g; M) \xrightarrow{p_{1!}} H^{*-2}(ST\Sigma_g; M).$$

Here the latter fiber integral  $p_{1!}$  is the usual one induced by the first projection  $p_1 : ST\Sigma_g \times \Sigma_g \rightarrow ST\Sigma_g$ . Thus we have

$$\iota^* m_{1,1} = p_{1!} j^*(e\omega) \quad \text{and} \quad \iota^* m_{0,3} = p_{1!} j^*(\omega^3).$$

Now we have

$$\begin{aligned} j^*(e) &= p_2^* e' \in H^2(ST\Sigma_g \times \Sigma_g; \mathbb{Z}) \\ j^*(\omega) &= p_2^* 1_H - p_1^* 1_H \in H^1(ST\Sigma_g \times \Sigma_g; H), \end{aligned}$$

where  $p_2 : ST\Sigma_g \times \Sigma_g \rightarrow \Sigma_g$  is the second projection and

$$e' = e(T\Sigma_g) \in H^2(\Sigma_g; \mathbb{Z}).$$

Since  $e'1_H \in H^3(\Sigma_g; H) = 0$ , we have

$$\begin{aligned} \iota^* m_{1,1} &= p_{11} j^*(e\omega) = p_{11}(p_2^* e')(p_2^* 1_H - p_1^* 1_H) \\ &= -(p_{11} p_2^* e')1_H = -(2 - 2g)1_H. \end{aligned}$$

On the other hand, since  $(1_H)^3 \in H^3(\Sigma_g; \wedge^3 H) = 0$  and  $p_{11} p_2^* 1_H \in H^{-1}(ST\Sigma_g; H) = 0$ , we have

$$j^*(\omega^3) = (p_2^* 1_H - p_1^* 1_H)^3 = -3(p_2^*(1_H)^2)p_1^* 1_H + 3(p_2^* 1_H)p_1^*(1_H)^2$$

and

$$p_{11} j^*(\omega^3) = -3(p_{11} p_2^*(1_H)^2)1_H + 3(p_{11} p_2^* 1_H)(1_H)^2 = -3 \langle (1_H)^2, [\Sigma_g] \rangle 1_H,$$

where we denote by  $[\Sigma_g] \in H_2(\Sigma_g; \mathbb{Z})$  the fundamental class. From a similar calculation to Proposition 4.1 follows  $\langle (1_H)^2, [\Sigma_g] \rangle = 2\Omega$ . Therefore

$$\iota^* m_{0,3} = p_{11} j^*(\omega^3) = -6\Omega \wedge 1_H.$$

This completes the proof of Assertion 4.8 and so those of Theorems 4.3 and 4.4.

*Remark 4.9.* The crossed homomorphism  $\tilde{k} = \frac{1}{2}2\tilde{k} : \Gamma_{g,1} \rightarrow \frac{1}{2} \wedge^3 H$  in (4.7) is the Johnson homomorphism extended to the whole mapping class group by Morita [Mo3]. Hence Theorem 4.4 implies the Johnson homomorphism  $\tilde{k}$  is represented by  $m_{0,3}$  and  $m_{0,2}m_{1,1}$  over  $\mathbb{Q}$ . The author, however, doesn't know the explicit representation of  $\tilde{k}$  by  $m_{0,3}$  and  $m_{0,2}m_{1,1}$ .

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