THE DISTANCE FUNCTION AND DEFECT ENERGY

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1. Introduction

It is important to measure the energy of jump discontinuities of a unit length gradient field $\nabla\varphi$ in a bounded Lipschitz domain in $\mathbb{R}^n$.

Such problems arise in the modelling of smectic liquid crystals [SK], [AG1] or of the blistering of thin films [OG]. The quantity measuring the energy of the jump discontinuities, the defect of $\nabla\varphi$, is

$$J^\beta(\varphi) = \int_\Sigma |\nabla\varphi^+ - \nabla\varphi^-|^\beta d\mathcal{H}^{n-1}$$

where $\beta > 0$; we call it a defect energy. Here Σ is the set of jump discontinuities of $\nabla\varphi$ and $\nabla\varphi^\pm$ is the trace of $\nabla\varphi$ of each side of Σ; $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure which is the surface element when Σ is smooth.

There may be a lot of Lipschitz solutions of the eikonal equation

$$|\nabla\varphi| = 1 \quad \text{in} \quad \Omega \quad \text{with} \quad \varphi = 0 \quad \text{on} \quad \partial\Omega,$$

but the distance function

$$d = d(x, \partial\Omega) = \inf\{|x-y|;\ y \in \partial\Omega\}$$

is the unique viscosity solution of the problem [CIL]. In other words the theory of viscosity solutions selects a solution of the eikonal equation. There is a fundamental question whether the distance function minimizes J^β among all *nonnegative* solutions of the eikonal equation.

If the space dimension n equals one, J^β just measures a constant multiple of the number of jumps of $\nabla\varphi$. There is no solution of the eikonal equation having no defect satisfying the zero boundary condition. Thus, the distance function is a (unique) minimizer of J^β since it has only one jump of the derivative of φ. However, for multidimensional case, the situation is different.

In this paper, we focus on the case $\beta = 1$ because of independent interest related to the total variation of the Hessian

$$I(\varphi) = \int_\Omega |\nabla^2\varphi|.$$

This integral is closely related to J^1. Indeed, if φ is piecewise linear, more precisely, $\nabla^2\varphi = 0$ (as a measure) outside Σ, then

$$I(\varphi) = \int_\Sigma |\nabla^2\varphi| = \int_\Sigma |\nabla^+\varphi\cdot\nu - \nabla\varphi^-\cdot\nu| d\mathcal{H}^{n-1}$$

where ν is the approximate normal of Σ [G]. Since the tangential component of $\nabla\varphi$ is approximately continuous, $|\nabla\varphi^+\cdot\nu - \nabla\varphi^-\cdot\nu| = |\nabla\varphi^+ - \nabla\varphi^-|$ if $|\nabla\varphi| = 1$. Thus, $I(\varphi) = J^1(\varphi)$ for piecewise linear φ. Our principal results are

(i) the distance function is the unique minimizer of $I(\varphi)$ among all nonnegative (Lipschitz) solutions of the eikonal equation $|\nabla\varphi| = 1$ in Ω with $\varphi = 0$ on $\partial\Omega$ provided that Ω is convex and $n = 2$. The values of minimum equals $\mathcal{H}^{n-1}(\partial\Omega)$.

(ii) there is a simply connected nonconvex domain Ω in $\mathbb{R}^2$ such that the distance function is not a minimizer of J^1 nor I.

This suggests that the selection mechanism of the ground state by I or J^1 is different from that in the theory of viscosity solutions, in general.

To show (i) we first observe that

$$|\Delta\varphi| = |\nabla^2\varphi|$$

as measures if φ solves $|\nabla\varphi| = 1$ and $n = 2$. This depends on the fact that $\nabla^2\varphi$ has rank one which is easy to observe heuristically. Differentiating $|\nabla\varphi| = 1$ implies that one of eigenvalues of $\nabla^2\varphi$ always equal zero. To carry out this idea we appeal to the theory of functions of bounded variation [G]. Note that the singular part (w.r.t. the Lebesgue measure) of $\nabla^2\varphi$ always has rank one [Al], [AG2]. Another key observation is

$$\int_\Omega |\Delta\varphi| \geq \int_\Omega -\Delta\varphi = \mathcal{H}^{n-1}(\partial\Omega)$$

if $|\nabla\varphi| = 1$, $\varphi \geq 0$ in Ω with $\varphi = 0$. The last equality formally follows from integration by parts and the fact that $|\nabla\varphi|$ agrees with inward normal derivative of φ on $\partial\Omega$. In section 2 we state these observations in a rigorous way allowing that $\nabla^2\varphi$ is a measure. If Ω is convex, the distance function d is concave in Ω so that $-\Delta d \geq 0$ in Ω (in the distribution senese). Thus

$$\int_\Omega |\Delta d| = \int_\Omega -\Delta d = \mathcal{H}^{n-1}(\partial\Omega)$$

so that d minimizes I as well as $\int_\Omega |\Delta\varphi|$. It turns out that d is a unique minimizer among all φ, $|\nabla\varphi| = 1$, $\varphi \geq 0$ in Ω with $\varphi = 0$ on $\partial\Omega$. The inequality

$$\int_\Omega |\Delta\varphi| \geq \int_\Omega -\Delta\varphi$$

is not sharp unless Ω is convex. In other words the minimum of I is strictly greater than $\mathcal{H}^{n-1}(\partial\Omega)$ (Theorem 2.5.) The proof of (ii) depends on an explicit construction of the domain Ω.

As a corollary of (i) we get: if d is piecewise linear, more precisely, $\nabla^2 d = 0$ outside the defect as a measure, then d also minimizes J^1 (among all nonnegative solutions of the eikonal equations) provided that the domain is convex. Note that such d exist if and only if the domain in a convex polygon as shown in Remark in §2.1.

Our counterexamples are interesting for the study of minimizers of singular perturbed variational problem

$$E_\varepsilon(\varphi) = \int_\Omega W(\nabla\varphi) + \varepsilon^2 \int_\Omega |\nabla^2\varphi|^2, \quad W(p) = (1 - |p|^2)^\sigma, \quad \sigma > 0$$

in a plane domain Ω with $\varphi = 0$ on $\partial\Omega$. Since the Euler-Lagrange equation is fourth order, we are entitled to impose another boundary condition. The natural choice seems to be $\partial u/\partial\nu = -1$, where ν is the unit outward normal of $\partial\Omega$. We divide E_ε by ε so that we hope that the energy has a nonzero limit as $\varepsilon \to 0$.

Formal analysis for $\sigma = 2$ done by [AG1], [OG] suggests that this problem has Gamma-limit

$$\tilde{J} = 2\int_\Sigma |\nabla\varphi^+ \cdot \nu - \nabla\varphi^- \cdot \nu| \int_{-b}^{b} |1 - (a^2 + \tau^2)|^{\sigma/2} d\tau d\mathcal{H}^{n-1}, \quad a = |(\nabla\varphi)_{\tan}|,\ b = (1-a)^{1/2},$$

where $(\nabla\varphi)_{\tan}$ denotes the tangential component of $\nabla\varphi^+$ (or $\nabla\varphi^-$). Since $|\nabla\varphi| = 1$, we see $\tilde{J}$ is a positive constant times $J^{\sigma+1}$. This $\tilde{J}$ (or $J^{\sigma+1}$) is to be minimized subject to the same boundary condition as for E_ε, and the interior condition $|\nabla\varphi| = 1$ a.e. in Ω.

It is tempting to think that the minimizer φ_ε of E_ε tends to

$$\varphi_0(x) = d(x, \partial\Omega).$$

as $\varepsilon \to 0$. Similarly, it is tempting to think that this function might by a mininizer of $\tilde{J}$ (or $J^{\sigma+1}$). These conjectures are more or less explicit in [OG] (cf. [AG1] for $\sigma = 2$).

An extended version of our examples (Theorem 3.5) says that the second conjecture is false for some nonconvex domain at least for $\sigma < \beta_0 - 1$ with some $\beta_0 > 1$ close to one. Unfortunately in our examples β_0 is less than 3 so they do not solve the original conjecture for $\sigma = 2$. However, they are important because they show some possible pitfalls. In particular, they show that if these conjectures are true for $\sigma = 2$, then the reasons must be subtle since other equally reasonable-sounding statements are false.

The limiting process of E_ε as $\varepsilon \to 0$ is not at all clear compared with the case when $\nabla\varphi$ is a scalar function. Such a convergence problem is studied in [KM1], [KM2] for $\nabla\varphi$ when W has isolated equal minimums.

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2. Estimate of total variations of gradient field of length one

We are concerned with total variation of $\nabla^2\psi$ in a bounded two dimensional domain Ω when $|\nabla\psi| = 1$, $\psi \geq 0$ on Ω and $\psi = 0$ on the boundary $\partial\Omega$. Our principal result in this section is that the minimum of the total variation is attained (uniquely) at the distance function provided that Ω is convex.

2.1. Notation. Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. For a Lebesgue integrable function φ, i.e. $\varphi \in L^1(\Omega)$, let $\nabla\varphi = (\partial_i\varphi)_{i=1}^n$ and $\nabla^2\varphi = (\partial_i\partial_j\varphi)$ $(1 \leq i, j \leq n)$ a distributional gradient and Hessian of φ, respectively. Let X be the space of $\varphi \in L^1(\Omega)$ such that $\partial_i\varphi \in L^1(\Omega)$ $(1 \leq i \leq n)$ and $\partial_i\partial_j\varphi$ is a finite Radon measure on Ω $(1 \leq i, j \leq n)$. In other words, $\partial_i\varphi$ is a function of (essentially) bounded variation, i.e. $\partial_i\varphi \in BV(\Omega)$. Let us recall fundamental decomposition of $\nabla^2\varphi$ for $\varphi \in X$; see e.g. [AG2]. Let Ω_0 be the largest subset in Ω such that $\nabla^2\varphi$ is absolutely continuous in Ω_0 and let Σ be the set of jump discontinuities of $\nabla\varphi$. Then

$$\nabla^2\varphi = \nabla^2\varphi\lfloor\Omega_0 + \nabla^2\varphi\lfloor(\Omega - \Omega_0 - \Sigma) + \nu \otimes (\nabla\varphi^+ - \nabla\varphi^-)\mathcal{H}^{n-1}\lfloor\Sigma.$$

Here for a set Z and measure μ we associate a new measure $\mu\lfloor Z$ by

$$(\mu\lfloor Z)(B) = \mu(Z \cap B), B \subset \Omega.$$

The vector field ν is the approximate unit normal of Σ and $\nabla\varphi^{\pm}$ is the trace of $\nabla\varphi$ on Σ in the direction of $\pm\nu$; $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. The first term $(\nabla^2\varphi)^{ab} = \nabla^2\varphi\lfloor\Omega_0$ is often called the *absolutely continuous* part of $\nabla^2\varphi$. We always identify $(\nabla^2\varphi)^{ab}$ with corresponding locally Lebesgue integrable function in Ω_0. The second term is often called the *mild* part and it lies on a non rectifiable set $\Omega - \Omega_0 - \Sigma$ of Lebesgue measure zero. The sum of last two terms is called the *singular* part of $\nabla^2\varphi$. Let Y be the space of $\varphi \in X$ such that $\nabla\varphi$ has no absolute continuous part and no mild part. In other words

$$Y = \{\varphi \in X; \nabla^2\varphi = \nu \otimes (\nabla\varphi^+ - \nabla\varphi^-)\mathcal{H}^{n-1}\lfloor\Sigma\}.$$

Let A be the space of $\varphi \in X$ such that $|\nabla\varphi| = 1$ a.e. in Ω with $\varphi = 0$ on $\partial\Omega$. We needs three subclasses of A

$$A_+ = \{\varphi \in A; \varphi \geq 0 \text{ in } \Omega\}, \quad A^0 = A \cap Y, \quad A_+^0 = A_+ \cap A^0.$$

We consider two integrals for $\varphi \in X$ which measure jumps of $\nabla\varphi$.

$$I(\varphi) = \int_\Omega |\nabla^2\varphi|,$$
$$J^\beta(\varphi) = \int_\Sigma |\nabla\varphi^+ - \nabla\varphi^-|^\beta d\mathcal{H}^{n-1} \quad \text{for} \quad \beta > 0.$$

Since $\nabla^2\varphi$ is a finite Radon measure, in the representation

$$\int_\Omega |\nabla^2\varphi| = \sup\{\sum_{1\leq i,j\leq n} \int_\Omega \theta^{ij}\partial_i\partial_j\varphi; \sum_{i,j} |\theta_{ij}|^2 \leq 1\}$$

the test function θ_{ij} is allowed to be $\theta_{ij} \in C^1(\overline{\Omega})$ not necessarily compactly supported.

Remark. The set A_+^0 and even A^0 may be empty. In fact, A_+^0 (and A^0) is empty if $\partial\Omega$ has a 'curved' part. Conversely, A_+^0 is nonempty if Ω is a polygon. The proof is by the induction of numbers of verteces of Ω. If Ω is a triangle, the distance function d is certainly piecewise linear. If Ω is a polygon of m (> 3) verteces, we set

$$\rho(x, \partial\Omega) = \min\{d(x, L(S)); L(S) \text{ is a straight line containing an edge } S \text{ of } \partial\Omega\}$$

$$d_* = \inf\{\rho(x, \partial\Omega); \text{there is at least three edges } S_1, S_2, S_3 \text{ of } \partial\Omega$$
$$\text{such that } \rho(x, \partial\Omega) = d(x, L(S_i)), i = 1, 2, 3 \text{ and } x \in \Omega\}.$$

It is easy to see that ρ is piecewise linear in

$$\Omega_* = \{x \in \Omega, d(x) < d_*\}.$$

By the choice of d_* the set $K = \Omega - \Omega_*$ is a closed polygon with at most $m-1$ verteces; it may have no interior so that K is a set of points or segments. Let Ω' be the interior of K so that it is a polygon with at most $m-1$ verteces. By the induction the function $\rho(x, \partial\Omega')$ is piecewise linear in Ω'. Since

$$\rho(x, \partial\Omega) = \rho(x, \partial\Omega') + d_* \quad \text{in } \Omega',$$

and $\rho = \rho(x, \partial\Omega)$ is piecewise linear in Ω_*, we see ρ is piecewise linear in Ω. This shows that $\rho \in A_+^0$ so that A_+^0 is nonempty.

Note that ρ is the distance function d if and only if Ω is a convex polygon. If Ω is nonconvex, d is not piecewise linear, so $d \in A_+^0$ if and only if Ω is a convex polygon.

We conclude this remark by pointing out that there is a domain Ω whose boundary is piecewise linear with infinite verteces such that $\rho \in A_+^0$. For example if we consider

$$\Omega = \{(x, y); |x| < 1, 1 > y > h(x)\}$$

$$h(x) = \begin{cases} \frac{1}{2^{\ell+1}} - |x - \frac{3}{2^{\ell+1}}| & \frac{1}{2^\ell} < x \leq \frac{1}{2^{\ell-1}}, \ell = 1, 2, \cdots \\ 0 & x \leq 0 \end{cases}$$

then $\rho \in A_+^0$. See figure 1.

2.2. Comparison Lemma of Hessian and Laplacian measure. *Assume that $n = 2$. Then for $\varphi \in A$,*

$$|\Delta\varphi| = |\nabla^2\varphi| \quad \textit{(as measures)}.$$

This is formally true since $\nabla^2\varphi$ is rank one. Indeed, differentiating $|\nabla\varphi|^2 = 1$ yields

$$\sum_{j=1}^{n}(\partial_i\partial_j\varphi)\partial_j\varphi = 0.$$

We shall justify this observation for general Hessian measure $\nabla^2\varphi$ of $\varphi \in A$. We say that for $\varphi \in X$ the rank of matrix of the Radon-Nikodym derivative

$$F(x) = \lim_{r\downarrow 0} \nabla^2\varphi(B_r(x))/|\nabla^2\varphi|(B_r(x))$$

is the rank of $\nabla^2\varphi$, where $B_r(x)$ denotes the closed ball of radius r centered at $x \in \Omega$. The rank of $\nabla\varphi$ is defined for $|\nabla^2\varphi|$-almost every point x of Ω. Since $\nabla^2\varphi$ is absolutely continuous with respect to $|\nabla^2\varphi|$,

$$|\Delta\varphi|(Z) = \int_Z |\text{trace } F|d\mu, \ |\nabla^2\varphi|(Z) = \int_Z |F|d\mu$$

with $\mu = |\nabla^2\varphi|$, where $|F|$ is the Hilbert-Schmidt norm of F, i.e., $|F|^2 = \Sigma_{ij}|F_{ij}|^2$. If F is rank 1, then

$$|\text{trace } F| = |F|$$

so that $|\Delta\varphi| = |\nabla^2\varphi|$. Lemma 2.2 rigorously follows from the following two lemmas.

Lemma [Aℓ]. *If $\varphi \in X$, then the rank of the singular part of $\nabla^2\varphi$ (i.e. $\nabla^2\varphi - (\nabla^2\varphi)^{ab}$) is one.*

This is clear if $\varphi \in Y$ because of representation of $\nabla^2\varphi$. Such property was proved for an important subset of the singular part by the authors [AG2] and conjectured there for all singular part. This difficult problem was solved by Alberti [Al]. We do not need to assume $|\nabla\varphi| = 1$.

Lemma. *If $\varphi \in A$, then rank of the absolutely continuous part $(\nabla^2\varphi)^{ab}$ is less than or equal to $n - 1$.*

Proof. For $\varphi \in X$ and j, $1 \leq j \leq n$ there is a representative of $u = \nabla\varphi$ ($\mathcal{L}^n$ – a.e.) so that the pointwise derivative $\partial u/\partial x_j$ exists $\mathcal{L}^n$-a.e. and

$$\frac{\partial u}{\partial x_j}(x) = (\partial_j \nabla\varphi)^{ab}(x) \quad \Omega_0 \quad (\mathcal{L}^n - \text{a.e.})\ (1 \leq j \leq n),$$

where $\mathcal{L}^n$ is the Lebesgues measure; see [AG3]. Note that the choice of u may depend on j. Differentiating $|\nabla\varphi|^2 = 1$ in the j-th direction yields

$$\begin{aligned} 0 &= 2\sum_{i=1}^{n} \frac{\partial u_i(x)}{\partial x_j} \cdot u_i(x) = 2\sum_{i=1}^{n} (\partial_j(\partial_i\varphi))^{ab}(x) u_i(x) \\ &= 2\sum_{i=1}^{n} (\partial_j(\partial_i\varphi)(x))^{ab}(\partial_i\varphi)(x) \quad \text{for} \quad \mathcal{L}^n\text{-a.e. } x \in \Omega_0, \end{aligned}$$

where $u = (u_i)_{i=1}^n$. Since $|\nabla\varphi| \neq 0$ for a.e. x, this implies that $(\nabla^2\varphi)^{ab}$ has a kernel for a.e. x so that rank of $(\nabla^2\varphi)^{ab}$ is less than or equal to $n-1$. □

2.3. A key lemma. *For $\varphi \in X$ assume that $|\nabla\varphi| = 1$ in Ω ($\mathcal{L}^n$ – a.e.) and $\varphi \geq 0$ in Ω with $\varphi = 0$ on $\partial\Omega$, i.e., $\varphi \in A_+$. Then*

$$(-\Delta\varphi)(\Omega) = \int_\Omega -\Delta\varphi \geq \mathcal{H}^{n-1}(\partial\Omega).$$

If $\varphi(x) = dist(x, \partial\Omega)$ near $\partial\Omega$, then the equality holds.

This is easy if φ is regular so that φ is a distance function $d(x, \partial\Omega)$ near $\partial\Omega$. Indeed, integrating by parts yields

$$\int_\Omega (-\Delta\varphi) d\mathcal{L}^n = -\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} d\mathcal{H}^{n-1},$$

where ν is the unit outward normal of $\partial\Omega$. Since $\varphi = 0$ on $\partial\Omega$ and $|\Delta\varphi| = 1$ so that $\nu = -\nabla\varphi/|\nabla\varphi|$, the derivative $-\partial\varphi/\partial\nu = 1$. Thus, the equality

$$\int_\Omega (-\Delta\varphi) d\mathcal{L}^n = \mathcal{H}^{n-1}(\partial\Omega)$$

is proved.

Proof. 1. For $\varphi \in X$ set

$$E_\alpha = \{x \in \Omega; \varphi(x) \geq \alpha\}, \alpha \geq 0.$$

Since $\varphi \geq 0$ near $\partial\Omega$, $\bigcup_{\alpha>0} E_\alpha = \Omega$ and E_α is decreasing in α. Since φ is continuous, E_α is closed for small $\alpha \geq 0$. Since $\Delta\varphi$ is a finite Radon measure in Ω, we obtain

$$\int_\Omega (-\Delta\varphi) = \lim_{\alpha\downarrow 0} \int_{E_\alpha} (-\Delta\varphi).$$

In particular

$$\int_\Omega (-\Delta\varphi) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\varphi).$$

2. We shall prove the identity

$$\int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\varphi) = \mathcal{L}^n(\Omega - E_\varepsilon) \quad \text{for small } \varepsilon > 0. \tag{2.1}$$

Let $\psi \in C^2(\Omega)$ with bounded gradient in Ω. Since φ is Lipschitz near $\partial\Omega$, the co-area formula [G, S] yields

$$\int_{\Omega - E_\varepsilon} \nabla\psi \cdot \nabla\varphi d\mathcal{L}^n = \int_0^\varepsilon d\alpha \int_{L_\alpha} (\nabla\psi, \frac{\nabla\varphi}{|\nabla\varphi|}) d\mathcal{H}^{n-1} \quad \text{with } L_\alpha = \{x \in \Omega; \varphi(x) = \alpha\}. \tag{2.2}$$

Since φ is differentiable and $|\nabla\varphi(x)| = 1$ for $\mathcal{L}^n$-a.e. x (near $\partial\Omega$), by Fubini's theorem, for small ($\mathcal{L}^1$-)a.e. $\alpha > 0$,

$$|\nabla\varphi(x_0)| = 1 \quad \text{for} \quad \mathcal{H}^{n-1}\text{-a.e. } x_0 \text{ of } L_\alpha.$$

For such $\alpha > 0$ we may assume that the level set L_α is countably $n-1$ rectifiable so that at x_0 the approximate outer unit normal $\nu_\alpha(x_0) = -\nabla\varphi(x_0)$. We may also assume that E_α is a set of finite parameter so that the gradient $\nabla\chi_{E_\alpha}$ of the characteristic function χ_{E_α} of E_α is a finite Radon measure and that

$$\nabla\chi_{E_\alpha} = -\nu_\alpha \mathcal{H}^{n-1} \lfloor L_\alpha.$$

For these properties the reader is referred to the monographs [G, S]. For the above selected $\alpha > 0$ we observe that

$$\int_{L_\alpha} (\nabla\psi \cdot \frac{\nabla\varphi}{|\nabla\varphi|}) d\mathcal{H}^{n-1} = \int_\Omega \nabla\psi \cdot \nabla\chi_{E_\alpha} = \int_{E_\alpha} (-\Delta\psi) d\mathcal{L}^n$$

by integration by parts. This together with (2.2) yields

$$\int_{\Omega - E_\varepsilon} \nabla\psi \cdot \nabla\varphi d\mathcal{L}^n = \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\psi) d\mathcal{L}^n. \tag{2.3}$$

We would like to take $\psi = \varphi$. Since φ is not C^2, we need to approximate. Mollifying φ by a standard approximation as in [G, 1.17] we see that there is a sequence $\psi_j \in C^2(\Omega)$ such that

$$\lim_{j\to\infty}\int_\Omega |\nabla\psi_j - \nabla\varphi| d\mathcal{L}^n = 0, \quad \sup_{j\geq 1}\sup_{\Omega}|\nabla\psi_j| < \infty$$
$$\lim_{j\to\infty}\int_\Omega |\triangle\psi_j| d\mathcal{L}^n = \int_\Omega |\triangle\varphi|$$

since $|\nabla\varphi|$ is bounded. In particular, $\nabla\psi_j \to \nabla\varphi$ for $\mathcal{L}^n$-a.e. x by taking a subsequence if necessary. Moreover,

$$|\triangle\psi_j| \to |\triangle\varphi|,\ \triangle\psi_j \to \triangle\varphi \quad \text{weakly as measures.}$$

We shall prove that $\int_{E_\alpha}(-\triangle\varphi)$ is approximated by $\int_{E_\alpha}(-\triangle\varphi_j)d\mathcal{L}^n$. Since $-\triangle\varphi$ is a nonnegative finite Radon mesure and since L_{α_1} and L_{α_2} is disjoint for $\alpha_1 \neq \alpha_2$, we see

$$(-\triangle\varphi)(L_{\alpha'}) = 0$$

except at most countably many values of $\alpha' > 0$. For the above selected α

$$\mathcal{H}^{n-1}(\partial E_\alpha - L_\alpha) = 0$$

since E_α is a set of finite perimeter [G].

Since $-\triangle\varphi$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$ [G,S] we see

$$(-\triangle\varphi)(\partial E_\alpha - L_\alpha) = 0.$$

Since E_α is closed so that ∂E_α contains L_α, we may assume

$$(-\triangle\varphi)(\partial E_\alpha) = \int_{\partial E_\alpha}(-\triangle\varphi) = 0$$

for the above selected α by excluding the values of α' with $(-\triangle\varphi)(L_{\alpha'}) > 0$. We now apply [G, Appendix A1] to get

$$\lim_{j\to\infty}\int_{E_\alpha}(-\triangle\psi_j)d\mathcal{L}^n = \int_{E_\alpha}(-\triangle\varphi).$$

Since $\sup_\Omega|\nabla\psi_j|$ is bounded and $\nabla\psi_j \to \nabla\psi$ a.e., we now obtain

$$\lim_{j\to\infty}\int_{\Omega-E_\varepsilon}(\nabla\psi_j\cdot\nabla\varphi)d\mathcal{L}^n = \int_{\Omega-E_\varepsilon}|\nabla\varphi|^2 d\mathcal{L}^n = \mathcal{L}^n(\Omega - E_\varepsilon).$$

Since

$$\sup_{\alpha,j} \int_{E_\alpha} |\Delta \psi_j|$$

is finite, the Lebesgue convergence theorem yields

$$\begin{aligned}\lim_{j\to\infty} \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\psi_j) d\mathcal{L}^n &= \int_0^\varepsilon d\alpha \lim_{j\to\infty} \int_{E_\alpha} (-\Delta\psi_j) d\mathcal{L}^n \\ &= \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\varphi).\end{aligned}$$

Plugging $\psi = \psi_j$ in (2.3) and letting $j \to \infty$ now yields (2.1).

3. Since $|\nabla\varphi| = 1$ in Ω, the set

$$\Omega - E_\varepsilon = \{x \in \Omega; 0 \le \varphi(x) < \varepsilon\}$$

includes

$$F_\varepsilon = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < \varepsilon\}.$$

Since $\partial\Omega$ is Lipschitz, we see

$$\lim_{\varepsilon\downarrow 0} \mathcal{L}^n(F_\varepsilon)/\varepsilon = \mathcal{H}^{n-1}(\partial\Omega).$$

By step 1 and 2,

$$\int_\Omega (-\Delta\varphi) = \lim_{\varepsilon\downarrow 0} \mathcal{L}^n(\Omega - E_\varepsilon)/\varepsilon \ge \lim_{\varepsilon\downarrow 0} \mathcal{L}^n(F_\varepsilon)/\varepsilon = \mathcal{H}^{n-1}(\partial\Omega).$$

The equality holds when $\varphi(x) = d(x, \partial\Omega)$ near $\partial\Omega$. □

2.4. Theorem on total variation of the Laplacian. *Let Ω be a bounded domain in R^n with Lipschitz boundary.*

(i) $\int_\Omega |\Delta\varphi| \ge \mathcal{H}^{n-1}(\partial\Omega)$ *for all* $\varphi \in A_+$

(ii) *If Ω is convex, the minimum of $\int_\Omega |\Delta\varphi|$ over A_+ is attained at $\varphi_0(x) = dist(x, \partial\Omega)$ and the minimal value is $\mathcal{H}^{n-1}(\partial\Omega)$.*

Proof. (i) This is a direct consequence of Lemma 2.3.

(ii) If Ω is convex, then φ_0 is a concave function in Ω. This is easy; similar result is proved in

[C, p.53 Lemma]. In particular, $-\Delta\varphi = |\Delta\varphi|$ as a measure. Thus, Lemma 2.3 yields

$$-\int_\Omega \Delta\varphi_0 = \mathcal{H}^{n-1}(\partial\Omega)$$

so φ_0 is a minimizer of I over A_+. □

2.5. Theorem on total variation of the Hessian. *Let Ω be a bounded domain in R^n with Lipschitz boundary.*

(i) $I(\varphi) = \int_\Omega |\nabla^2\varphi| dx \geq \mathcal{H}^{n-1}(\partial\Omega)$ *for all* $\varphi \in A_+$ *with* $n \leq 2$ *or for all* $\varphi \in A_+^0$ *for arbitrary* n.

(ii) *If Ω is convex, the minimum of I over A_+ (with $n \leq 2$) or over A_+^0 is uniquely attained at $\varphi_0(x) = dist(x, \partial\Omega)$ and*

$$I(\varphi_0) = \mathcal{H}^{n-1}(\partial\Omega).$$

(iii) *If Ω is not convex, $I(\varphi) > \mathcal{H}^{n-1}(\partial\Omega)$ for all $\varphi \in A_+$, (with $n \leq 2$) or for all $\varphi \in A_+^0$.*

Proof. (i) By Lemma 2.2 we see

$$|\Delta\varphi| = |\nabla^2\varphi| \quad \text{for } \varphi \in A \quad \text{with } n = 2;$$

this equality is also true for $\varphi \in A^0 = A \cap Y$ or for $\varphi \in A$ with $n = 1$, since $\nabla^2\varphi$ is rank one, Theorem 2.4 (i) now yields (i).

(ii) Since $|\Delta\varphi| = |\nabla^2\varphi|$, this follows from Theorem 2.4 (ii) except the uniqueness of the minimizer. Suppose that $I(\varphi) = \mathcal{H}^{n-1}(\partial\Omega)$, then

$$\int_\Omega |\nabla^2\varphi| = \int_\Omega |\Delta\varphi| = \int_\Omega -\Delta\varphi.$$

In particular $-\Delta\varphi \geq 0$ as a measure. Since $\nabla^2\varphi$ is rank 1, this means that φ is concave. A concave function φ in A_+ is a viscosity solution of $|\nabla\varphi| = 1$; see e.g. [L] so it is the distance function φ_0.

(iii) If there is φ such that $I(\varphi) = \mathcal{H}^{n-1}(\partial\Omega)$, we see φ is a concave distance function φ_0 as in (ii). However, such a distance function φ_0 is concave in Ω if and only if Ω is convex. Thus, the strict inequality holds for a nonconvex domain. $\square$

Remark. The uniqueness of minimizers of Theorem 2.4 (ii) would be true if $\varphi \in A_+$ satisfying $-\Delta\varphi \geq 0$ would be a viscosity solutions of $|\nabla\varphi| = 1$ (without assuming that φ is concave.) However, we do not attempt to discuss this problem here.

3. Counterexample

We shall construct a simply connected but nonconvex domain Ω in $\mathbb{R}^2$ such that the distance function does not minimize neither J^1 nor I among the classes A, A_+, A^0, A^0_+ defined in § 2.2.

3.1. Choice of Domain. For a positive constant ℓ let D_ℓ be a square of the form

$$D_\ell = \{(x,y); |y| < \ell, 0 < x < 2\ell\}.$$

Let D be a unit square of the form

$$D = \{(x,y); |y| < 1/2, -1 < x < 0\}.$$

Let Ω_ℓ be the interior of the union of $\bar{D}$ and $\bar{D}_\ell$, where the bar denotes the closure. We shall always assume that $\ell > 1/2$. Clearly, Ω is a bounded simply connected but nonconvex domain.

3.2. Defects. We consider three solutions of the eikonal equation

$$|\nabla\varphi| = 1 \quad \text{in } \Omega_\ell \quad \text{with } \varphi = 0 \quad \text{on } \partial\Omega_\ell$$

of the form:

$$\varphi_0(x) = \text{dist}(x, \partial\Omega_\ell)$$

$$\varphi_1(x) = \begin{cases} -\text{dist}(x, \partial D) & \text{for } x \in \bar{D} \\ \\ \text{dist}(x, \partial D_\ell) & \text{for } x \in \bar{D}_\ell \end{cases}$$

$$\varphi_2(x) = \begin{cases} \operatorname{dist}(x, \partial D) & \text{for } x \in \bar{D} \\ \operatorname{dist}(x, \partial D_\ell) & \text{for } x \in \bar{D}_\ell. \end{cases}$$

The set of jump discontinuities of $\nabla\varphi_1$ consists of

$$L_1 : y = \ell - x, 0 < x < 2\ell, \qquad L_2 : y = x - \ell, 0 < x < 2\ell$$

$$L_3 : y = -\frac{1}{2} - x, -1 < x < 0, \quad L_4 : y = \frac{1}{2} + x, -1 < x < 0.$$

The magnitude of jumps

$$j = |\nabla\varphi^+ - \nabla\varphi^-|$$

on each defects are $\sqrt{2}$. For later convenience we decompose defects of φ_1:

$$\Sigma_1^+ = L_3 \cap R_1, \qquad \Sigma_1^- = L_4 \cap R_1$$

$$S_1^+ = L_4 \cap R_2, \qquad S_1^- = L_3 \cap R_2$$

$$\Sigma_2^+ = L_1 \cap R_3, \qquad \Sigma_2^- = L_2 \cap R_3$$

$$S_2^+ = L_1 \cap R_4, \qquad S_2^- = L_2 \cap R_4$$

$$\Sigma_3^+ = L_2 \cap R_5, \qquad \Sigma_3^- = L_1 \cap R_5$$

with

$$R_1 = \{-1 < x < -1/2\}, \quad R_2 = \{-1/2 < x < 0\}$$

$$R_3 = \{0 < x < \ell - 1/2\}, \quad R_4 = \{\ell - 1/2 < x < \ell\}$$

$$R_5 = \{\ell < x < 2\ell\}.$$

The defect of φ_0 consists of

$$\Sigma_i^\pm \quad i = 1, 2, 3 \quad \text{and}$$

$$\Gamma_1 = \{y = 0\} \cap R_2, \quad \Gamma_2 = \{y = 0\} \cap (R_3 \cup R_4),$$

$$\Gamma^+ = \{x^2 + (y - \frac{1}{2})^2 = (\ell - y)^2, y > 0\} \cap R_4$$

$$\Gamma^- = \{x^2 + (y + \frac{1}{2})^2 = (\ell + y)^2, y < 0\} \cap R_4.$$

see figure 2. The defect of φ_2 consists of $\Sigma_i^\pm$ $(i = 1,2,3)$ $S_i^\pm$ $(i = 1,2)$ and

$$C = \{x = 0, |y| < 1/2\}$$

with jump $j = 2$ on C.

3.3. Computation of defect energy. We shall estimate the difference $J^\beta(\varphi_0) - J^\beta(\varphi_1)$. By symmetry with respect to $y = 0$ we observe that

$$J^\beta(\varphi_0) = 2\sum_{i=1}^{3} J^\beta(\varphi_0, \Sigma_i^+) + \sum_{i=1}^{2} J^\beta(\varphi_0, \Gamma_i) + 2J^\beta(\varphi_0, \Gamma^+)$$
$$J^\beta(\varphi_1) = 2\sum_{i=1}^{3} J^\beta(\varphi_1, \Sigma_i^+) + 2\sum_{i=1}^{2} J^\beta(\varphi_1, S_i^+)$$

where

$$J^\beta(\varphi, B) = \int_{\Sigma\cap B} j^\beta d\mathcal{H}^{n-1} \quad \text{with } j = |\nabla\varphi^+ - \nabla\varphi^-|.$$

Since jump j is the same both for φ_0 and φ_1 on Σ_i^+,

$$J^\beta(\varphi_0, \Sigma_i^+) = J^\beta(\varphi_1, \Sigma_i^+) \quad i = 1,2,3.$$

Thus

$$J^\beta(\varphi_0) - J^\beta(\varphi_1) = \sum_{i=1}^{2} J^\beta(\varphi_0, \Gamma_i) + 2J^\beta(\varphi_0, \Gamma^+) - 2\sum_{i=1}^{2} J^\beta(\varphi_1, S_i^+).$$

Proposition. (i) $J^\beta(\varphi_0, \Gamma_1) = 2^{\beta-1}$.

(ii) $\lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) = \int_0^\infty \frac{dx}{(x^2+1/4)^{\beta/2}} \geq \frac{1}{\beta-1}2^{\beta-1}$ *for* $\beta > 1$.

For $\beta \leq 1$, $\lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) = \infty$.

(iii) $J^\beta(\varphi_0, \Gamma^+) \to 2^{(\beta-1)/2}$ *as* $\ell \to \infty$.

(iv) $J^\beta(\varphi_1, S_i^+) = 2^{(\beta-1)/2}$, $i = 1,2$.

Proof. (i) Clearly, $j = 2$ on Γ_1. Since the length of Γ_1 is $1/2$, $J^\beta = 2^{\beta-1}$.

(ii) Since the level curve of φ_0 intersecting $(x,0)$ $(0 < x < \ell)$ is the circle centered $(0,1/2)$ for

$y > 0$, the normal component of $\nabla\varphi_0^+$ equals

$$\frac{1}{2}\frac{1}{(x^2+1/4)^{1/2}}.$$

Thus

$$J^\beta(\varphi_0,\Gamma_2) = \int_0^\ell \frac{dx}{(x^2+1/4)^{\beta/2}} \geq \int_0^\ell \frac{dx}{(x+1/2)^\beta} = \frac{1}{\beta-1}(x+\frac{1}{2})^{1-\beta}|_\ell^0$$

and letting $\ell \to \infty$ completes the proof.

(iii) Recall that Γ^+ is a set of points whose distance to $(0,1/2)$ equals the distance to the line $y=\ell$. The curve Γ^+ is a parabola given by

$$(2\ell-1)y = -x^2+\ell^2-1/4.$$

The jump

$$j(x) = 2\frac{1}{(y'(x)^2+1)^{1/2}}$$

and the length element equals $(y'(x)^2+1)^{1/2}dx$. If p is the largest zero of $y(x)$, i.e.

$$p = (\ell^2-1/4)^{1/2},$$

then

$$J^\beta(\varphi_0,\Gamma^+) = \int_{\ell-1/2}^p (2\frac{1}{((y')^2+1)^{1/2}})^\beta((y')^2+1)^{1/2}dx.$$

Notice that

$$\begin{aligned} y'(x)^2+1 &= (\frac{-2x}{2\ell-1})^2+1, \quad \ell-1/2<x<\ell \\ &\to 2 \quad \text{as} \quad \ell\to\infty \\ p-(\ell-1/2) = (\ell^2-1/4)^{1/2}-(\ell-1/2) &= \frac{1}{2}+\frac{-2/4}{(\ell^2-1/4)^{1/2}+\ell} \\ &\to \frac{1}{2} \quad \text{as} \quad \ell\to\infty. \end{aligned}$$

We thus conclude that

$$J^\beta(\varphi_0,\Gamma^+) = \int_{\ell-1}^p 2^\beta((y')^2+1)^{\frac{1-\beta}{2}}dx \to 2^\beta 2^{(1-\beta)/2}\cdot\frac{1}{2} = 2^{(\beta-1)/2} \quad \text{as} \quad \ell\to\infty.$$

(iv) Since $j=\sqrt{2}$ and the length of S_i^+ equal $1/\sqrt{2}$, the result follows immediately. $\square$

3.4. Proposition. *Let Ω be the domain Ω_ℓ defined in §3.1. Let φ_0 be the distance function of $\partial\Omega$ and let φ_1 and φ_2 be solutions of the eikonal equation defined in §3.2.*

(i) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) = \infty$ *for* $0 < \beta \le 1$.

(ii) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_2)) = \infty$ *for* $0 < \beta \le 1$.

(iii) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) > 0$ *for all* $\beta > 0$.

(iv) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_2)) > 0$ *for all* $0 < \beta < \beta_0$ *with some* $\beta_0 > 4/3$.

Proof. Applying the previous Proposition to the formula $J^\beta(\varphi_0) - J^\beta(\varphi_1)$ yields

$$\begin{aligned} T = \lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) &= 2^{\beta-1} + \lim_{\ell\to\infty} J^\beta(\varphi_0,\Gamma_2) + 2\cdot 2^{(\beta-1)/2} \\ &\quad - 2(2^{(\beta-1)/2} + 2^{(\beta-1)/2}) \\ &= 2^{\beta-1} - 2\cdot 2^{(\beta-1)/2} + \lim_{\ell\to\infty} J^\beta(\varphi_0,\Gamma_2). \end{aligned}$$

If $\beta \le 1$, then this formula yield (i). Note that

$$J^\beta(\varphi_2) = J^\beta(\varphi_1) + J^\beta(\varphi_2, C) \quad \text{with} \quad C = \bar{D} \cap \bar{D}_\ell.$$

Since $J^\beta(\varphi_2, C) = 1\cdot 2^\beta$, the proof of (ii) is now complete.

To show (iii) we may assume $\beta > 1$ and use the estimate

$$\lim_{\ell\to\infty} J^\beta(\varphi_0,\Gamma_2) \ge \frac{1}{\beta-1}2^{\beta-1}.$$

to get

$$T \ge 2\cdot 2^{(\beta-1)2}(f(\beta) - 1)$$

with $f(\beta) = 2^{(\beta-3)/2}\frac{\beta}{\beta-1}$. An elementary calculation shows that f takes the only minimum at $\beta = \beta_1$ over all $\beta > 0$. The number β_1 is the solution of

$$-\frac{1}{\beta_1 - 1} + \frac{1}{2}(\log 2)\beta_1 = 0 \quad \text{so that} \quad \beta_1 \ge 1.$$

Since $\log 2 > 1/2$, $(\beta_1 - 1)\beta_1 \le 1/4$ so that $\beta_1 - 1 \le 1/2$. Since $\beta_1 \ge 1$ we now obtain

$$\begin{aligned} f(\beta) \ge f(\beta_1) &= 2^{(\beta_1-3)/2}(1 + \frac{1}{\beta_1 - 1}) \\ &> 2^{-1}(1+2). \end{aligned}$$

We thus conclude

$$J^\beta(\varphi_0) > J^\beta(\varphi_1) + 2 \cdot 2^{(\beta-1)/2} \cdot 1/2.$$

It remains to prove (iv). We may assume $\beta > 1$. Notice that $J^\beta(\varphi_2, C) = 2^\beta$ to get

$$\begin{aligned} T &\ge \frac{\beta}{\beta-1} 2^{\beta-1} - 2 \cdot 2^{(\beta-1)/2} - 2^\beta \\ &= 2 \cdot 2^{(\beta-1)/2}((\frac{1}{\beta-1} - 1)2^{(\beta-3)/2} - 1). \end{aligned}$$

The right hand side is positive if $\beta \le 4/3$. □

3.5. Theorem. *Assume that $\Omega = \Omega_\ell$.*

(i) ($\beta \le 1$) *For each $M > 0$, there is a constant $\ell_0 = \ell_0(\beta)$ such that if $\ell > \ell_0(\beta)$ then*

$$J^\beta(\varphi_0) \ge J^\beta(\varphi_2) + M \ge J^\beta(\varphi_1) + M.$$

Moreover, φ_0 does not minimize neither I nor J^β in A_+, A.

(ii) ($\beta > 1$) *There is a constant $\ell_1 = \ell_1(\beta)$ such that φ_0 does not minimize J^β in A for $\ell > \ell_1(\beta)$. Moreover, if $\beta \le 4/3$ (or $\beta < \beta_0$), then φ_0 does not minimize J^β in A_+.*

Proof. (i) The first inequality follows from Proposition 3.4 (ii) and

$$J^\beta(\varphi_2) = J^\beta(\varphi_1) + 2^\beta$$

for sufficiently large ℓ. Since $\varphi_0 \in A_+ \subset A$, $\varphi_1 \in A^0$ and $\varphi_2 \in A^0_+$, we now observe that

$$I(\varphi_0) \ge J^1(\varphi_0) > I(\varphi_2) = J^1(\varphi_2) > I(\varphi_1) = J^1(\varphi_1).$$

and

$$J^\beta(\varphi_0) > J^\beta(\varphi_2) > J^\beta(\varphi_1) \quad \text{for} \quad \beta \le 1.$$

(ii) Since $\varphi_0 \in A_+$, $\varphi_1 \in A^0$ and $\varphi_2 \in A^0_+$, Proposition 3.4 yields the desired conclusion. □

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Figure 1

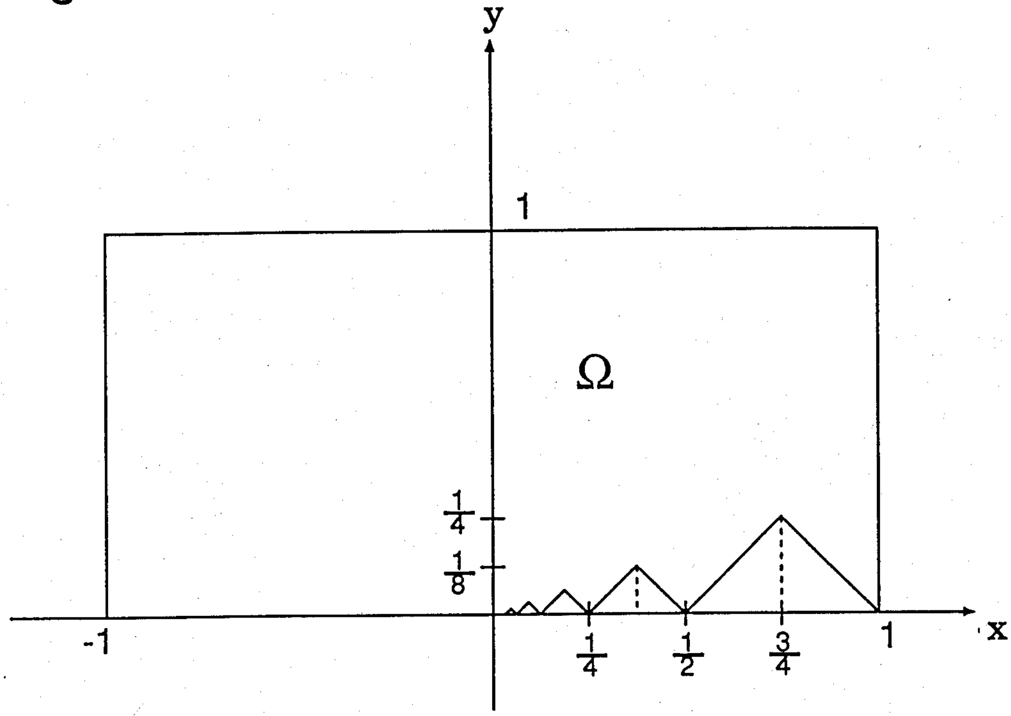
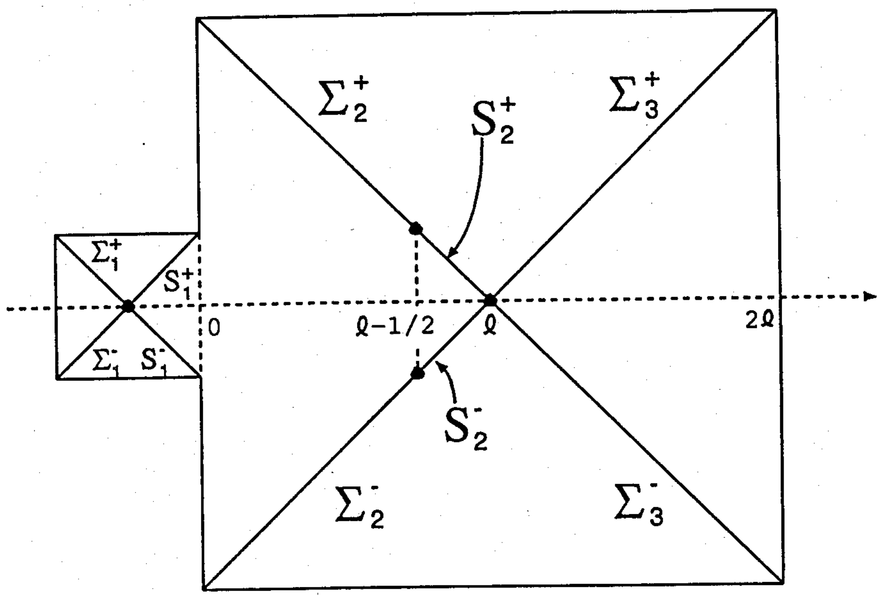
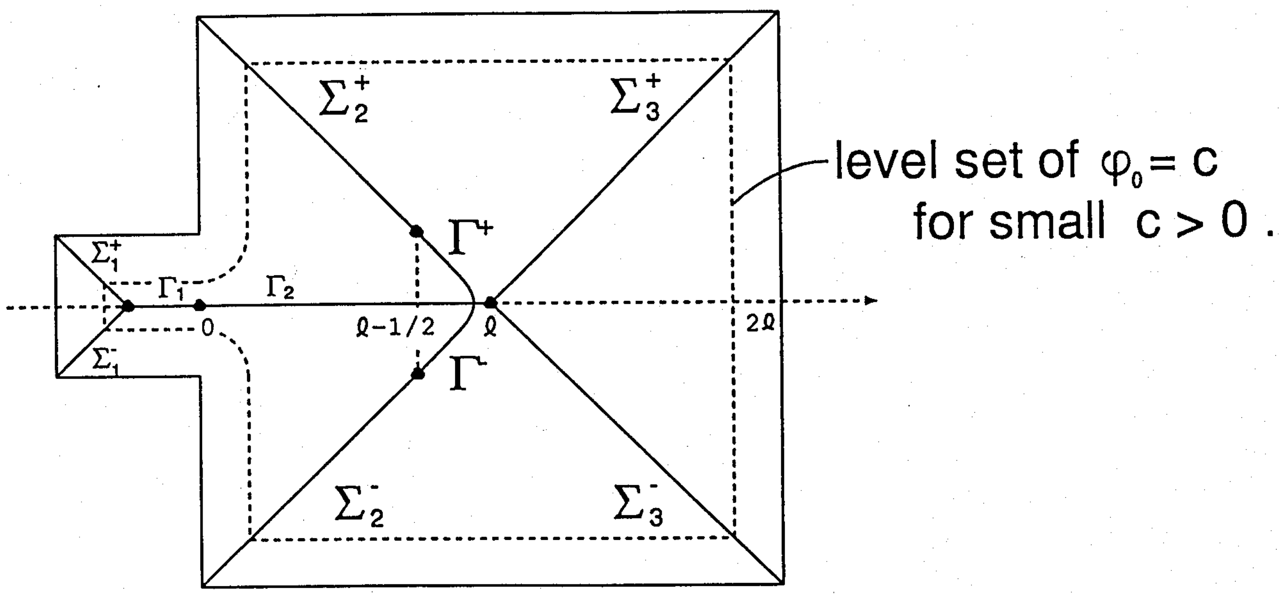


Figure 2



defects of φ_1



defects of φ_0