THE DISTANCE FUNCTION AND DEFECT ENERGY

P. Aviles and Y. Giga

Series ♯293. June 1995

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PATRICIO AVILES

DEPARTMENT OF MATHEMATICS AND PHYSICS

OXFORD UNIVERSITY, U.K.

OR

ETH. CH-8592, ZURICH

SWITZERLAND

YOSHIKAZU GIGA

DEPARTMENT OF MATHEMATICS

HOKKAIDO UNIVERSITY

SAPPORO 060, JAPAN

1. Introduction

It is important to measure the energy of jump discontinuities of a unit length gradient field $\nabla\varphi$ in a bounded Lipschitz domain in $\mathbb{R}^n$.

Such problems arise in the modelling of smectic liquid crystals [SK], [AG1] or of the blistering of thin films [OG]. The quantity measuring the energy of the jump discontinuities, the defect of $\nabla\varphi$, is

$$J^\beta(\varphi) = \int_\Sigma |\nabla\varphi^+ - \nabla\varphi^-|^\beta d\mathcal{H}^{n-1}$$

where $\beta > 0$; we call it a defect energy. Here Σ is the set of jump discontinuities of $\nabla\varphi$ and $\nabla\varphi^\pm$ is the trace of $\nabla\varphi$ of each side of Σ; $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure which is the surface element when Σ is smooth.

There may be a lot of Lipschitz solutions of the eikonal equation

$$|\nabla\varphi| = 1 \quad \text{in} \quad \Omega \quad \text{with} \quad \varphi = 0 \quad \text{on} \quad \partial\Omega,$$

but the distance function

$$d = d(x, \partial\Omega) = \inf\{|x - y|;\ y \in \partial\Omega\}$$

is the unique viscosity solution of the problem [CIL]. In other words the theory of viscosity solutions selects a solution of the eikonal equation. There is a fundamental question whether the distance function minimizes J^β among all *nonnegative* solutions of the eikonal equation.

If the space dimension n equals one, J^β just measures a constant multiple of the number of jumps of $\nabla\varphi$. There is no solution of the eikonal equation having no defect satisfying the zero boundary condition. Thus, the distance function is a (unique) minimizer of J^β since it has only one jump of the derivative of φ. However, for multidimensional case, the situation is different.

In this paper, we focus on the case $\beta = 1$ because of independent interest related to the total variation of the Hessian

$$I(\varphi) = \int_\Omega |\nabla^2\varphi|.$$

This integral is closely related to J^1. Indeed, if φ is piecewise linear, more precisely, $\nabla^2\varphi = 0$ (as a measure) outside Σ, then

$$I(\varphi) = \int_\Sigma |\nabla^2\varphi| = \int_\Sigma |\nabla^+\varphi \cdot \nu - \nabla\varphi^- \cdot \nu| d\mathcal{H}^{n-1}$$

where ν is the approximate normal of Σ [G]. Since the tangential component of $\nabla\varphi$ is approximately continuous, $|\nabla\varphi^+ \cdot \nu - \nabla\varphi^- \cdot \nu| = |\nabla\varphi^+ - \nabla\varphi^-|$ if $|\nabla\varphi| = 1$. Thus, $I(\varphi) = J^1(\varphi)$ for piecewise linear φ. Our principal results are

(i) the distance function is the unique minimizer of $I(\varphi)$ among all nonnegative (Lipschitz) solutions of the eikonal equation $|\nabla\varphi| = 1$ in Ω with $\varphi = 0$ on $\partial\Omega$ provided that Ω is convex and $n = 2$. The values of minimum equals $\mathcal{H}^{n-1}(\partial\Omega)$.

(ii) there is a simply connected nonconvex domain Ω in $\mathbb{R}^2$ such that the distance function is not a minimizer of J^1 nor I.

This suggests that the selection mechanism of the ground state by I or J^1 is different from that in the theory of viscosity solutions, in general.

To show (i) we first observe that

$$|\Delta\varphi| = |\nabla^2\varphi|$$

as measures if φ solves $|\nabla\varphi| = 1$ and $n = 2$. This depends on the fact that $\nabla^2\varphi$ has rank one which is easy to observe heuristically. Differentiating $|\nabla\varphi| = 1$ implies that one of eigenvalues of $\nabla^2\varphi$ always equal zero. To carry out this idea we appeal to the theory of functions of bounded variation [G]. Note that the singular part (w.r.t. the Lebesgue measure) of $\nabla^2\varphi$ always has rank one [Al], [AG2]. Another key observation is

$$\int_\Omega |\Delta\varphi| \geq \int_\Omega -\Delta\varphi = \mathcal{H}^{n-1}(\partial\Omega)$$

if $|\nabla\varphi| = 1$, $\varphi \geq 0$ in Ω with $\varphi = 0$. The last equality formally follows from integration by parts and the fact that $|\nabla\varphi|$ agrees with inward normal derivative of φ on $\partial\Omega$. In section 2 we state these observations in a rigorous way allowing that $\nabla^2\varphi$ is a measure. If Ω is convex, the distance function d is concave in Ω so that $-\Delta d \geq 0$ in Ω (in the distribution senese). Thus

$$\int_\Omega |\Delta d| = \int_\Omega -\Delta d = \mathcal{H}^{n-1}(\partial\Omega)$$

so that d minimizes I as well as $\int_\Omega |\Delta\varphi|$. It turns out that d is a unique minimizer among all φ, $|\nabla\varphi| = 1$, $\varphi \geq 0$ in Ω with $\varphi = 0$ on $\partial\Omega$. The inequality

$$\int_\Omega |\Delta\varphi| \geq \int_\Omega -\Delta\varphi$$

is not sharp unless Ω is convex. In other words the minimum of I is strictly greater than $\mathcal{H}^{n-1}(\partial\Omega)$ (Theorem 2.5.) The proof of (ii) depends on an explicit construction of the domain Ω.

As a corollary of (i) we get: if d is piecewise linear, more precisely, $\nabla^2 d = 0$ outside the defect as a measure, then d also minimizes J^1 (among all nonnegative solutions of the eikonal equations) provided that the domain is convex. Note that such d exist if and only if the domain in a convex polygon as shown in Remark in §2.1.

Our counterexamples are interesting for the study of minimizers of singular perturbed variational problem

$$E_\varepsilon(\varphi) = \int_\Omega W(\nabla\varphi) + \varepsilon^2 \int_\Omega |\nabla^2\varphi|^2, \quad W(p) = (1-|p|^2)^\sigma, \quad \sigma > 0$$

in a plane domain Ω with $\varphi = 0$ on $\partial\Omega$. Since the Euler-Lagrange equation is fourth order, we are entitled to impose another boundary condition. The natural choice seems to be $\partial u/\partial\nu = -1$, where ν is the unit outward normal of $\partial\Omega$. We divide E_ε by ε so that we hope that the energy has a nonzero limit as $\varepsilon \to 0$.

Formal analysis for $\sigma = 2$ done by [AG1], [OG] suggests that this problem has Gamma-limit

$$\tilde{J} = 2\int_\Sigma |\nabla\varphi^+ \cdot \nu - \nabla\varphi^- \cdot \nu| \int_{-b}^{b} |1-(a^2+\tau^2)|^{\sigma/2} d\tau d\mathcal{H}^{n-1}, \quad a = |(\nabla\varphi)_{\tan}|,\ b = (1-a)^{1/2},$$

where $(\nabla\varphi)_{\tan}$ denotes the tangential component of $\nabla\varphi^+$ (or $\nabla\varphi^-$). Since $|\nabla\varphi| = 1$, we see $\tilde{J}$ is a positive constant times $J^{\sigma+1}$. This $\tilde{J}$ (or $J^{\sigma+1}$) is to be minimized subject to the same boundary condition as for E_ε, and the interior condition $|\nabla\varphi| = 1$ a.e. in Ω.

It is tempting to think that the minimizer φ_ε of E_ε tends to

$$\varphi_0(x) = d(x, \partial\Omega).$$

as $\varepsilon \to 0$. Similarly, it is tempting to think that this function might by a mininizer of $\tilde{J}$ (or $J^{\sigma+1}$). These conjectures are more or less explicit in [OG] (cf. [AG1] for $\sigma = 2$).

An extended version of our examples (Theorem 3.5) says that the second conjecture is false for some nonconvex domain at least for $\sigma < \beta_0 - 1$ with some $\beta_0 > 1$ close to one. Unfortunately in our examples β_0 is less than 3 so they do not solve the original conjecture for $\sigma = 2$. However, they are important because they show some possible pitfalls. In particular, they show that if these conjectures are true for $\sigma = 2$, then the reasons must be subtle since other equally reasonable-sounding statements are false.

The limiting process of E_ε as $\varepsilon \to 0$ is not at all clear compared with the case when $\nabla\varphi$ is a scalar function. Such a convergence problem is studied in [KM1], [KM2] for $\nabla\varphi$ when W has isolated equal minimums.

The authors are grateful to Professor Robert Kohn for his interest and valuable comments on this paper. Much of this work was done while the second author visited the Japan-American Mathematics Institute of the Johns Hopkins University. It hospitality is gratefully acknowledged, as is support from Japan Society for the Promotion of Science. This work is partly supported by THE SUHARA MEMORIAL FOUNDATION and YAMADA SCIENCE FOUNDATION.

2. Estimate of total variations of gradient field of length one

We are concerned with total variation of $\nabla^2\psi$ in a bounded two dimensional domain Ω when $|\nabla\psi| = 1$, $\psi \geq 0$ on Ω and $\psi = 0$ on the boundary $\partial\Omega$. Our principal result in this section is that the minimum of the total variation is attained (uniquely) at the distance function provided that Ω is convex.

2.1. Notation. Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. For a Lebesgue integrable function φ, i.e. $\varphi \in L^1(\Omega)$, let $\nabla\varphi = (\partial_i\varphi)_{i=1}^n$ and $\nabla^2\varphi = (\partial_i\partial_j\varphi)$ $(1 \leq i,j \leq n)$ a distributional gradient and Hessian of φ, respectively. Let X be the space of $\varphi \in L^1(\Omega)$ such that $\partial_i\varphi \in L^1(\Omega)$ $(1 \leq i \leq n)$ and $\partial_i\partial_j\varphi$ is a finite Radon measure on Ω $(1 \leq i,j \leq n)$. In other words, $\partial_i\varphi$ is a function of (essentially) bounded variation, i.e. $\partial_i\varphi \in BV(\Omega)$. Let us recall fundamental decomposition of $\nabla^2\varphi$ for $\varphi \in X$; see e.g. [AG2]. Let Ω_0 be the largest subset in Ω such that $\nabla^2\varphi$ is absolutely continuous in Ω_0 and let Σ be the set of jump discontinuities of $\nabla\varphi$. Then

$$\nabla^2\varphi = \nabla^2\varphi\lfloor\Omega_0 + \nabla^2\varphi\lfloor(\Omega - \Omega_0 - \Sigma) + \nu \otimes (\nabla\varphi^+ - \nabla\varphi^-)\mathcal{H}^{n-1}\lfloor\Sigma.$$

Here for a set Z and measure μ we associate a new measure $\mu\lfloor Z$ by

$$(\mu\lfloor Z)(B) = \mu(Z \cap B), B \subset \Omega.$$

The vector field ν is the approximate unit normal of Σ and $\nabla\varphi^{\pm}$ is the trace of $\nabla\varphi$ on Σ in the direction of $\pm\nu$; $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. The first term $(\nabla^2\varphi)^{ab} = \nabla^2\varphi\lfloor\Omega_0$ is often called the *absolutely continuous* part of $\nabla^2\varphi$. We always identify $(\nabla^2\varphi)^{ab}$ with corresponding locally Lebesgue integrable function in Ω_0. The second term is often called the *mild* part and it lies on a non rectifiable set $\Omega - \Omega_0 - \Sigma$ of Lebesgue measure zero. The sum of last two terms is called the *singular* part of $\nabla^2\varphi$. Let Y be the space of $\varphi \in X$ such that $\nabla\varphi$ has no absolute continuous part and no mild part. In other words

$$Y = \{\varphi \in X; \nabla^2\varphi = \nu \otimes (\nabla\varphi^+ - \nabla\varphi^-)\mathcal{H}^{n-1}\lfloor\Sigma\}.$$

Let A be the space of $\varphi \in X$ such that $|\nabla\varphi| = 1$ a.e. in Ω with $\varphi = 0$ on $\partial\Omega$. We needs three subclasses of A

$$A_+ = \{\varphi \in A; \varphi \geq 0 \text{ in } \Omega\}, \quad A^0 = A \cap Y, \quad A_+^0 = A_+ \cap A^0.$$

We consider two integrals for $\varphi \in X$ which measure jumps of $\nabla\varphi$.

$$I(\varphi) = \int_\Omega |\nabla^2\varphi|,$$
$$J^\beta(\varphi) = \int_\Sigma |\nabla\varphi^+ - \nabla\varphi^-|^\beta d\mathcal{H}^{n-1} \quad \text{for} \quad \beta > 0.$$

Since $\nabla^2\varphi$ is a finite Radon measure, in the representation

$$\int_\Omega |\nabla^2\varphi| = \sup\{\sum_{1\leq i,j\leq n}\int_\Omega \theta^{ij}\partial_i\partial_j\varphi; \sum_{i,j}|\theta_{ij}|^2 \leq 1\}$$

the test function θ_{ij} is allowed to be $\theta_{ij} \in C^1(\overline{\Omega})$ not necessarily compactly supported.

Remark. The set A_+^0 and even A^0 may be empty. In fact, A_+^0 (and A^0) is empty if $\partial\Omega$ has a 'curved' part. Conversely, A_+^0 is nonempty if Ω is a polygon. The proof is by the induction of numbers of verteces of Ω. If Ω is a triangle, the distance function d is certainly piecewise linear. If Ω is a polygon of m (> 3) verteces, we set

$$\rho(x, \partial\Omega) = \min\{d(x, L(S)); L(S) \text{ is a straight line containing an edge } S \text{ of } \partial\Omega\}$$

$$d_* = \inf\{\rho(x, \partial\Omega); \text{there is at least three edges } S_1, S_2, S_3 \text{ of } \partial\Omega$$
$$\text{such that } \rho(x, \partial\Omega) = d(x, L(S_i)), i = 1, 2, 3 \text{ and } x \in \Omega\}.$$

It is easy to see that ρ is piecewise linear in

$$\Omega_* = \{x \in \Omega, d(x) < d_*\}.$$

By the choice of d_* the set $K = \Omega - \Omega_*$ is a closed polygon with at most $m-1$ verteces; it may have no interior so that K is a set of points or segments. Let Ω' be the interior of K so that it is a polygon with at most $m-1$ verteces. By the induction the function $\rho(x, \partial\Omega')$ is piecewise linear in Ω'. Since

$$\rho(x, \partial\Omega) = \rho(x, \partial\Omega') + d_* \quad \text{in } \Omega',$$

and $\rho = \rho(x, \partial\Omega)$ is piecewise linear in Ω_*, we see ρ is piecewise linear in Ω. This shows that $\rho \in A^0_+$ so that A^0_+ is nonempty.

Note that ρ is the distance function d if and only if Ω is a convex polygon. If Ω is nonconvex, d is not piecewise linear, so $d \in A^0_+$ if and only if Ω is a convex polygon.

We conclude this remark by pointing out that there is a domain Ω whose boundary is piecewise linear with infinite verteces such that $\rho \in A^0_+$. For example if we consider

$$\Omega = \{(x, y); |x| < 1, 1 > y > h(x)\}$$

$$h(x) = \begin{cases} \frac{1}{2^{\ell+1}} - |x - \frac{3}{2^{\ell+1}}| & \frac{1}{2^\ell} < x \le \frac{1}{2^{\ell-1}}, \ell = 1, 2, \cdots \\ 0 & x \le 0 \end{cases}$$

then $\rho \in A^0_+$. See figure 1.

2.2. Comparison Lemma of Hessian and Laplacian measure. *Assume that $n = 2$. Then for $\varphi \in A$,*

$$|\triangle\varphi| = |\nabla^2\varphi| \quad \textit{(as measures).}$$

This is formally true since $\nabla^2\varphi$ is rank one. Indeed, differentiating $|\nabla\varphi|^2 = 1$ yields

$$\sum_{j=1}^{n}(\partial_i\partial_j\varphi)\partial_j\varphi = 0.$$

We shall justify this observation for general Hessian measure $\nabla^2\varphi$ of $\varphi \in A$. We say that for $\varphi \in X$ the rank of matrix of the Radon-Nikodym derivative

$$F(x) = \lim_{r\downarrow 0}\nabla^2\varphi(B_r(x))/|\nabla^2\varphi|(B_r(x))$$

is the rank of $\nabla^2\varphi$, where $B_r(x)$ denotes the closed ball of radius r centered at $x \in \Omega$. The rank of $\nabla\varphi$ is defined for $|\nabla^2\varphi|$-almost every point x of Ω. Since $\nabla^2\varphi$ is absolutely continuous with respect to $|\nabla^2\varphi|$,

$$|\triangle\varphi|(Z) = \int_Z |\text{trace } F|d\mu, \ |\nabla^2\varphi|(Z) = \int_Z |F|d\mu$$

with $\mu = |\nabla^2\varphi|$, where $|F|$ is the Hilbert-Schmidt norm of F, i.e., $|F|^2 = \Sigma_{ij}|F_{ij}|^2$. If F is rank 1, then

$$|\text{trace } F| = |F|$$

so that $|\triangle\varphi| = |\nabla^2\varphi|$. Lemma 2.2 rigorously follows from the following two lemmas.

Lemma [Aℓ]. *If $\varphi \in X$, then the rank of the singular part of $\nabla^2\varphi$ (i.e. $\nabla^2\varphi - (\nabla^2\varphi)^{ab}$) is one.*

This is clear if $\varphi \in Y$ because of representation of $\nabla^2\varphi$. Such property was proved for an important subset of the singular part by the authors [AG2] and conjectured there for all singular part. This difficult problem was solved by Alberti [Al]. We do not need to assume $|\nabla\varphi| = 1$.

Lemma. *If $\varphi \in A$, then rank of the absolutely continuous part $(\nabla^2\varphi)^{ab}$ is less than or equal to $n - 1$.*

Proof. For $\varphi \in X$ and j, $1 \leq j \leq n$ there is a representative of $u = \nabla\varphi$ ($\mathcal{L}^n$ – a.e.) so that the pointwise derivative $\partial u/\partial x_j$ exists $\mathcal{L}^n$-a.e. and

$$\frac{\partial u}{\partial x_j}(x) = (\partial_j \nabla\varphi)^{ab}(x) \quad \Omega_0 \quad (\mathcal{L}^n - \text{a.e.})\ (1 \leq j \leq n),$$

where $\mathcal{L}^n$ is the Lebesgues measure; see [AG3]. Note that the choice of u may depend on j. Differentiating $|\nabla\varphi|^2 = 1$ in the j-th direction yields

$$\begin{aligned} 0 &= 2\sum_{i=1}^{n} \frac{\partial u_i(x)}{\partial x_j} \cdot u_i(x) = 2\sum_{i=1}^{n} (\partial_j(\partial_i\varphi))^{ab}(x)u_i(x) \\ &= 2\sum_{i=1}^{n} (\partial_j(\partial_i\varphi)(x))^{ab}(\partial_i\varphi)(x) \quad \text{for} \quad \mathcal{L}^n\text{-a.e. } x \in \Omega_0, \end{aligned}$$

where $u = (u_i)_{i=1}^n$. Since $|\nabla\varphi| \neq 0$ for a.e. x, this implies that $(\nabla^2\varphi)^{ab}$ has a kernel for a.e. x so that rank of $(\nabla^2\varphi)^{ab}$ is less than or equal to $n-1$. □

2.3. A key lemma. *For $\varphi \in X$ assume that $|\nabla\varphi| = 1$ in Ω ($\mathcal{L}^n$ – a.e.) and $\varphi \geq 0$ in Ω with $\varphi = 0$ on $\partial\Omega$, i.e., $\varphi \in A_+$. Then*

$$(-\Delta\varphi)(\Omega) = \int_\Omega -\Delta\varphi \geq \mathcal{H}^{n-1}(\partial\Omega).$$

If $\varphi(x) = dist(x, \partial\Omega)$ near $\partial\Omega$, then the equality holds.

This is easy if φ is regular so that φ is a distance function $d(x, \partial\Omega)$ near $\partial\Omega$. Indeed, integrating by parts yields

$$\int_\Omega (-\Delta\varphi)d\mathcal{L}^n = -\int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu} d\mathcal{H}^{n-1},$$

where ν is the unit outward normal of $\partial\Omega$. Since $\varphi = 0$ on $\partial\Omega$ and $|\Delta\varphi| = 1$ so that $\nu = -\nabla\varphi/|\nabla\varphi|$, the derivative $-\partial\varphi/\partial\nu = 1$. Thus, the equality

$$\int_\Omega (-\Delta\varphi)d\mathcal{L}^n = \mathcal{H}^{n-1}(\partial\Omega)$$

is proved.

Proof. 1. For $\varphi \in X$ set

$$E_\alpha = \{x \in \Omega; \varphi(x) \geq \alpha\}, \alpha \geq 0.$$

Since $\varphi \geq 0$ near $\partial\Omega$, $\bigcup_{\alpha>0} E_\alpha = \Omega$ and E_α is decreasing in α. Since φ is continuous, E_α is closed for small $\alpha \geq 0$. Since $\Delta\varphi$ is a finite Radon measure in Ω, we obtain

$$\int_\Omega (-\Delta\varphi) = \lim_{\alpha\downarrow 0} \int_{E_\alpha} (-\Delta\varphi).$$

In particular

$$\int_\Omega (-\Delta\varphi) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\varphi).$$

2. We shall prove the identity

$$\int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\varphi) = \mathcal{L}^n(\Omega - E_\varepsilon) \quad \text{for small } \varepsilon > 0. \tag{2.1}$$

Let $\psi \in C^2(\Omega)$ with bounded gradient in Ω. Since φ is Lipschitz near $\partial\Omega$, the co-area formula [G, S] yields

$$\int_{\Omega - E_\varepsilon} \nabla\psi \cdot \nabla\varphi d\mathcal{L}^n = \int_0^\varepsilon d\alpha \int_{L_\alpha} (\nabla\psi, \frac{\nabla\varphi}{|\nabla\varphi|}) d\mathcal{H}^{n-1} \quad \text{with } L_\alpha = \{x \in \Omega; \varphi(x) = \alpha\}. \tag{2.2}$$

Since φ is differentiable and $|\nabla\varphi(x)| = 1$ for $\mathcal{L}^n$-a.e. x (near $\partial\Omega$), by Fubini's theorem, for small ($\mathcal{L}^1$-)a.e. $\alpha > 0$,

$$|\nabla\varphi(x_0)| = 1 \quad \text{for} \quad \mathcal{H}^{n-1}\text{-a.e. } x_0 \text{ of } L_\alpha.$$

For such $\alpha > 0$ we may assume that the level set L_α is countably $n-1$ rectifiable so that at x_0 the approximate outer unit normal $\nu_\alpha(x_0) = -\nabla\varphi(x_0)$. We may also assume that E_α is a set of finite parameter so that the gradient $\nabla\chi_{E_\alpha}$ of the characteristic function χ_{E_α} of E_α is a finite Radon measure and that

$$\nabla\chi_{E_\alpha} = -\nu_\alpha \mathcal{H}^{n-1} \lfloor L_\alpha.$$

For these properties the reader is referred to the monographs [G, S]. For the above selected $\alpha > 0$ we observe that

$$\int_{L_\alpha} (\nabla\psi \cdot \frac{\nabla\varphi}{|\nabla\varphi|}) d\mathcal{H}^{n-1} = \int_\Omega \nabla\psi \cdot \nabla\chi_{E_\alpha} = \int_{E_\alpha} (-\Delta\psi) d\mathcal{L}^n$$

by integration by parts. This together with (2.2) yields

$$\int_{\Omega - E_\varepsilon} \nabla\psi \cdot \nabla\varphi d\mathcal{L}^n = \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\Delta\psi) d\mathcal{L}^n. \tag{2.3}$$

We would like to take $\psi = \varphi$. Since φ is not C^2, we need to approximate. Mollifying φ by a standard approximation as in [G, 1.17] we see that there is a sequence $\psi_j \in C^2(\Omega)$ such that

$$\lim_{j\to\infty}\int_\Omega |\nabla\psi_j - \nabla\varphi| d\mathcal{L}^n = 0, \quad \sup_{j\geq 1}\sup_{\Omega}|\nabla\psi_j| < \infty$$
$$\lim_{j\to\infty}\int_\Omega |\triangle\psi_j| d\mathcal{L}^n = \int_\Omega |\triangle\varphi|$$

since $|\nabla\varphi|$ is bounded. In particular, $\nabla\psi_j \to \nabla\varphi$ for $\mathcal{L}^n$-a.e. x by taking a subsequence if necessary. Moreover,

$$|\triangle\psi_j| \to |\triangle\varphi|, \ \triangle\psi_j \to \triangle\varphi \quad \text{weakly as measures.}$$

We shall prove that $\int_{E_\alpha}(-\triangle\varphi)$ is approximated by $\int_{E_\alpha}(-\triangle\varphi_j)d\mathcal{L}^n$. Since $-\triangle\varphi$ is a nonnegative finite Radon mesure and since L_{α_1} and L_{α_2} is disjoint for $\alpha_1 \neq \alpha_2$, we see

$$(-\triangle\varphi)(L_{\alpha'}) = 0$$

except at most countably many values of $\alpha' > 0$. For the above selected α

$$\mathcal{H}^{n-1}(\partial E_\alpha - L_\alpha) = 0$$

since E_α is a set of finite perimeter [G].

Since $-\triangle\varphi$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$ [G,S] we see

$$(-\triangle\varphi)(\partial E_\alpha - L_\alpha) = 0.$$

Since E_α is closed so that ∂E_α contains L_α, we may assume

$$(-\triangle\varphi)(\partial E_\alpha) = \int_{\partial E_\alpha}(-\triangle\varphi) = 0$$

for the above selected α by excluding the values of α' with $(-\triangle\varphi)(L_{\alpha'}) > 0$. We now apply [G, Appendix A1] to get

$$\lim_{j\to\infty}\int_{E_\alpha}(-\triangle\psi_j)d\mathcal{L}^n = \int_{E_\alpha}(-\triangle\varphi).$$

Since $\sup_\Omega |\nabla\psi_j|$ is bounded and $\nabla\psi_j \to \nabla\psi$ a.e., we now obtain

$$\lim_{j\to\infty}\int_{\Omega - E_\varepsilon}(\nabla\psi_j\cdot\nabla\varphi)d\mathcal{L}^n = \int_{\Omega - E_\varepsilon}|\nabla\varphi|^2 d\mathcal{L}^n = \mathcal{L}^n(\Omega - E_\varepsilon).$$

Since

$$\sup_{\alpha,j} \int_{E_\alpha} |\triangle \psi_j|$$

is finite, the Lebesgue convergence theorem yields

$$\begin{aligned} \lim_{j\to\infty} \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\triangle \psi_j) d\mathcal{L}^n &= \int_0^\varepsilon d\alpha \lim_{j\to\infty} \int_{E_\alpha} (-\triangle \psi_j) d\mathcal{L}^n \\ &= \int_0^\varepsilon d\alpha \int_{E_\alpha} (-\triangle \varphi). \end{aligned}$$

Plugging $\psi = \psi_j$ in (2.3) and letting $j \to \infty$ now yields (2.1).

3. Since $|\nabla\varphi| = 1$ in Ω, the set

$$\Omega - E_\varepsilon = \{x \in \Omega; 0 \le \varphi(x) < \varepsilon\}$$

includes

$$F_\varepsilon = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) < \varepsilon\}.$$

Since $\partial\Omega$ is Lipschitz, we see

$$\lim_{\varepsilon\downarrow 0} \mathcal{L}^n(F_\varepsilon)/\varepsilon = \mathcal{H}^{n-1}(\partial\Omega).$$

By step 1 and 2,

$$\int_\Omega (-\triangle\varphi) = \lim_{\varepsilon\downarrow 0} \mathcal{L}^n(\Omega - E_\varepsilon)/\varepsilon \ge \lim_{\varepsilon\downarrow 0} \mathcal{L}^n(F_\varepsilon)/\varepsilon = \mathcal{H}^{n-1}(\partial\Omega).$$

The equality holds when $\varphi(x) = d(x, \partial\Omega)$ near $\partial\Omega$. □

2.4. Theorem on total variation of the Laplacian. *Let Ω be a bounded domain in R^n with Lipschitz boundary.*

(i) $\int_\Omega |\triangle\varphi| \ge \mathcal{H}^{n-1}(\partial\Omega)$ *for all* $\varphi \in A_+$

(ii) *If Ω is convex, the minimum of $\int_\Omega |\triangle\varphi|$ over A_+ is attained at $\varphi_0(x) = dist(x, \partial\Omega)$ and the minimal value is $\mathcal{H}^{n-1}(\partial\Omega)$.*

Proof. (i) This is a direct consequence of Lemma 2.3.

(ii) If Ω is convex, then φ_0 is a concave function in Ω. This is easy; similar result is proved in

[C, p.53 Lemma]. In particular, $-\Delta\varphi = |\Delta\varphi|$ as a measure. Thus, Lemma 2.3 yields

$$-\int_\Omega \Delta\varphi_0 = \mathcal{H}^{n-1}(\partial\Omega)$$

so φ_0 is a minimizer of I over A_+. $\square$

2.5. Theorem on total variation of the Hessian. *Let Ω be a bounded domain in R^n with Lipschitz boundary.*

(i) $I(\varphi) = \int_\Omega |\nabla^2\varphi| dx \geq \mathcal{H}^{n-1}(\partial\Omega)$ *for all* $\varphi \in A_+$ *with* $n \leq 2$ *or for all* $\varphi \in A_+^0$ *for arbitrary* n.

(ii) *If Ω is convex, the minimum of I over A_+ (with $n \leq 2$) or over A_+^0 is uniquely attained at $\varphi_0(x) = dist(x, \partial\Omega)$ and*

$$I(\varphi_0) = \mathcal{H}^{n-1}(\partial\Omega).$$

(iii) *If Ω is not convex, $I(\varphi) > \mathcal{H}^{n-1}(\partial\Omega)$ for all $\varphi \in A_+$, (with $n \leq 2$) or for all $\varphi \in A_+^0$.*

Proof. (i) By Lemma 2.2 we see

$$|\Delta\varphi| = |\nabla^2\varphi| \quad \text{for } \varphi \in A \quad \text{with } n = 2;$$

this equality is also true for $\varphi \in A^0 = A \cap Y$ or for $\varphi \in A$ with $n = 1$, since $\nabla^2\varphi$ is rank one, Theorem 2.4 (i) now yields (i).

(ii) Since $|\Delta\varphi| = |\nabla^2\varphi|$, this follows from Theorem 2.4 (ii) except the uniqueness of the minimizer. Suppose that $I(\varphi) = \mathcal{H}^{n-1}(\partial\Omega)$, then

$$\int_\Omega |\nabla^2\varphi| = \int_\Omega |\Delta\varphi| = \int_\Omega -\Delta\varphi.$$

In particular $-\Delta\varphi \geq 0$ as a measure. Since $\nabla^2\varphi$ is rank 1, this means that φ is concave. A concave function φ in A_+ is a viscosity solution of $|\nabla\varphi| = 1$; see e.g. [L] so it is the distance function φ_0.

(iii) If there is φ such that $I(\varphi) = \mathcal{H}^{n-1}(\partial\Omega)$, we see φ is a concave distance function φ_0 as in (ii). However, such a distance function φ_0 is concave in Ω if and only if Ω is convex. Thus, the strict inequality holds for a nonconvex domain. □

Remark. The uniqueness of minimizers of Theorem 2.4 (ii) would be true if $\varphi \in A_+$ satisfying $-\Delta\varphi \geq 0$ would be a viscosity solutions of $|\nabla\varphi| = 1$ (without assuming that φ is concave.) However, we do not attempt to discuss this problem here.

3. Counterexample

We shall construct a simply connected but nonconvex domain Ω in $\mathbb{R}^2$ such that the distance function does not minimize neither J^1 nor I among the classes A, A_+, A^0, A^0_+ defined in § 2.2.

3.1. Choice of Domain. For a positive constant ℓ let D_ℓ be a square of the form

$$D_\ell = \{(x,y); |y| < \ell, 0 < x < 2\ell\}.$$

Let D be a unit square of the form

$$D = \{(x,y); |y| < 1/2, -1 < x < 0\}.$$

Let Ω_ℓ be the interior of the union of $\bar{D}$ and $\bar{D}_\ell$, where the bar denotes the closure. We shall always assume that $\ell > 1/2$. Clearly, Ω is a bounded simply connected but nonconvex domain.

3.2. Defects. We consider three solutions of the eikonal equation

$$|\nabla\varphi| = 1 \quad \text{in } \Omega_\ell \quad \text{with } \varphi = 0 \quad \text{on } \partial\Omega_\ell$$

of the form:

$$\varphi_0(x) = \operatorname{dist}(x, \partial\Omega_\ell)$$

$$\varphi_1(x) = \begin{cases} -\operatorname{dist}(x, \partial D) & \text{for } x \in \bar{D} \\ \operatorname{dist}(x, \partial D_\ell) & \text{for } x \in \bar{D}_\ell \end{cases}$$

$$\varphi_2(x) = \begin{cases} \mathrm{dist}(x, \partial D) & \text{for } x \in \bar{D} \\ \mathrm{dist}(x, \partial D_\ell) & \text{for } x \in \bar{D}_\ell. \end{cases}$$

The set of jump discontinuities of $\nabla\varphi_1$ consists of

$$L_1 : y = \ell - x, 0 < x < 2\ell, \qquad L_2 : y = x - \ell, 0 < x < 2\ell$$

$$L_3 : y = -\frac{1}{2} - x, -1 < x < 0, \quad L_4 : y = \frac{1}{2} + x, -1 < x < 0.$$

The magnitude of jumps

$$j = |\nabla\varphi^+ - \nabla\varphi^-|$$

on each defects are $\sqrt{2}$. For later convenience we decompose defects of φ_1:

$$\Sigma_1^+ = L_3 \cap R_1, \qquad \Sigma_1^- = L_4 \cap R_1$$

$$S_1^+ = L_4 \cap R_2, \qquad S_1^- = L_3 \cap R_2$$

$$\Sigma_2^+ = L_1 \cap R_3, \qquad \Sigma_2^- = L_2 \cap R_3$$

$$S_2^+ = L_1 \cap R_4, \qquad S_2^- = L_2 \cap R_4$$

$$\Sigma_3^+ = L_2 \cap R_5, \qquad \Sigma_3^- = L_1 \cap R_5$$

with

$$R_1 = \{-1 < x < -1/2\}, \quad R_2 = \{-1/2 < x < 0\}$$

$$R_3 = \{0 < x < \ell - 1/2\}, \quad R_4 = \{\ell - 1/2 < x < \ell\}$$

$$R_5 = \{\ell < x < 2\ell\}.$$

The defect of φ_0 consists of

$$\Sigma_i^\pm \quad i = 1, 2, 3 \quad \text{and}$$

$$\Gamma_1 = \{y = 0\} \cap R_2, \quad \Gamma_2 = \{y = 0\} \cap (R_3 \cup R_4),$$

$$\Gamma^+ = \{x^2 + (y - \frac{1}{2})^2 = (\ell - y)^2, y > 0\} \cap R_4$$

$$\Gamma^- = \{x^2 + (y + \frac{1}{2})^2 = (\ell + y)^2, y < 0\} \cap R_4.$$

see figure 2. The defect of φ_2 consists of $\Sigma_i^\pm$ $(i = 1,2,3)$ $S_i^\pm$ $(i = 1,2)$ and

$$C = \{x = 0, |y| < 1/2\}$$

with jump $j = 2$ on C.

3.3. Computation of defect energy. We shall estimate the difference $J^\beta(\varphi_0) - J^\beta(\varphi_1)$. By symmetry with respect to $y = 0$ we observe that

$$J^\beta(\varphi_0) = 2\sum_{i=1}^{3} J^\beta(\varphi_0, \Sigma_i^+) + \sum_{i=1}^{2} J^\beta(\varphi_0, \Gamma_i) + 2J^\beta(\varphi_0, \Gamma^+)$$
$$J^\beta(\varphi_1) = 2\sum_{i=1}^{3} J^\beta(\varphi_1, \Sigma_i^+) + 2\sum_{i=1}^{2} J^\beta(\varphi_1, S_i^+)$$

where

$$J^\beta(\varphi, B) = \int_{\Sigma\cap B} j^\beta d\mathcal{H}^{n-1} \quad \text{with } j = |\nabla\varphi^+ - \nabla\varphi^-|.$$

Since jump j is the same both for φ_0 and φ_1 on Σ_i^+,

$$J^\beta(\varphi_0, \Sigma_i^+) = J^\beta(\varphi_1, \Sigma_i^+) \quad i = 1,2,3.$$

Thus

$$J^\beta(\varphi_0) - J^\beta(\varphi_1) = \sum_{i=1}^{2} J^\beta(\varphi_0, \Gamma_i) + 2J^\beta(\varphi_0, \Gamma^+) - 2\sum_{i=1}^{2} J^\beta(\varphi_1, S_i^+).$$

Proposition. (i) $J^\beta(\varphi_0, \Gamma_1) = 2^{\beta-1}$.

(ii) $\lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) = \int_0^\infty \frac{dx}{(x^2+1/4)^{\beta/2}} \geq \frac{1}{\beta-1}2^{\beta-1}$ *for* $\beta > 1$.

For $\beta \leq 1$, $\lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) = \infty$.

(iii) $J^\beta(\varphi_0, \Gamma^+) \to 2^{(\beta-1)/2}$ *as* $\ell \to \infty$.

(iv) $J^\beta(\varphi_1, S_i^+) = 2^{(\beta-1)/2}$, $i = 1,2$.

Proof. (i) Clearly, $j = 2$ on Γ_1. Since the length of Γ_1 is $1/2$, $J^\beta = 2^{\beta-1}$.

(ii) Since the level curve of φ_0 intersecting $(x,0)$ $(0 < x < \ell)$ is the circle centered $(0,1/2)$ for

$y > 0$, the normal component of $\nabla\varphi_0^+$ equals

$$\frac{1}{2}\frac{1}{(x^2+1/4)^{1/2}}.$$

Thus

$$J^\beta(\varphi_0,\Gamma_2) = \int_0^\ell \frac{dx}{(x^2+1/4)^{\beta/2}} \geq \int_0^\ell \frac{dx}{(x+1/2)^\beta} = \frac{1}{\beta-1}(x+\frac{1}{2})^{1-\beta}|_\ell^0$$

and letting $\ell \to \infty$ completes the proof.

(iii) Recall that Γ^+ is a set of points whose distance to $(0,1/2)$ equals the distance to the line $y=\ell$. The curve Γ^+ is a parabola given by

$$(2\ell-1)y = -x^2+\ell^2-1/4.$$

The jump

$$j(x) = 2\frac{1}{(y'(x)^2+1)^{1/2}}$$

and the length element equals $(y'(x)^2+1)^{1/2}dx$. If p is the largest zero of $y(x)$, i.e.

$$p = (\ell^2-1/4)^{1/2},$$

then

$$J^\beta(\varphi_0,\Gamma^+) = \int_{\ell-1/2}^p (2\frac{1}{((y')^2+1)^{1/2}})^\beta((y')^2+1)^{1/2}dx.$$

Notice that

$$\begin{aligned} y'(x)^2+1 &= (\frac{-2x}{2\ell-1})^2+1, \quad \ell-1/2<x<\ell \\ &\to 2 \quad \text{as} \quad \ell\to\infty \\ p-(\ell-1/2) = (\ell^2-1/4)^{1/2}-(\ell-1/2) &= \frac{1}{2}+\frac{-2/4}{(\ell^2-1/4)^{1/2}+\ell} \\ &\to \frac{1}{2} \quad \text{as} \quad \ell\to\infty. \end{aligned}$$

We thus conclude that

$$J^\beta(\varphi_0,\Gamma^+) = \int_{\ell-1}^p 2^\beta((y')^2+1)^{\frac{1-\beta}{2}}dx \to 2^\beta 2^{(1-\beta)/2}\cdot\frac{1}{2} = 2^{(\beta-1)/2} \quad \text{as} \quad \ell\to\infty.$$

(iv) Since $j=\sqrt{2}$ and the length of S_i^+ equal $1/\sqrt{2}$, the result follows immediately. □

3.4. Proposition. *Let Ω be the domain Ω_ℓ defined in §3.1. Let φ_0 be the distance function of $\partial\Omega$ and let φ_1 and φ_2 be solutions of the eikonal equation defined in §3.2.*

(i) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) = \infty$ *for* $0 < \beta \le 1$.

(ii) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_2)) = \infty$ *for* $0 < \beta \le 1$.

(iii) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) > 0$ *for all* $\beta > 0$.

(iv) $\lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_2)) > 0$ *for all* $0 < \beta < \beta_0$ *with some* $\beta_0 > 4/3$.

Proof. Applying the previous Proposition to the formula $J^\beta(\varphi_0) - J^\beta(\varphi_1)$ yields

$$\begin{aligned} T = \lim_{\ell\to\infty}(J^\beta(\varphi_0) - J^\beta(\varphi_1)) &= 2^{\beta-1} + \lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) + 2\cdot 2^{(\beta-1)/2} \\ &\quad - 2(2^{(\beta-1)/2} + 2^{(\beta-1)/2}) \\ &= 2^{\beta-1} - 2\cdot 2^{(\beta-1)/2} + \lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2). \end{aligned}$$

If $\beta \le 1$, then this formula yield (i). Note that

$$J^\beta(\varphi_2) = J^\beta(\varphi_1) + J^\beta(\varphi_2, C) \quad \text{with} \quad C = \bar{D} \cap \bar{D}_\ell.$$

Since $J^\beta(\varphi_2, C) = 1\cdot 2^\beta$, the proof of (ii) is now complete.

To show (iii) we may assume $\beta > 1$ and use the estimate

$$\lim_{\ell\to\infty} J^\beta(\varphi_0, \Gamma_2) \ge \frac{1}{\beta - 1} 2^{\beta-1}.$$

to get

$$T \ge 2\cdot 2^{(\beta-1)2}(f(\beta) - 1)$$

with $f(\beta) = 2^{(\beta-3)/2}\frac{\beta}{\beta-1}$. An elementary calculation shows that f takes the only minimum at $\beta = \beta_1$ over all $\beta > 0$. The number β_1 is the solution of

$$-\frac{1}{\beta_1 - 1} + \frac{1}{2}(\log 2)\beta_1 = 0 \quad \text{so that} \quad \beta_1 \ge 1.$$

Since $\log 2 > 1/2$, $(\beta_1 - 1)\beta_1 \leq 1/4$ so that $\beta_1 - 1 \leq 1/2$. Since $\beta_1 \geq 1$ we now obtain

$$\begin{aligned} f(\beta) \geq f(\beta_1) &= 2^{(\beta_1-3)/2}(1 + \frac{1}{\beta_1 - 1}) \\ &> 2^{-1}(1+2). \end{aligned}$$

We thus conclude

$$J^\beta(\varphi_0) > J^\beta(\varphi_1) + 2 \cdot 2^{(\beta-1)/2} \cdot 1/2.$$

It remains to prove (iv). We may assume $\beta > 1$. Notice that $J^\beta(\varphi_2, C) = 2^\beta$ to get

$$\begin{aligned} T &\geq \frac{\beta}{\beta - 1} 2^{\beta-1} - 2 \cdot 2^{(\beta-1)/2} - 2^\beta \\ &= 2 \cdot 2^{(\beta-1)/2}((\frac{1}{\beta-1} - 1)2^{(\beta-3)/2} - 1). \end{aligned}$$

The right hand side is positive if $\beta \leq 4/3$. □

3.5. Theorem. *Assume that $\Omega = \Omega_\ell$.*

(i) ($\beta \leq 1$) *For each $M > 0$, there is a constant $\ell_0 = \ell_0(\beta)$ such that if $\ell > \ell_0(\beta)$ then*

$$J^\beta(\varphi_0) \geq J^\beta(\varphi_2) + M \geq J^\beta(\varphi_1) + M.$$

Moreover, φ_0 does not minimize neither I nor J^β in A_+, A.

(ii) ($\beta > 1$) *There is a constant $\ell_1 = \ell_1(\beta)$ such that φ_0 does not minimize J^β in A for $\ell > \ell_1(\beta)$. Moreover, if $\beta \leq 4/3$ (or $\beta < \beta_0$), then φ_0 does not minimize J^β in A_+.*

Proof. (i) The first inequality follows from Proposition 3.4 (ii) and

$$J^\beta(\varphi_2) = J^\beta(\varphi_1) + 2^\beta$$

for sufficiently large ℓ. Since $\varphi_0 \in A_+ \subset A$, $\varphi_1 \in A^0$ and $\varphi_2 \in A^0_+$, we now observe that

$$I(\varphi_0) \geq J^1(\varphi_0) > I(\varphi_2) = J^1(\varphi_2) > I(\varphi_1) = J^1(\varphi_1).$$

and

$$J^\beta(\varphi_0) > J^\beta(\varphi_2) > J^\beta(\varphi_1) \quad \text{for} \quad \beta \leq 1.$$

(ii) Since $\varphi_0 \in A_+$, $\varphi_1 \in A^0$ and $\varphi_2 \in A^0_+$, Proposition 3.4 yields the desired conclusion. □

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Figure 1

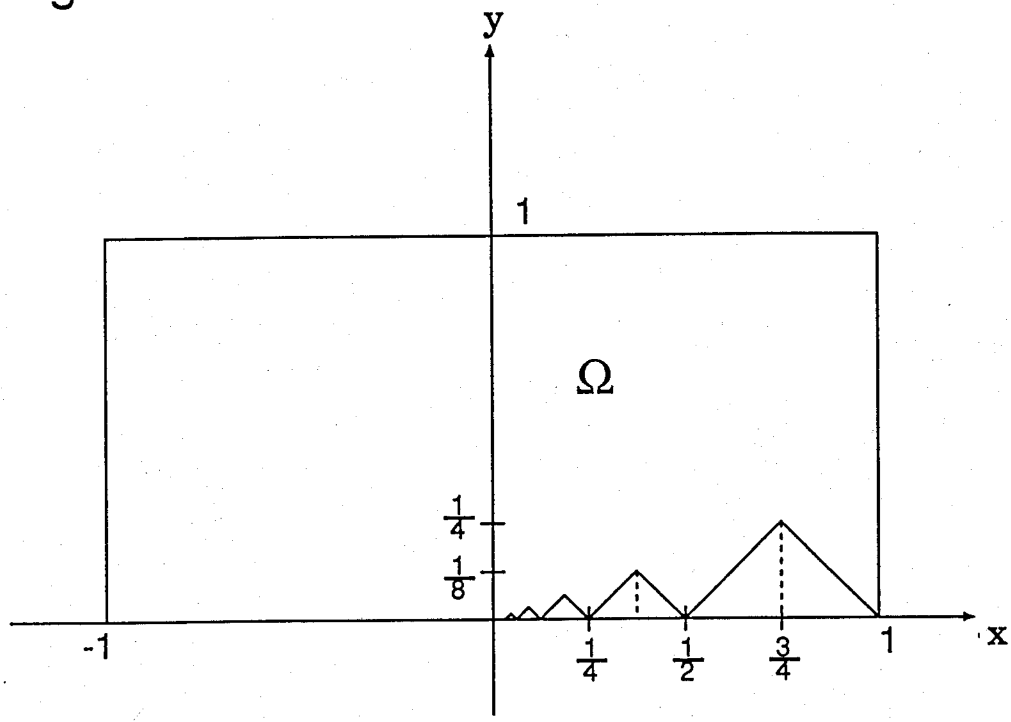
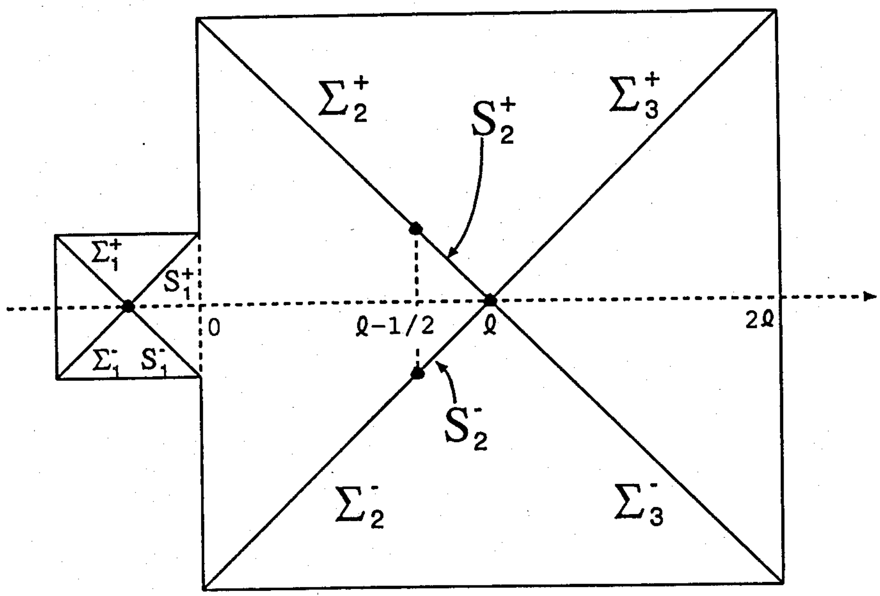
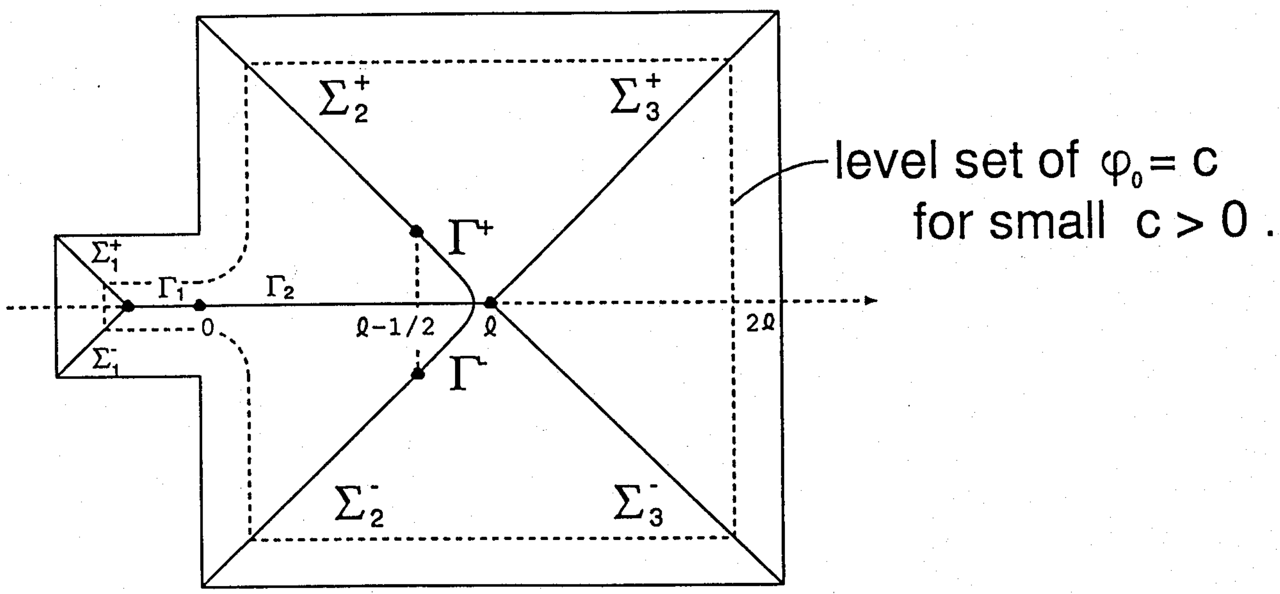


Figure 2



defects of φ_1



defects of φ_0