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A TIME-LIKE SURFACE IN MINKOWSKI 3-SPACE WHICH CONTAINS PSEUDOCIRCLES

S. Izumiya and A. Takiyama

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A TIME-LIKE SURFACE IN MINKOWSKI 3-SPACE WHICH CONTAINS PSEUDOCIRCLES

SHYUICHI IZUMIYA AND AKIHIRO TAKIYAMA

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan

ABSTRACT. Simple characterizations of a pseudosphere or a plane in Minkowski 3-space by the existence of pseudocircles are given.

1. INTRODUCTION

One of the simple characterizations of sphere is Euclidian 3-space $\mathbb{E}^3$ is as follows:
(*) A circle in $\mathbb{E}^3$ of arbitrary given radius can be pressed entirely on an arbitray position of a surface.

The condition (*) is quite natural because an observer is an inhabitant of an ambient space, however, (*) requires a very large quantity of information because of its condition "an arbitrary position". In [2], Ogiue and Takagi give a much practical condition for a compact surface to be a sphere.

Theorem ([2], Ogiue-Takagi). *Let S be a surface in Euclidian 3-space $\mathbb{E}^3$. Suppose that, through each point $p \in S$, there exist two circles of $\mathbb{E}^3$ such that*
(1) *they are contained in S in a neighbourhood of p,*
(2) *they are tangent to each other at p.*
Then S is locally a plane or a sphere.

In this paper we consider characterizations for a pseudosphere $S_1^2(r, a)$ in Minkowski 3-space $\mathbb{M}^3$ (for definition, see Section 2) analogous to the Ogiue-Takagi's result. The normal vector field on a pseudosphere in $\mathbb{M}^3$ is space-like, so that we stick to a surface S in $\mathbb{M}^3$ such that the normal vector field on S is space-like. Such a surface is called *a time-like surface.* There is another important class of surfaces in Minkowski 3-space $\mathbb{M}^3$ which is called *space-like surfaces.* In this case the induced metric on the surface is positively definite, so that we can give characterizations of a hyperbolic surface $H_1^2(r, a)$ by exactly the same arguments as those in [2]. Thus we do not consider this case in this paper.

The notion of psuedocircles in Minkowski 3-space is given analogous to that of circles in Euclidian 3-space. Let $\gamma(s)$ be a time-like or a space-like curve (cf., Section 2). We can define the principal normal $\mathbf{N}(s)$, the binormal $\mathbf{B}(s)$, the curvature $k(s)$ and the torsion $\tau(s)$, so that the Frenet-Serret type formula for such a curve holds (cf., Section 2). We say that $\gamma(s)$ is *a pseudocircle* if the torsion is equal to zero and the curvature is positive constant along the curve.

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Firstly, we have a simple characterization for pseudosphere in $\mathbb{M}^3$ which is peculiar to pseudosphere in Minkowski 3-space.

Theorem A. *Let S be a time-like surface in Minkowski 3-space $\mathbb{M}^3$. Suppose that, through each point $p \in S$, there exist two pseudocircles γ_1, γ_2 of $\mathbb{M}^3$ such that*
(1) *γ_1, γ_2 are contained in S in a neighbourhood of p,*
(2) *γ_1, γ_2 are tangent to each other at p,*
(3) *$\delta(\gamma_1(s))\delta(\gamma_2(s)) = -1$, where $\delta(\gamma_1(s)) =< \mathbf{N}(s), \mathbf{N}(s) >$.*
Then S is locally a pseudosphere.

If we remove the condition (3) in the assumption in Theorem A, we have the following theorem analogous to the Ogiue-Takagi's result.

Theorem B. *Let S be a time-like surface in Minkowski 3-space $\mathbb{M}^3$. Suppose that, through each point $p \in S$, there exist two pseudocircles γ_1, γ_2 of $\mathbb{M}^3$ such that*
(1) *γ_1, γ_2 are contained in S in a neighbourhood of p,*
(2) *γ_1, γ_2 are tangent to each other at p.*
Then S is locally a time-like-plane or a pseudosphere.

We remark that an intersection of a psudosphere with an arbitray plane is a pseudocircle or a pair of light-like lines. We have given characterizations of a pseudosphere by the existence of light-like lines in [1]. The method of the proofs of the theorems we used here is analogous to that of Theorem 1 in [2].

All surfaces and maps considered here are of class C^∞ unless stated otherwise.

2. Basic notions

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3)|x_1, x_2, x_3 \in \mathbb{R}\}$ be the usual oriented 3-dimensional vector space and differential manifold, which is oriented by $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ and given the euclidian differentiable structure. *Minkowski 3-space* is defined by $\mathbb{M}^3 = \{\mathbb{R}^3, I_{(2,1)}\}$, where $I_{(2,1)} = dx_1^2 + dx_2^2 - dx_3^2$. Thus the metric tensor is given by $< \mathbf{X}, \mathbf{Y} >= x_1y_1 + x_2y_2 - x_3y_3$, where $\mathbf{X}= (x_1, x_2, x_3)$ and $\mathbf{Y}= (y_1, y_2, y_3)$. A vector $\mathbf{X}$ in $\mathbb{M}^3$ is called *light-like* if $< \mathbf{X}, \mathbf{X} >= 0$, *space-like* if $< \mathbf{X}, \mathbf{X} >> 0$ and *time-like* if $< \mathbf{X}, \mathbf{X} >< 0$. A curve γ is called *light-like* if its tangent vector field $\dot{\gamma}$ is always light-like. We also say that a curve γ is *non-light-like* if its tangent vector field $\dot{\gamma}$ is always space-like or time-like. *The pseudosphere* is defined to be

$$S_1^2(r, a) = \{(x_1, x_2, x_3)|(x_1 - a_1)^2 + (x_2 - a_2)^2 - (x_3 - a_3)^2 = r^2\},$$

where $a = (a_1, a_2, a_3)$ is the center and $r > 0$ is the radial of $S_1^2(r, a)$.

The Levi-Civita connection of $\mathbb{M}^3$ is denoted by $\widetilde{\nabla}$. Let S be a surface in $\mathbb{M}^3$. We say that S is *time-like* if the normal vector field to S is space-like. On the time-like surface S we have the Levi-Civita connection corresponding to the induced Lorenzian metric on S. We denote it ∇. *The normal connection* on S is denoted by $\nabla^\perp$. Let σ be a second fundamental form of S in $\mathbb{M}^3$ which is given by

$$\sigma(\mathbf{X}, \mathbf{Y}) = \widetilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{X}}\mathbf{Y},$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector field tangent to S. Since S is a time-like surface, there exists (at least locally) a unit normal vector field ξ on S. We have the following formula:

$$< \sigma(\mathbf{X},\mathbf{Y}), \xi >=< -\widetilde{\nabla}_{\mathbf{X}}\xi, \mathbf{Y} >,$$

so that $-\widetilde{\nabla}_{\mathbf{X}}\xi$ is *the shape operator* in this case. We denote that $A\mathbf{X} = -\widetilde{\nabla}_{\mathbf{X}}\xi$. It is well-known that the shape operator is self-adjoint with respect to $<,>$ (i.e., $< A\mathbf{X}, \mathbf{Y} >=< \mathbf{X}, A\mathbf{Y} >$). *The covariant derivative* $\nabla'_{\mathbf{X}}\sigma$ of σ is defined by

$$\nabla'_{\mathbf{X}}\sigma(\mathbf{Y},\mathbf{Z}) = \nabla^{\perp}_{\mathbf{X}}\sigma(\mathbf{Y},\mathbf{Z}) - \sigma(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) - \sigma(\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}).$$

We say that a point $p \in S$ is *umbilic* if there is a normal vector $\mathbf{Z} \in T_pS^{\perp}$ such that $\sigma(\mathbf{X},\mathbf{Y}) =< \mathbf{X},\mathbf{Y} > \mathbf{Z}$ for all $\mathbf{X},\mathbf{Y} \in T_pS$. A time-like surface S is *totally umbilic* provided every point of S is umbilic. It is well-known that S is totally umbilic if and only if its shape operator is scalar (cf., §4, Lemma 21 in [3]). The following result gives a characterization of a pseudo sphere or a time-like plane in $\mathbb{M}^3$ (cf., §4, Propositions 13, 36 in [3]).

Proposition 2.1. *If S is totally umbilic, then S is an open set in a pseudosphere or a time-like plane.*

In order to define the notion of pseudocircles, we introduce the notion of *Lorenzian exterior products.* For any vectors $\mathbf{X} = (x_1, x_2, x_3), \mathbf{Y} = (y_1, y_2, y_3) \in \mathbb{M}^3$, *the Lorenzian exterior product (with respect to $<,>$)* is defined to be

$$\mathbf{X} \wedge \mathbf{Y} = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

Let γ be a non-light-like curve. We may assume that γ is parameterized by arclength s. Thus we have $< \dot{\gamma}(s), \dot{\gamma}(s) >= \varepsilon(\gamma(s)) = \pm 1$. We call $\varepsilon(\gamma(s))$ *the causal character* of γ. We can also define the notion of *the curvature* $k(s)$ of the curve γ by $k(s) = \sqrt{| < \ddot{\gamma}(s), \ddot{\gamma}(s) > |}$. We define *the principal normal vector* $\mathbf{N}(s)$ by $\ddot{\gamma}(s) = k(s)\mathbf{N}(s)$ except the point where $k(s) = 0$. We call $\delta(\gamma(s)) =< \mathbf{N}(s), \mathbf{N}(s) >$ *the second causal character* of $\gamma(s)$. We also define *the binormal vector* $\mathbf{B}(s)$ by $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$, where $\mathbf{T}(s) = \dot{\gamma}(s)$. We can show that $\dot{\mathbf{B}}(s) = \tau(s)\mathbf{N}(s)$ for some function $\tau(s)$. We call $\tau(s)$ *the torsion* of γ at s. By the exactly same arguments as in the case for a curve in Euclidian 3-space, we have the following Frenet-Serret formula (cf., [4]):

$$(2.1) \qquad \begin{cases} \widetilde{\nabla}_{\mathbf{T}(s)}\mathbf{T}(s) = k(s)\mathbf{N}(s) \\ \widetilde{\nabla}_{\mathbf{T}(s)}\mathbf{N}(s) = -\varepsilon(\gamma(s))\delta(\gamma(s))k(s)\mathbf{T}(s) + \varepsilon(\gamma(s))\tau(s)\mathbf{B}(s) \\ \widetilde{\nabla}_{\mathbf{T}(s)}\mathbf{B}(s) = \tau(s)\mathbf{N}(s). \end{cases}$$

We remark that if the curvature k never vanish then the second causal character $\delta(s)$ is constant along the curve γ. When the γ is a plane curve, then we have $\tau(s) \equiv 0$.

We now define the notion of pseudocircle analogous to the notion of circles in Euclidian space $\mathbb{E}^3$. Let γ be a non-light-like curve in $\mathbb{M}^3$. We say that γ is *a pseudocircle* if the torsion τ is always vanish and the curvature k is positive constant along the curve. By Frenet-Serret formula, the curve γ is a pseudocircle if and only if there exists a positive real number k such that

$$\tag{2.2} \begin{cases} \widetilde{\nabla}_{\mathbf{T}(s)}\mathbf{T}(s) = k\mathbf{N}(s) \\ \widetilde{\nabla}_{\mathbf{T}(s)}\mathbf{N}(s) = -\varepsilon(\gamma(s))\delta(\gamma(s))k\mathbf{T}(s). \end{cases}$$

We call γ ***a space-like pseudocircle*** if $\varepsilon(\gamma) = 1$ and ***a time-like pseudocircl*** if $\varepsilon(\gamma) = -1$. We also call γ ***a positive pseudocircle*** if $\delta(\gamma) = 1$ and γ ***a negative pseudocircle*** if $\delta(\gamma) = 1$. On a time-like plane, the causal characters of the tangent vector and the normal vector are different, so that we only have three cases those are the positive space-like pseudocircles, the positive time-like pseudocircles or the negative space-like pseudocircles. We have the following characterization of a pseudocircle in $\mathbb{M}^3$.

Lemma 2.2. *Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength s. Then $\gamma = \gamma(s)$ is a pseudocircle of curvature k if and only if*

$$\tag{2.3} \widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma = 0$$

Proof. Suppose that γ is a pseudocircle of curvature k. It follows from (2.2) that

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \widetilde{\nabla}_{\dot\gamma}(k\mathbf{N}) = k\widetilde{\nabla}_{\dot\gamma}\mathbf{N} = -\varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma.$$

If γ satisfies the condition (2.3), by the Frenet-Serret formula (2.1), we have

$$\begin{aligned} \widetilde{\nabla}_{\dot\gamma(s)}\widetilde{\nabla}_{\dot\gamma(s)}\dot\gamma(s) =&\widetilde{\nabla}_{\dot\gamma(s)}(k(s)\mathbf{N}(s)) \\ =&(\widetilde{\nabla}_{\dot\gamma(s)}k(s))\mathbf{N}(s) + k(s)\widetilde{\nabla}_{\dot\gamma(s)}\mathbf{N}(s) \\ =&(\widetilde{\nabla}_{\dot\gamma(s)}k(s))\mathbf{N}(s) - \varepsilon(\gamma)\delta(\gamma(s))k(s)^2\dot\gamma(s) + \varepsilon(\gamma)k(s)\tau(s)\mathbf{B}(s). \end{aligned}$$

It follows from (2.3) that

$$\widetilde{\nabla}_{\dot\gamma(s)}\nabla_{\dot\gamma(s)}\dot\gamma(s) = -\varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma(s).$$

Thus we have

$$\begin{aligned} \widetilde{\nabla}_{\dot\gamma(s)}k(s) &= 0, \\ -\varepsilon(\gamma)\delta(\gamma(s))k(s)^2 &= -\varepsilon(\gamma)\delta(\gamma(s))k^2, \\ \varepsilon(\gamma)k(s)\tau(s) &= 0, \end{aligned}$$

so that, $k(s) = k$ and $\tau(s) \equiv 0$.

We have the following result which is useful for the proof of theorems.

Lemma 2.3. *Let $\gamma = \gamma(s)$ be a non-light-like curve in $\mathbb{M}^3$ parameterized by arclength s. The $\gamma = \gamma(s)$ is a pseudocircle of curvature k if and only if it satisfies*

$$\tag{2.4} \begin{cases} \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma + \widetilde\nabla_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) - \nabla^{\perp}_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) = 0 \\ \nabla'_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + 3\sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = 0. \end{cases}$$

Proof. For any γ, we have

$$\begin{aligned} &\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma + \widetilde\nabla_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) - \nabla^{\perp}_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) \\ &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma + \widetilde\nabla_{\dot\gamma}(\widetilde\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma) - \nabla^{\perp}_{\dot\gamma}(\widetilde\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma) \\ &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma + \widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma - \nabla^{\perp}_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma \\ &= \widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma - \nabla^{\perp}_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma. \end{aligned}$$

If γ is a pseudocircle, it follows from (2.3) that $\nabla^{\perp}_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma = 0$, so that the first equation of (2.4) holds.

We also have

$$\begin{aligned} &\nabla'_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + 3\sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) \\ &= \nabla^{\perp}_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)) - 2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) + 3\sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) \\ &= \nabla^{\perp}_{\dot\gamma}(\sigma(\dot\gamma,\dot\gamma)) + \sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) \\ &= \nabla^{\perp}_{\dot\gamma}(\widetilde\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma) + \widetilde\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma \\ &= \nabla^{\perp}_{\dot\gamma}(\widetilde\nabla_{\dot\gamma}\dot\gamma - \nabla_{\dot\gamma}\dot\gamma) + \nabla^{\perp}_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma \\ &= \nabla^{\perp}_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma. \end{aligned}$$

It follows from (2.3) that $\nabla^{\perp}_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma = 0$.

Suppose that the equalities (2.4) hold for some positive number k, then we have

$$\begin{aligned} &\widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}\dot\gamma + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma \\ &= \widetilde\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma + \sigma(\dot\gamma,\dot\gamma)) + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma \\ &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) + \widetilde\nabla_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) - \nabla^{\perp}_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + \nabla^{\perp}_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + \varepsilon(\gamma)\delta(\gamma)k^2\dot\gamma \\ &= \sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) + \nabla^{\perp}_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) \\ &= \sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) + \nabla'_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma,\dot\gamma) \\ &= \nabla'_{\dot\gamma}\sigma(\dot\gamma,\dot\gamma) + 3\sigma(\dot\gamma,\nabla_{\dot\gamma}\dot\gamma) = 0 \end{aligned}$$

It follows from Lemma 2.2 that γ is a pseudocircle of curvature k.

We can show that the intersection of a pseudosphere $S^2_1(r,a)$ with a non-degenerate plane is a pseudocircle or a pair of light-like lines.

3. Proof of results

In this section we give proofs of the theorems. The method of the proofs almost follows along the line of the proof of Theorem 1 in [2], however, there are some differences from the original proof caused by the index of the metric tensor. There are common parts of the proofs for Theorems A and B, so we firstly start to prove Theorem A.

Proof of Theorem A. Let k_i be the curvature of γ_i $(i=1,2)$. Let X_p be the common unit tangent vector of γ_1 and γ_2 at p. Then $X: p \mapsto X_p$ defines a vector field on S, which may not be continuous. Let Y be a unit vector field on S which satisfies $<\mathbf{X},\mathbf{Y}>=0$ and $<\mathbf{X}\wedge\mathbf{Y},\xi>>0$, where ξ is the unit normal vector field on S. Let $\mathbf{X}_1,\mathbf{X}_2$ be smooth unit vector fields on a neighbourhood of $\gamma_1\cup\gamma_2$ such that $\mathbf{X}_i$ is equal to the unit tangent vector of γ_i on γ_i and $\mathbf{X}_i(p)=X_p$ $(i=1,2)$. It follows that $<\nabla_{\mathbf{X}_i}\mathbf{X}_i,\mathbf{X}_i>=0$. Thus there exist real numbers c_i $(i=1,2)$ such that $\nabla_{\mathbf{X}_i}\mathbf{X}_i(p)=c_i\mathbf{Y}(p)$. Suppose that $c_1=c_2$. Since $\widetilde{\nabla}_{\mathbf{X}_i}\mathbf{X}_i=\nabla_{\mathbf{X}_i}\mathbf{X}_i+\sigma(\mathbf{X}_i,\mathbf{X}_i)$ $(i=1,2)$, and $\mathbf{X}_i(p)=\mathbf{X}_p$, we have $\widetilde{\nabla}_{\mathbf{X}_1}\mathbf{X}_1(p)=\widetilde{\nabla}_{\mathbf{X}_2}\mathbf{X}_2(p)$. The last equation means that $k_1=k(\gamma_1(p))=k(\gamma_2(p))=k_2$, so that $\gamma_1=\gamma_2$ on a neighbourhood of p. This contradicts to the assumption (i.e., $c_1\neq c_2$).

It follows from Lemma 2.3 that

$$\left\{\begin{array}{l}\nabla'_{\mathbf{X}}\sigma(\mathbf{X},\mathbf{X})+3\sigma(\mathbf{X},c_1\mathbf{Y})=0\\ \nabla'_{\mathbf{X}}\sigma(\mathbf{X},\mathbf{X})+3\sigma(\mathbf{X},c_2\mathbf{Y})=0\end{array}\right. \tag{3.1}$$

at each point. Therefore we have $\sigma(\mathbf{X},\mathbf{Y})=0$. It follows that $<A\mathbf{X},\mathbf{Y}>=<\sigma(\mathbf{X},\mathbf{Y}),\xi>=0$. Since $\dim S=2$, we see that $A\mathbf{X}$ is parallel to $\mathbf{X}$. Thus $\mathbf{X}$ is (and hence $\mathbf{Y}$ is also) a principal vector at each point. Let λ and μ be principal curvature and η_1 and η_2 the corresponding principal vectors so that $A\eta_1=\lambda\eta_1$ and $A\eta_2=\mu\eta_2$. Put $S_0=\{p\in S|\lambda(p)\neq\mu(p)\}$. If $S_0=\emptyset$, then S is totally umbilic. Therefore we suppose that $S_0\neq\emptyset$. Then η_1 and η_2 are smooth vector fields in some neighbourhood of each point of S_0. Since $<\eta_1,\eta_2>=0$, we may assume that η_1 is space-like and η_2 is time-like. Since $<\eta_1,\eta_2>=1$, we have

$$<\nabla_{\eta_i}\eta_1,\eta_1>=0\quad(i=1,2), \tag{3.2}$$

so that we may put

$$\nabla_{\eta_1}\eta_1=\alpha\eta_2 \text{ and } \nabla_{\eta_2}\eta_1=\beta\eta_2. \tag{3.3}$$

It follows from $<\eta_1,\eta_2>=0$ that

$$<\nabla_{\eta_i}\eta_1,\eta_2>+<\eta_1,\nabla_{\eta_i}\eta_2>=0\quad(i=1,2). \tag{3.4}$$

We substitute (3.30 to (4.3), then we have $<\eta_1,\nabla_{\eta_1}\eta_2>\alpha$ and $<\eta_1,\nabla_{\eta_2}\eta_2>=\beta$. Since η_1 is space-like, we have

$$\nabla_{\eta_1}\eta_2=\alpha\eta_1 \text{ and } \nabla_{\eta_2}\eta_2=\beta\eta_1. \tag{3.5}$$

Put $S_{0,i}=\{p\in S|\mathbf{X}(p)=\eta_i(p)\}$ $(i=1,2)$. Then $S_0=S_{o,1}\cup S_{0,2}$, and it is easily seen that $S_0\subset\overline{S_{0,1}}\cup\operatorname{Int}S_{0,2}$ and $S_0\subset\overline{S_{0,2}}\cup\operatorname{Int}S_{0,1}$ and hence that $S_{0,1}$ or $S_{0,2}$ has interior points, or $S_{0,1}$ or $S_{0,2}$ is dense in S_0. We now distinguish the following two cases.

Case 1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in S_0.
Case 2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in S_0.

The situation is different between two cases. In the case 1) both of two pseudo circles through p are space-like. On the other hand, both of the pseudocircles are positive time-like in the case 2), so we only consider the case 1).

In this case $X(p)$ is space-like. Since $\sigma(\mathbf{X}, \mathbf{Y}) = 0$, we obtain

$$\begin{aligned}\nabla'_{\eta_1}\sigma(\eta_1, \eta_1) =& \nabla^{\perp}_{\eta_1}\sigma(\eta_1, \eta_1) - 2\sigma(\eta_1, \nabla_{\eta_1}\eta_1)\\ =& \nabla^{\perp}_{\eta_1}(\lambda\xi) - 2\alpha\sigma(\eta_1, \eta_2)\\ =& \nabla^{\perp}_{\eta_1}(\lambda\xi)\\ =& (\nabla^{\perp}_{\eta_1}\lambda)\xi + \lambda\nabla^{\perp}_{\eta_1}\xi\\ =& (\nabla^{\perp}_{\eta_1}\lambda)\xi.\end{aligned}$$

It follows from (3.1) and $\sigma(\mathbf{X}, \mathbf{Y}) = 0$ that

$$\nabla'_{\mathbf{X}}\sigma(\mathbf{X}, \mathbf{X}) = 0. \tag{3.6}$$

Therefore we have $\nabla_{\mathbf{X}}\lambda = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$, then

$$\nabla_{\eta_1}\lambda = 0 \tag{3.7}$$

holds in some neighbourhood of p. If $S_{0,1}$ is dense in S_0, then, by continuity, (3.7) holds on S_0.

We choose an orthonormal frame field e_1, e_2 in a sufficiently small tubular neighbourhood of γ_1 in such a way that $e_1 = \dot{\gamma}_1$ along γ_1, and put $\nabla_{e_1}e_1 = ae_2$ and $\nabla_{e_2}e_1 = be_2$ so that $\nabla_{e_1}e_2 = ae_1$ and $\nabla_{e_2}e_2 = be_1$. Let (h_{ij}) be the matrix of the shape operator with respect to e_1 and e_2 (i.e., $-\nabla_{e_i}\xi = h_{i1}e_1 + h_{i2}e_2 \quad (i = 1, 2)$).

On the other hand, we have

$$\widetilde{\nabla}_{e_1}\sigma(e_1, e_1) - \nabla^{\perp}_{e_1}\sigma(e_1, e_1) = -(h_{11}^2 e_1 + h_{11}h_{12}e_2)$$

and

$$\nabla_{e_1}\nabla_{e_1}e_1 = a^2 e_1 + (\nabla_{e_1}a)e_2.$$

By Lemma 2.3, we have

$$\{a^2 e_1 + (\nabla_{e_1}a)e_2\} + \delta(\gamma_1)k^2 e_1 - \{h_{11}^2 e_1 + h_{11}h_{12}e_2\} = 0,$$

so that, along γ_1,

$$\delta(\gamma_1)k^2 = -a^2 + h_{11}^2 \tag{3.8}$$

$$\nabla_{e_1}a = h_{11}h_{12}. \tag{3.9}$$

It also follows from the second equation of Lemma 2.3 that

$$\nabla^{\perp}_{e_1}\sigma(e_1,e_1)+\sigma(e_1,\nabla_{e_1}e_1)=0.$$

Since

$$\nabla^{\perp}_{e_1}\sigma(e_1,e_1)=\nabla^{\perp}_{e_1}\{<-\nabla_{e_1}\xi,e_1>\xi\}=(\nabla^{\perp}_{e_1}h_{11})\xi$$

and

$$\sigma(e_1,\nabla_{e_1}e_1)=\sigma(e_1,ae_2)=-ah_{12}\xi,$$

we have

$$\nabla_{e_1}h_{11}-ah_{12}=0. \tag{3.10}$$

We remark that ξ_1, e_1 are space-like and ξ_2, e_2 are time-like. We define θ such that

$$\begin{cases} e_1 =\eta_1\cosh\theta+\eta_2\sinh\theta\\ e_2 =\eta_1\sinh\theta+\eta_2\cosh\theta. \end{cases} \tag{3.11}$$

So we have

$$\begin{cases} Ae_1 =\lambda\eta_1\cosh\theta+\mu\eta_2\sinh\theta\\ Ae_2 =\lambda\eta_1\sinh\theta+\mu\eta_2\cosh\theta, \end{cases}$$

we get

$$\begin{cases} h_{11} =\lambda\cosh^2\theta-\mu\sinh^2\theta\\ h_{12} =-(\lambda-\mu)\cosh\theta\sinh\theta\\ h_{22} =-\lambda\sinh^2\theta+\mu\cosh^2\theta. \end{cases} \tag{3.12}$$

By differentiating (3.11) with respect to e_1, we have

$$\nabla_{e_1}e_1=\eta_1\{\alpha\cosh\theta+\beta\sinh\theta+\nabla_{e_1}\theta\}\sinh\theta+\eta_2\{\alpha\cosh\theta+\beta\sinh\theta+\nabla_{e_1}\theta\}\cosh\theta.$$

Since $\nabla_{e_1}e_1=ae_2=a\eta_1\sinh\theta+a\eta_2\cosh\theta$, we obtain

$$a=\alpha\cosh\theta+\beta\sinh\theta+\nabla_{e_1}\theta. \tag{3.13}$$

We also have $(\nabla_{\eta_1}A)(\eta_2)=\nabla_{\eta_1}(A\eta_2)-A\nabla_{\eta_1}\eta_2=(\nabla_{\eta_1}\mu)\eta_2+\alpha(\mu-\lambda)\eta_1$ and $(\nabla_{\eta_2}A)(\eta_1)=(\nabla_{\eta_2}\lambda)\eta_1+\beta(\lambda-\mu)\eta_2$. By the equation of Coddazzi $(\nabla_{\eta_1}A)(\eta_2)-(\nabla_{\eta_2}A)(\eta_1)=0$, we have

$$\begin{cases} \alpha(\lambda-\mu) =\nabla_{\eta_2}\lambda\\ \beta(\lambda-\mu) =\nabla_{\eta_1}\mu. \end{cases} \tag{3.14}$$

Therefore by (3.7),(3.10),(3.12),(3.13) and (3.14) we obtain

$$\{3(\lambda-\mu)\cosh\theta\cdot\nabla_{e_1}\theta-\sinh^2\theta\cdot\nabla_{\eta_2}\mu\}\sinh\theta=0. \tag{3.15}$$

We remark that $\theta = 0$ at p. Then the point p has one of the following properties:
(A) There exists no sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $\lim p_n = p$.
(B) There exists a sequence $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$ with $\lim p_n = p$.

If p is a point which has the property (A), then it is clear that the integral curve of η_1 through p coincides with γ_1 on the connected component of $\{q \in \gamma_1 | \theta(q) = 0\}$ containing p.

If p is a point with the property (B), it follows from (3.15) that

$$3(\lambda - \mu)\cosh\theta \cdot \nabla_{e_1}\theta - \sinh^2\theta \cdot \nabla_{\eta_2}\mu = 0. \tag{3.16}$$

Since $\lambda \neq \mu$, taking to the limit of (3.16), we obtain

$$\nabla_{e_1}\theta = 0. \tag{3.17}$$

Applying ∇_{e_1} to (3.15) and substituting the relation (3.16) into the equation on the set $\{p_n \in \gamma_1 | \theta(p_n) \neq 0\}$, obtain

$$\begin{aligned}
&\frac{\sinh\theta \cdot \nabla_{\eta_2}\mu}{3(\lambda - \mu)}\{3(\lambda - \mu)\cosh\theta \cdot \nabla_{e_1}\theta - \sinh^2\theta \cdot \nabla_{\eta_2}\mu\} \\
&+\{3(\nabla_{e_1}\lambda - \nabla_{e_1}\mu)\cosh\theta \cdot \nabla_{e_1}\theta + 3(\lambda - \mu)\sinh\theta \cdot (\nabla_{e_1}\theta)^2 \\
&\quad +3(\lambda - \mu)\cosh\theta \cdot \nabla_{e_1}\nabla_{e_1}\theta - 2\sinh\theta\cosh\theta \cdot \nabla_{e_1}\nabla_{\eta_2}\mu - \sinh^2\theta \cdot \nabla_{e_1}\nabla_{\eta_2}\mu\} = 0.
\end{aligned}$$

Taking the limit of the above equation we have

$$\nabla_{e_1}\nabla_{e_1}\theta = 0, \tag{3.18}$$

since $\sinh\theta = 0$ and $\cosh\theta = 1$. It is clear that the (3.17) and (3.18) hold even if p has the property (A).

It follows from (3.13) and (3.17) that $a(p) = \alpha(p)$. This, together with (3.8) and (3.12), yields

$$\delta(\gamma_1)k^2 = -(\alpha(p))^2 + (\lambda(p))^2, \tag{3.19}$$

which means that the curvature of γ_1 is $\sqrt{\delta\{-(\alpha(p))^2 + (\lambda(p))^2\}}$.

For γ_2, by exactly the same arguments as those of γ_1, we obtain

$$\delta(\gamma_2)k_2^2 = -(\alpha(p))^2 + (\lambda(p))^2, \tag{3.20}$$

where k_2 is the curvature of γ_2. Until here we have not used the condition that $\delta(\gamma_1)\delta(\gamma_2) = -1$. It follows from (3.19), (3.18) and the assumption $\delta(\gamma_1)\delta(\gamma_2) = -1$ that we have

$$(-(\alpha(p))^2 + (\gamma(\lambda(p))^2)^2 < 0$$

This is a contradiction, so that $S_0 = \emptyset$ and S is totally umbilic. Since the time-like plane does not satisfy the condition that $\delta(\gamma_1)\delta(\gamma_2) = -1$, S is an open set of a pseudosphere.

Proof of Theorem B. Since we already proved Theorem A, we may consider the case that $\delta(\gamma_1)\delta(\gamma_2) = 1$. We use the same notations as those of the proof of Theorem A. So our purpose is to show that $S_0 = \emptyset$. Therefore we suppose that $S_0 \neq \emptyset$, so that we also distinguish the following two cases:

Case 1) $S_{0,1}$ has interior points or $S_{0,1}$ is dense in S_0.

Case 2) $S_{0,2}$ has interior points or $S_{0,2}$ is dense in S_0.

Both of the pseudocircles through p are space-like in the case 1) and are positive time-like in the case 2). For a positive time-like curve, we have $\varepsilon\delta = -1$, so that the equations (2.3) and (2.4) have the same form as equations for a negative space-like curve. It follows that the almost all arguments for positive time-like curves are same as those for negative space-like curve. Hence, we only consider the case 1).

We may use the equations (3.1)-(3.20) for the proof of Theorem B. Since $\delta(\gamma_1)\delta(\gamma_2) = 1$, γ_1 and γ_2 are space-like pseudocircles with the common same curvature $\sqrt{-(\alpha(p))^2 + (\lambda(p))^2}$.

Applying ∇_{e_1} to (3.13), we obtain $\nabla_{\eta_1} a = \nabla_{\eta_1}\alpha$ at p because of (3.17) and (3.18).

On the other hand, from (3.9) we get $\nabla_{\eta_1} a = \nabla_{e_1} a = h_{11}h_{12} = 0$ at p. Therefore we have $\nabla_{\eta_1}\alpha = 0$ at p. Since p is arbitrary, we get $\nabla_{\eta_1}\alpha = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$, then

$$(3.21) \qquad \nabla_{\eta_1}\alpha = 0$$

holds in some neighbourhood of p. If $S_{0,1}$ is dense in S_0, then, by continuity, (3.21) holds on S_0. It follows from (3.21) that

$$\nabla_{\eta_1}\nabla_{\eta_1}\eta_1 = \nabla_{\eta_1}(\alpha\eta_2) = \alpha^2\eta_1.$$

Moreover, since η_1 is a principal vector, we get

$$\begin{aligned}
&< \tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1) - \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1), V >=< \tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1), V > - < \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1), V > \\
&=< \tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1), V >= < \tilde{\nabla}_{\eta_1}\xi, V >=< (\nabla_{\eta_1}\lambda)\xi + \lambda\tilde{\nabla}_{eta_1}\xi, V >= \lambda < \tilde{\nabla}_{\eta_1}\xi, V > \\
&= \lambda < -A\eta_1, V >= -\lambda^2 < \eta_1, V >
\end{aligned}$$

for any tangent vector field V on S. Therefore we obtain

$$(3.22) \qquad \nabla_{\eta_1}\nabla_{\eta_1}\eta_1 + (-\alpha^2 + \lambda^2)\eta_1 + \tilde{\nabla}_{\eta_1}\sigma(\eta_1,\eta_1) - \nabla^{\perp}_{\eta_1}\sigma(\eta_1,\eta_1) = 0.$$

Furthermore, since $\nabla_{\eta_1}\eta_1 = \alpha\eta_2$, it follows from $\sigma(\mathbf{X},\mathbf{Y}) = 0$ and (3.6) that $(\nabla'_{\eta_1}\sigma)(\eta_1,\eta_1) + 3\sigma(\eta_1,\nabla_{\eta_1}\eta_1) = 0$ on $S_{0,1}$. If p is an interior point of $S_{0,1}$,

$$(3.23) \qquad (\nabla'_{\eta_1}\sigma)(\eta_1,\eta_1) + 3\sigma(\eta_1,\nabla_{\eta_1}\eta_1) = 0$$

holds in some neighbourhood of p. If $S_{0,1}$ is dense in S_0, then, by continuity, (3.23) holds on S_0. By (3.22), (3.23) and Lemma 2.3, we see that the integral curve of η_1 through p is a space-like pseudocircle of curvature $\sqrt{\delta(\eta_1)\{-(\alpha(p))^2 + (\lambda(p))^2\}}$, where $\delta(\eta_1) =< \eta_1, \eta_1 >$. By (3.7) and (3.21), the above curvature is constant along the curve. It is clear that $\delta(\eta_1) = \delta(\gamma_1)$.

Since we can apply the same arguments to γ_2, letting θ_i $(i = 1, 2)$ be the function defined the same as the function of (3.11), we consider the following cases:
$(A)_i$: There exists no sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.
$(B)_i$: There exists a sequence $\{p_n \in \gamma_i | \theta_i(p_n) \neq 0\}$ with $p = \lim p_n$.

It is clear that $(A)_1$ and $(A)_2$ does not occur. If $(B)_1$ and $(A)_2$, then γ_1 and γ_2 have the same curvature and the integral curve of η_1 thorough p coincide with γ_2. Then we have

$$\begin{aligned}\widetilde{\nabla}_{\eta_1}\eta_1 &= \nabla_{\eta_1}\eta_1 + \sigma(\eta_1, \eta_1) = \alpha\eta_2 + \lambda\xi. \\ \widetilde{\nabla}_{e_1}e_1 &= \nabla_{e_1}e_1 + \sigma(e_1, e_1) = ae_2 + h_{11}\xi.\end{aligned}$$

Since $d = a$, $\lambda = h_{11}$ and $\eta_2 = e_2$ at p, we obtain $\widetilde{\nabla}_{\eta_1}\eta_1 = \widetilde{\nabla}_{e_1}e_1$. This equation means that γ_1 and γ_2 coincide in a neighbourhood of p. This contradicts the assumption. By the same arguments as the above case, the case $(A)_1$ and $(B)_2$ contradicts the assumption. If $(B)_1$ and $(B)_2$, then γ_1, γ_2 and the integral curve of η_1 through p have the same curvature and hence, we can show that normal vectors for these vectors are the same. It follows that γ_1 and γ_2 coincide in a neighbourhood of p. This contradicts the assumption, so that $S_0 = \emptyset$. This completes the proof.

References

1. S. Izumiya and A. Takiyama, *A time-like surface in Minkowski 3-space which contains light-like lines*, preprint (1995).
2. K. Ogiue and R. Takagi, *A Submanifold which contains Many Extrinsic Circles*, Tsukuba J. Math. **8** (1984), 171–182.
3. B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.