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Series ♯295. June 1995

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Shyuichi IZUMIYA

1 Introduction

We consider the following partial differential equation which is called *a quasilinear first order partial differential equation* (briefly, a quasilinear equation):

$$\sum_{i=1}^{n} a_i(x,y)\frac{\partial y}{\partial x_i} - b(x,y) = 0$$

where $a_i(x,y)$ and $b(x,y)$ are C^∞-functions. This equation is well studied in several articles ([3],[5],[6],[7],[9],[11], etc.).

One of the most interesting qusilinear equations are scalar conservatin laws. The notion of entropy solutions has provided the right weak setting for the study of the Cauchy problem for scalar conservation laws. Existence and uniqueness of the entropy solution have been established in [7].

On the other hand, if we solve the eqation by the classical method of characteristics, in generally, we have multi-valued solutions. The multi-valued solution is classical solution near the Cauchy data and it coinicdes with the entropy solution, however, around *the singular point* of the multi-valued solution, the entropy solution has *shock waves* (i.e., surfaces across which the entropy solution is discontinuous). In order to understand the process how shock waves appear, interact and bifurcate, we need to classify sinuglar points of multi-valued solutions. After that we can solve Riemann problem for each normal form of the classification. In [5] we have studied a classification of multi-valued solutions of time-dependent quasilinear equations in order to study shock waves of the Cauchy problem for scalar conservation laws. Most of the techniques we use here have been developed in [5], however, the situation here is slightly different. For time-dependent equations, we distinguish the time parameter and the space parameters, so the bifurcations of singularities of multi-valued solutions along the time parameter is most interesting. Here we consider stationary quasilinear equations, so that there are no reason to distinguish a special parameter from the space parameters. There is another motivation to consider the stationary quasilinear equations. Recently, Tsuji[10] studied the singularities of solutions for hyperbolic Monge-Ampère equations, in which the notion of intermediate integral is quite important. One of the examples of such first order partial differential equations are quasilinear equations. We can show that each quasilinear equation can be realized as an intermediate integral of a Monge-Ampère equation (cf., [10]). So our result gives a classification of multi-valued solutions of hyperbolic Monge-Ampère equations whose intermediate integral are quasilinear equations.

All maps are differentiable of class C^∞ unless stated otherwise.

2 Geometry of quasilinear equations

We construct the geometric framework of equations in the projective cotangent bundle $\pi : PT^*(\mathbf{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$. Consider the tangent bundle $\tau : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to PT^*(\mathbb{R}^n \times \mathbb{R})$ and the differential map $d\pi : TPT^*(\mathbb{R}^n \times \mathbb{R}) \to T(\mathbb{R}^n \times \mathbb{R})$ of π.

For any $X \in TPT^*(\mathbb{R}^n \times \mathbb{R})$, there exists an element $\alpha \in T^*_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$ such that $\tau(X) = [\alpha]$. For an element $V \in T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(\mathbb{R}^n \times \mathbb{R})$ by

$$K = \{X \in TPT^*(\mathbb{R}^n \times \mathbb{R}) | \tau(X)(d\pi(X)) = 0\}.$$

Because of the trivialization $PT^*(\mathbb{R}^n \times \mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}) \times P(\mathbb{R}^n \times \mathbb{R})^*$, we call

$$((x_1, \ldots, x_n, y), [\xi_1; \ldots; \xi_n; \eta])$$

a homogeneous coordinate, where $[\xi_1; \ldots; \xi_n; \eta]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^n \times \mathbb{R})^*$.

It is easy to show that $X \in K_{((x,y),[\xi;\eta])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i + \lambda\eta = 0$, where $d\tilde{\pi}(X) = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \lambda \frac{\partial}{\partial y}$.

We remark that $PT^*(\mathbb{R}^n \times \mathbb{R})$ is a fibrewise compactification of the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ as follows : We consider an open subset $U_\eta = \{((x,y),[\xi;\eta]) | \eta \neq 0\}$ of $PT^*(\mathbb{R}^n \times \mathbb{R})$. For any $((x,y),[\xi;\eta]) \in U_\eta$, we have

$$((x_1, \ldots, x_n, y), [\xi_1; \ldots; \xi_n; \eta]) = ((x_1, \ldots, x_n, y), [-\frac{\xi_1}{\eta}; \ldots; -\frac{\xi_n}{\eta}; -1])$$

so that we may adopt the corresponding *affine coordinates*

$$((x_1, \ldots, x_n, y), (p_1, \ldots, p_n)),$$

where $p_i = -\frac{\xi_i}{\eta}$. On U_η we can easily show that $\theta^{-1}(0) = K|U_\eta$, where $\theta = dy - \sum_{i=1}^n p_i dx_i$. This means that U_η may be identified with the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$. We call the above coordinate *a system of canonical coordinates.* Throughout the remainder of this paper, we use this identification so that we have

$$J^1(\mathbb{R}^n,\mathbb{R}) \subset PT^*(\mathbb{R}^n \times \mathbb{R}).$$

A quasilinear equation is defined to be a hypersurface

$$E(a_1,\ldots,a_n,b) = \{((x,y),[\xi;\eta]) \in PT^*(\mathbb{R}^n \times \mathbb{R}) | \sum_{i=1}^n a_i(x,y)\xi_i + b(x,y)\eta = 0\}$$

where $a_i(x,y), b(x,y)$ are C^∞ -function.

We usually assume that $(a_1(x,y),\ldots,a_n(x,y)) \neq (0,\ldots,0)$.

We remark that $\sum_{i=1}^n a_i(x,y)\xi + b(x,y)\eta = 0$ if and only if $\sum_{i=1}^n a_i(x,y)p_i - b(x,y) = 0$ under the canonical coordinate of $U_\eta = J^1(\mathbb{R}^n,\mathbb{R})$. So the above definition is consistent with the classical definition of quasilinear equations.

In accordance with the philosophy of Lie, we may define the notion of geometric solutions as follows: An immersion $i : L \to PT^*(\mathbb{R}^n \times \mathbb{R})$ is *a Legendrian immersion* if $\dim \mathrm{L} = \mathrm{n}$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. *A geometric solution* of $E(a_1,\ldots,a_n,b)$ is defined to be a Legendrian immersion $i : L \to PT^*(\mathbb{R}^n \times \mathbb{R})$ such that $\mathrm{Image}(L) \subset E(a_1,\ldots,a_n,b)$. We also say that the geometric solution is *smooth* if $\pi|L : L \to \mathbb{R}^n \times \mathbb{R}$ is an immersion.

We consider the meaning of the notion of geometric solutions. Let S be a smooth hypersurface in $\mathbb{R}^n \times \mathbb{R}$, then we have a unique Legendrian submanifold $\hat{S}$ in $PT^*(\mathbb{R}^n \times \mathbb{R})$ such that $\pi(\hat{S}) = S$ which is given as follows :

$$\hat{S} = \{((x,y),[\xi;\eta]) | \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial y} \in T_{(x,y)}S^\perp\}.$$

It follows that if L is a smooth geometric solution of $E(a_1,\ldots,a_n,b)$, then we have $L = \widehat{\pi(L)}$.

We consider the condition on a smooth hypersurface S that $\hat{S} \subset E(a_1,\ldots,a_n,b)$. For any $(x_0,y_0) \in S$, there exists a smooth submersion germ $f : (\mathbb{R}^n \times \mathbb{R},(x_0,y_0)) \to (\mathbb{R},0)$ such that $(f^{-1}(0),(x_0,y_0)) = (S,(x_0,y_0))$. A vector $\sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \lambda \frac{\partial}{\partial y}$ is tangent to S at $(x,y) \in (S,(x_0,y_0))$ if and only if it satisfies $\sum_{i=1}^n \mu_i \frac{\partial f}{\partial x_i} + \lambda \frac{\partial f}{\partial y} = 0$ at (x,y). Then we have the following representation of $\hat{S}$:

$$(\hat{S},((x_0,y_0);[\xi_0;\eta_0])) = \{((x,y),[\frac{\partial f}{\partial x};\frac{\partial f}{\partial y}]) | (x,y) \in (S,(x_0,y_0))\}.$$

We now consider the following vector field on $\mathbb{R}^n \times \mathbb{R}$ associated with the quasilinear equation $E(a_1,\ldots,a_n,b)$:

$$X(a_1,\ldots,a_n,b) = \sum_{i=1}^n a_i(x,y)\frac{\partial}{\partial x_i} + b(x,y)\frac{\partial}{\partial y}.$$

Then we have the following proposition.

Proposition 2.1. *Let S be a smooth hypersurface in $\mathbb{R}^n \times \mathbb{R}$. Then $\hat{S}$ is a smooth geometric solution of $E(a_1, \ldots, a_n, b)$ if and only if the vector field $X(a_1, \ldots, a_n, b)$ is tangent to S.*

Proof. We have a local representation

$$(\hat{S}, ((x_0, y_0); [\xi_0; \eta_0])) = \{((x, y), [\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}]) | (x, y) \in (S, (x_0, y_0))\},$$

where $f : (\mathbb{R}^n \times \mathbb{R}, (x_0, y_0)) \to (\mathbb{R}, 0)$ is a submersion germ with $f^{-1}(0) = S$. It follows that $\hat{S} \subset E(a_1, \ldots, a_n, b)$ if and only if

$$\sum_{i=1}^{n} a_i(x, y) \frac{\partial f}{\partial x_i} + b(x, y) \frac{\partial f}{\partial y} = 0$$

for $(x, y) \in (S, (x_0, y_0))$. The last condition is equivalent to the condition that $X(a_1, \ldots, a_n, b)$ is tangent to S at (x, y). qed

The Cauchy problem can be solved by the method of characteristics. We say that *a geometric Cauchy problem (GCP)* for an equation $E(a_1, \ldots, a_n, b)$ is a given $(n-1)$-dimensional submanifold $i' : S' \subset \mathbb{R}^n \times \mathbb{R}$ such that the characteristic vector field $X(a_1, \ldots, a_n, b)$ is not tangent to S'. The following result is well-known (cf. [1]), however it clarifies the geometric situation in this paper.

Theorem 2.2 (Local existence theorem). *A GCP $i' : S' \subset \mathbb{R}^n \times \mathbb{R}$ has a unique smooth geometric solution, that is, there exists a smooth hypersurface $S \subset \mathbb{R}^n \times \mathbb{R}$, $S' \subset S$ such that $\hat{S} \subset E(a_1, \ldots, a_n, b)$, and any two such smooth hypersurfaces coincide in a neighborhood of S'.*

Proof. Consider the embedding $i : S \subset \mathbb{R}^n \times \mathbb{R}$, where $S \subset (-\varepsilon, \varepsilon) \times S'$ is a neighborhood of $0 \times S'$, $\varepsilon > 0$; here $i(t, q) = T_t(q)$, $q \in S'$, $(t, q) \in S$, and T_t is an one-parameter group of translations along $X(a_1, \ldots, a_n, b)$. By the construction, we have $X(a_1, \ldots, a_n, b) \in TS$,so that $\hat{S} \subset E(a_1, \ldots, a_n, b)$ by Proposition 2.1.

On the other hand, for any geometric solution L of $E(a_1, \ldots, a_n, b)$, $\pi(L)$ must be invariant under $X(a_1, \ldots, a_n, b)$, so that it coincides with S in some neighborhood. qed

We also consider the notion of classical solutions of $E(a_1, \ldots, a_n, b)$. We say that a C^∞-function $y = g(x_1, \ldots, x_n)$ is *a classical solution* of $E(a_1, \ldots, a_n, b)$ if it satisfies

$$\sum_{i=1}^{n} a_i(x, g(x)) \frac{\partial g}{\partial x_i} - b(x, g(x)) = 0.$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, then we have a smooth hypersurface $S_g = \{(x, g(x)) | x \in \mathbb{R}^n\}$ in $\mathbb{R}^n \times \mathbb{R}$. Since $S_g = f^{-1}(0)$, where $f(x,y) = g(x) - y$, it follows from the previous arguments that $\widehat{S_g}$ is a smooth geometric solution of $E(a_1, \ldots, a_n, b)$ if and only if $y = g(x_1, \ldots, x_n)$ is a classical solution.

On the other hand, we consider the canonical projection $\pi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ given by $\pi_1(x,y) = x$. Let L be a smooth geometric solution of $E(a_1, \ldots, a_n, b)$. We say that $q \in L$ is *a singular point* if $\operatorname{rank} \mathrm{d}(\pi_1 \circ \pi | \mathrm{L}) < \mathrm{n}$. We remark that $q \in L$ is not a singular point if and only if there exists a smooth function germ $g : (\mathbb{R}^n, x_0) \to \mathbb{R}$ such that $(S, (x_0, y_0)) = (S_g, (x_0, y_0))$. Our problem in this paper is summarized as follows:

Problem. Classify the generic singularities of smooth geometric solutions of GCP for a quasilinear equation which is solved by the characteristic method.

3 Projections of smooth hypersurfaces

The smooth geometric solution of GCP for a quasilinear equation is a smooth hypersurface and the multi-valuedness of geometric solutions is caused by sinuglarities of the projection on to $\mathbb{R}^n$, so that we consider the classification of singularities of the projection of hypersurfaces. Since any smooth hypersurface is locally given as a zero point set of a smooth submersion, we consider the following. Let $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a submersion germ, then $F^{-1}(0)$ is a smooth hypersurface in $(\mathbb{R}^n \times \mathbb{R}, 0)$. For the study of the singularities of the projection $\pi_F : (F^{-1}(0), 0) \to \mathbb{R}^n$, we define the following equivalence relation. Let

$$F_i : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0), \ (i = 0, 1)$$

be submersion germs. We say that F_0 and F_1 are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$$

of the form

$$\Phi(x,y) = (\phi_1(x), \phi_2(x,y))$$

such that

$$\langle F_1 \circ \Phi \rangle_{\mathcal{E}_{(x,y)}} = \langle F_0 \rangle_{\mathcal{E}_{(x,y)}},$$

where $\langle F_0 \rangle_{\mathcal{E}_{(x,y)}}$ denotes the ideal generated by F_0 in the ring $\mathcal{E}_{(x,y)}$ of function germs of (x,y)-variables at the origin.

Our purpose in this section is to classify submersions F under the P-$\mathcal{K}$-equivalence. For this purpose we need some notions from singularity theory (cf., [4], [8]). For a submersion germ $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$, we define

$$T_e(\mathcal{K})(f) = \langle \frac{\partial f}{\partial y}, f \rangle_{\mathcal{E}_y},$$

where $f = F|0 \times \mathbb{R}$. We say that F is *a $\mathcal{K}$-versal deformation of f* if it satisfies

$$\mathcal{E}_y = \langle \frac{\partial F}{\partial x_1}|0 \times \mathbb{R}, \ldots, \frac{\partial F}{\partial x_n}|0 \times \mathbb{R} \rangle_{\mathbb{R}} + T_e(\mathcal{K})(f).$$

The above definiton is an infinitesimal version of the definition of the $\mathcal{K}$-versality, however it is known that these are equivalent (cf., [8]), so that we adopt the above definiton.

It is well known that the following classification theorem holds (cf.,[2],[8]).

Theorem 3.1. *Let $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a $\mathcal{K}$-versal deformation of f, then F is P-$\mathcal{K}$-equivalent to one of the following ${}^0A_k$-type germs $(0 \leq k \leq n)$:*

$$y^{k+1} + \sum_{i=1}^{k} x_i y^{i-1}.$$

Remarks. (1) The above theorem gives a "generic" classification of subersion germs under the P-$\mathcal{K}$-equivalence.
(2) If $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ is a $\mathcal{K}$-versal deformation of f, then π_F is an $\mathcal{A}$-stable map germ in the sense of Mather (cf., [9]).

The following theorem is the classification theorem of Mather (cf., [8]) and it is quite useful in the latter sections.

Theorem 3.2 (J. Mather). *Let $F_i : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be $\mathcal{K}$-versal deformations of $f_i = F_i|0 \times \mathbb{R}$, $i = 0, 1$. Then f_0 and f_1 are $\mathcal{K}$-equivalent if and only if F_0 and F_1 are* P-*$\mathcal{K}$-equivalent.*

The definiton of the $\mathcal{K}$-equivalence can be found in ([8]).

4 Realization theorems

In this section we identify geometric solutions of a (GCP) introduced in Section 2 with smooth hypersurfaces.

We have already observed that the local geometric solution of the geometric Cauchy problem for a quasilinear equation $E(a_1, \ldots, a_n, b)$ is a smooth surface germ in Theorem 2.2. Here, we prove that the converse is also true.

Theorem 4.1. *Let $(S, (x_0, y_0))$ be a smooth hypersurface germ. Then there exist function germs $a_i(x, y)$ $(i = 1, \ldots, n)$, $b(x, y)$ and codimension one submanifold germ $(S', (x_0, y_0)) \subset$*

$(S,(x_0,y_0))$ such that $(\widehat{S},(x_0,y_0))$ is a local geometric solution of the geometric Cauchy problem for the quasilinear equation $E(a_1,\ldots,a_n,b)$ with the Cauchy data $(S',(x_0,y_0))$.

Proof. Let $\phi : (\mathbb{R}^n,0) \to (S,(x_0,y_0))$ be a local parametrization of the smooth hyper surface $(S,(x_0,y_0))$ (i.e., $\phi : (\mathbb{R}^n,0) \to (\mathbb{R}^n \times \mathbb{R},(x_0,y_0))$ is an immersion germ with $(\mathrm{Image}(\phi),(x_0,y_0)) = (S,(x_0,y_0)))$. Then $d\phi(\frac{\partial}{\partial u_1})$ is a non-singular vector field on $(S,(x_0,y_0))$, where $(u_1,\ldots,u_n)$ are the canonical coordinates of $(\mathbb{R}^n,0)$. Since ϕ is an immersion germ, there exists a vector field germ X on $(\mathbb{R}^n \times \mathbb{R},(x_0,y_0))$ such that $X \circ \phi = d\phi(\frac{\partial}{\partial u_1})$. So there exist function germs $a_i(x,y)$ $(i=1,\ldots,n$ and $b(x,y)$ such that

$$X = \sum_{i=1}^{n} a_i(x,y)\frac{\partial}{\partial x_i} + b(x,y)\frac{\partial}{\partial y}.$$

By Proposition 2.1, $(\widehat{S},(x_0,y_0))$ is a local geometric solution of the quasilinear equation $E(a_1,\ldots,a_n,b)$. We consider the smooth submanifold $(S',(x_0,y_0))$ of codimension one in $(S,(x_0,y_0))$ which is transverse to the vector field X in $(S,(x_0,y_0))$, $(\widehat{S},(x_0,y_0))$ is the geometric solution of the geometric Cauchy problem where the Cauchy data is $(S',(x_0,y_0))$ qed

The above theorem guarantees that the class of smooth hypersurfaces supplies the correct class to describe the geometric solutions of GCP for quasilinear equations. We must also concern ourselves with which types of singularities the geometric solution to a given quasilinear equation may exhibit. Representation Theorems 4.2 and 4.3 address this question.

Theorem 4.2. *Let $E(a_1,\ldots,a_n,b)$ be a quasilinear equation. For a germ of ${}^0A_k$ $k<n$, there exists a submersion germ $G : (\mathbb{R}^n \times \mathbb{R},(x_0,y_0) \to (\mathbb{R},0)$ such that $\widehat{G^{-1}(0)}$ is a geometric solution of $E(a_1,\ldots,a_n,b)$ and G is P-$\mathcal{K}$-equivalent to the germ ${}^0A_k$.*

Proof. Without loss of generality, we may assume that $(x_0,y_0) = (0,0)$ and $a_n(0,0) \neq 0$.

We now consider the germ

$$H(x_1,\ldots,x_n,y) = y^{k+1} + \sum_{i=1}^{k} x_i y^{i-1} \quad (0 \leq k < n),$$

then $S' = (H|_{\{x_n=0\}})^{-1}(0))$ is a smooth hypersurface in $\mathbb{R}^{n-1} \times 0 \times \mathbb{R} = \{x_n = 0\}$. Since $a_n(0,0) \neq 0$, $X_{(a_1,\ldots,a_n,b)}$ is transverse to S', so that we have a hypersurface S such that $\hat{S}$ is a geometric solution of $E(a_1,\ldots,a_n,b)$ by Theorem 2.2.

Let $G : (\mathbb{R}^n \times \mathbb{R},(x_0,y_0)) \to (\mathbb{R},0)$ a submersion with $G^{-1}(0) = S$. By the uniqueness of the geometric solution, we have $(G|\mathbb{R}^{n-1} \times 0 \times \mathbb{R})^{-1}(0) = S'$, so that $G|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$ is P-$\mathcal{K}$-equivalent to $H|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$. It follows that $g = G|0 \times 0 \times \mathbb{R})$ is $\mathcal{K}$-equivalent to y^{k+1}.

Since $k < n$, $H|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$ is a $\mathcal{K}$-versal deformation of y^{k+1}. Therefore $G|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$ is a $\mathcal{K}$-versal deformation of g. It follows that G is also a $\mathcal{K}$-versal deformation of g. By Theorem 3.2, G is P-$\mathcal{K}$-equivalent to H. This completes the proof. qed

We remark that ${}^0A_k$-type germs in Theorem 3.1 can always be realized as geometric solutions for any quasilinear equation if $k < n$. However, if $k = n$, the situation is different. We fix $i \in \{1, \ldots, n\}$, let $E(a_1, \ldots, a_n, b)$ be a quasilinear equation with $a_i(x_0, y_0) \neq 0$. We say that it is *k-non-degenerated at* (x_0, y_0) $(1 \leq k \leq n)$ if

$$\frac{\partial^k \tilde{a}}{\partial y^k}(x_0, y_0) \neq 0,$$

where $\frac{\partial^k \tilde{a}}{\partial y^k}(x_0, y_0) = (\frac{\partial^k a_1}{\partial y^k}(x_0, y_0), \ldots, \frac{\partial^k a_{i-1}}{\partial y^k}(x_0, y_0), \frac{\partial^k a_{i+1}}{\partial y^k}(x_0, y_0), \frac{\partial^k a_n}{\partial y^k}(x_0, y_0))$. we simply say that $E(a_1, \ldots, a_n, b)$ is *non-degenerated* if it is 1-non-degenerated. Then we have the following realization theorem.

Theorem 4.3. *Let $E(a_1, \ldots, a_n, b)$ be a quasilinear equation and $F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a submersion germ such that F is a $\mathcal{K}$-versal deformation of f_0. If the equation is non-degenerated at (x_0, y_0), then there exists a submersion germ $G : (\mathbb{R}^n \times \mathbb{R}, (x_0, y_0)) \to (\mathbb{R}, 0)$ such that $\widehat{G^{-1}(0)}$ is a local geometric solution of $E(a_1, \ldots, a_n.b)$ and F, G are P-$\mathcal{K}$-equivalent.*

Proof. Without the loss of generality, we may assume that $(x_0, y_0) = (0, 0)$, $a_n(0, 0) \neq 0$ and $F(x, y) = y^{k+1} + \sum_{i=1}^{k} x_i y^{i-1}$. It follows that $S' = f^{-1}(0)$ is a smooth hypersurface in $\mathbb{R}^{n-1} \times 0 \times \mathbb{R} = \{x_n = 0\}$, where $f = F|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$. Since $a_n(0, 0) \neq 0$, $X(a_1, \ldots, a_n, b)$ is transverse to S', so that we have a hypersurface S such that $\hat{S}$ is a geometric solution of $E(a_1, \ldots, a_n, b)$ by Theorem 1.2. Let $G : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a submersion with $G^{-1}(0) = S$, then $g = G|\mathbb{R}^{n-1} \times 0 \times \mathbb{R}$ and f are $\mathcal{C}$-equivalent (i.e., there exists a function germ $\lambda(x_1, \ldots, x_{n-1}, y)$ with $\lambda(0) \neq 0$ such that $g = \lambda \cdot f$). If $k \leq n - 1$, then f is already a $\mathcal{K}$-versal deformation of $f_0 = f|0 \times \mathbb{R}$. Hence, by the same reason as in the proof of Theorem 3.2, F is P-$\mathcal{K}$-equivalent to G. We now assume that $k = n$.

We now consider a vector space

$$T(g_0) = \langle \frac{\partial g}{\partial x_1}|0 \times \mathbb{R}, \ldots, \frac{\partial g}{\partial x_{n-1}}|0 \times \mathbb{R} \rangle_{\mathbb{R}} + T_e(\mathcal{K})(g_0).$$

If $T(g_0) = \mathcal{E}_y$, then g is already a $\mathcal{K}$-versal deformation of g_0, so that we can get the required assertion by Theorem 2.2 as in the previous case.

Suppose that $T(g_0) \neq \mathcal{E}_y$ and $\frac{\partial G}{\partial x_n}|0 \times \mathbb{R} \in T(g_0)$ for any submersion germ G with $G^{-1}(0) = S$. Since $G^{-1}(0)$ is a geometric solution of $E(a_1, \ldots, a_n, b)$, we have a relation

$$\sum_{i=1}^{n} a_i(x, y) \frac{\partial G}{\partial x_i} + b(x, y) \frac{\partial G}{\partial y} = 0 \quad \text{on } \mathrm{G}^{-1}(0),$$

so that

$$-a_n(x,y)\frac{\partial G}{\partial x_n} \equiv \sum_{i=1}^{n-1} a_i(x,y)\frac{\partial G}{\partial x_i} + b(t,x,y)\frac{\partial G}{\partial y} \text{ mod } \langle \mathrm{G}\rangle_{\mathcal{E}_{(x,y)}}.$$

It follows that that

$$-a_n(0,y)\frac{\partial G}{\partial x_n}|0\times I\!R \equiv \sum_{i=1}^{n-1} a_i(0,y)\frac{\partial g}{\partial x_i}(0,y) + b(0,0,y)\frac{\partial g_0}{\partial y} \text{ mod } \langle \mathrm{g}_0\rangle_{\mathcal{E}_y}.$$

We may assume that $g_0 \in \mathcal{M}_y^k$ for $k \geq 3$, where $\mathcal{M}_y$ is the unique maximal ideal of $\mathcal{E}_y$.

We now consider the Taylor polynomial of $a_i(x,y)$ for sufficiently higher order at $(x,0)$ with respect to y-variables as follows :

$$a_i(x,y) = a_i(x,0) + \frac{\partial a_i}{\partial y}(x,0)y + \frac{1}{2}\frac{\partial^2 a_i}{\partial y^2}(x,0)y^2 + \text{higher terms}.$$

On the other hand, we have

$$\frac{\partial G}{\partial x_n}|0\times I\!R \in T(g_0),$$

so that

$$\begin{aligned}\sum_{i=1}^{n-1} a_i(0,y)\frac{\partial g}{\partial x_i}(0,y) \quad &+ \quad b(0,y)\frac{\partial g_0}{\partial y}(y) \\ &\in \quad \langle g_0(y), \frac{\partial g_0}{\partial y}(y)\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\ldots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\boldsymbol{R}},\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n-1}(a_i(0,0) + \frac{\partial a_i}{\partial y}(0,0)y \quad &+ \quad \frac{1}{2}\frac{\partial^2 a_i}{\partial y^2}(0,0)y^2 + \text{higher terms})\frac{\partial g}{\partial x_i}(0,y) \\ &\in \quad \langle g_0(y), \frac{\partial g_0}{\partial y}(y)\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\ldots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\boldsymbol{R}}.\end{aligned}$$

For any linear isomorphism $A : I\!R^{n-1} \to I\!R^{n-1}$, we have a relation

$$\frac{\partial f(Ax',0,y)}{\partial x_i} = \sum_{j=1}^{n-1} A_{ij}\frac{\partial f}{\partial x_j}(Ax',0,y),$$

where $x' = (x_1,\ldots,x_{n-1})$. We remark that the vector space

$$\langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\ldots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\boldsymbol{R}}$$

is an invariant under the action of the linear isomorphism A. Since

$$\sum_{i=1}^{n-1} a_i(0,0)\frac{\partial g}{\partial x_i}(0,y) \in \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}},$$

we have

$$\begin{aligned}(\sum_{i=1}^{n-1}\frac{\partial a_i}{\partial y}(0,0)\frac{\partial g}{\partial x_i}(0,y))y \quad &+ \quad \text{higher terms}\\ &\in \quad \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}}.\end{aligned}$$

By assumptions, $\frac{\partial a_i}{\partial y}(0,0) \neq 0$ and $\frac{\partial g}{\partial x_j}(0,0) \neq 0$ for some $i, j = 1,\dots,n-1$. If necessary, by applying some linear isomorphism A, we have

$$\sum_{i=1}^{n-1}\frac{\partial a_i}{\partial y}(0,0)\frac{\partial g}{\partial x_i}(0,0) \neq 0.$$

It follows that $\mathcal{M}_y \subset \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}}$ mod $\mathcal{M}_y^2$.

Since $\mathcal{K}-$cod (g_0) is finite (for the definition of $\mathcal{K}-$finiteness, see [18,22]), then there exists $r \in \mathbb{N}$ such that $\mathcal{M}_y^r \subset \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} \subset \mathcal{M}_y^{r-1}$. If $y \in \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y}$, we have $\mathcal{M}_y = \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y}$. Then this case is corresponding to the case $r = 1$.

We may assume that $y \in \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}}$mod $\mathcal{M}_y^2$. It follows that there exist real numbers λ_i $(i = 1,\dots,n-1)$ such that $y = \sum_{i=1}^{n-1}\lambda_i\frac{\partial g}{\partial x_i}(0,y)$. If necessary, applying a linear isomorphism A, we may assume that $y = \frac{\partial g}{\partial x_i}(0,y)$ for some $i = 1,\dots,n-1$. By the same arguments as those of previous paragraphs, we can asserts that $y^2 \in \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}}$mod $\mathcal{M}_y^3$. We can continue this procedure up to degree $r-1$. Eventually, every polynomial of degree $r-1$ is contained in the vector space, so that we ahve

$$\mathcal{M}_y \subset \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}} \text{ mod } \mathcal{M}_y^r.$$

Since $\mathcal{M}_y^r \subset \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y}$ and $\frac{\partial g}{\partial x_j}(0,0) \neq 0$ for some $i = 1,\dots n-1$, we have

$$\mathcal{E}_y = \langle g_0, \frac{\partial g_0}{\partial y}\rangle_{\mathcal{E}_y} + \langle \frac{\partial g}{\partial x_1}(0,y),\dots,\frac{\partial g}{\partial x_{n-1}}(0,y)\rangle_{\mathbb{R}} = T(g_0).$$

This contradicts to the assumption that $T(g_0) \neq \mathcal{E}_y$, so that $\frac{\partial G}{\partial x_n}|0 \times \mathbb{R} \notin T(g_0)$ Since f and g are $\mathcal{C}$-equivalent, it is P-$\mathcal{K}$-equivalent, so that

$$\dim_{\mathbb{R}}\frac{\mathcal{E}_{(x',y)}}{\langle g, \frac{\partial g}{\partial y}\rangle_{\mathcal{E}_{(x',y)}} + \langle \frac{\partial g}{\partial x_1},\dots\frac{\partial g}{\partial x_{n-1}}\rangle_{\mathcal{E}_{x'}}} = \dim_{\mathbb{R}}\frac{\mathcal{E}_{(x',y)}}{\langle f, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}_{(x',y)}} + \langle \frac{\partial f}{\partial x_1},\dots\frac{\partial f}{\partial x_{n-1}}\rangle_{\mathcal{E}_{x'}}}.$$

It is easy to calculate that the right hand side of the above equality is equal to 1.

We now consider the canonical inclusion $i : (\mathbb{R}, 0) \to (\mathbb{R}^{n-1} \times \mathbb{R}, 0)$ given by $i(y) = (0, y)$ and the induced epimorphism $i^* : \mathcal{E}_{(x',y)} \to \mathcal{E}_y$ Then we have $\mathrm{Ker}(i^*) = \mathcal{M}_{x'}\mathcal{E}_{(x',y)}$ and

$$i^*(\langle \frac{\partial g}{\partial x_1}, \dots \frac{\partial g}{\partial x_{n-1}} \rangle_{\mathcal{E}_{x'}}) = T(g_0).$$

It follows that

$$\dim_{\mathbb{R}} \frac{\mathcal{E}_y}{T(g_0)} = \dim_{\mathbb{R}} \frac{\mathcal{E}_{(x',y)}}{\langle g, \frac{\partial g}{\partial y} \rangle_{\mathcal{E}_{(x',y)}} + \langle \frac{\partial g}{\partial x_1}, \dots \frac{\partial g}{\partial x_{n-1}} \rangle_{\mathcal{E}_{x'}} + \mathcal{M}_{x'}\mathcal{E}_{(x',y)}} \leq 1.$$

Since $\frac{\partial G}{\partial x_n}|0 \times \mathbb{R} \notin T(g_0)$, we have

$$\mathcal{E}_y = \frac{\partial G}{\partial x_n}|0 \times \mathbb{R} + T(g_0).$$

This means that G is a $\mathcal{K}$-versal deformation of g_0. Since F is a $\mathcal{K}$-versal deformation of f_0 and the fact that f_0 and g_0 are $\mathcal{C}$-equivalent, F and G are P-$\mathcal{K}$-equivalent by Theorem 2.2. This completes the proof. qed

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