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SINGULARITIES OF SOLUTION SURFACES FOR QUASILINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

SHYUICHI IZUMIYA AND WEI-ZHI SUN

ABSTRACT. We study singularities of solution surfaces of characteristic Cauchy problem for quasilinear first order partial differential equations as an application of the previous result on vector fields near a generic submanifold.

0. INTRODUCTION

We consider the solution surface $h^{-1}(0)$ of the following first order partial differential equation :

$$(\mathrm{E}_{(a_1,\dots,a_{n+1})}) \qquad \sum_{i=1}^{n+1} a_i(x_1,\dots,x_{n+1})\frac{\partial h}{\partial x_i}(x_1,\dots,x_{n+1})=0$$

where $a_i(x_1,\dots,x_{n+1})$ are C^∞-functions.

The above equation looks a linear equation, however, the situation is slightly different if we consider solution surfaces in stead of solutions. If $\frac{\partial h}{\partial x_{n+1}} \neq 0$ at a point $x_0 \in h^{-1}(0)$, by the implicit function theorem, $h^{-1}(0)$ is locally represented as a graph $x_{n+1}=u(x_1,\dots,x_n)$, so that the solution surface of $\mathrm{E}_{(a_1,\dots,a_{n+1})}$ is a graph of a solution of a qusilinear equation

$$(\mathrm{Q}_{(a_1,\dots,a_{n+1})}) \qquad \sum_{i=1}^{n} a_i(x_1,\dots,x_n,u)\frac{\partial u}{\partial x_i} - a_{n+1}(x_1,\dots,x_n,u)=0.$$

Therefore it is a quasilinear problem. In [3] we have studied a classification of multi-valued solutions for the Cauchy problem of $\mathrm{Q}_{(a_1\dots,a_{n+1})}$ under the condition that the Cauchy data satisfies the non-characteristic conditions. In that case the solution surfaces are always smooth, however, the singularities appear when the Cauchy data does not satisfy the non-characteristic conditions. In this paper we give a generic classification of the singularities of surfaces which are the solutions of the above equation under the characteristic Cauchy data. There is a motivation to consider the solution surface. Recently, Tsuji[4] studies the singularities of solutions for hyperbolic Monge-Ampère type equations, in which the notion of intermediate integral is quite important which is a kind of first order partial

Key words and phrases. First order quasilinear partial differential equations, Singularities of solution surfaces, Characteristic Cauchy problem.

differential equations. One of the examples of such first order partial differential equations is a quasilinear equation. We can show that each quasilinear equation can be realized as an intermediate integral of a Monge-Ampère type equation (cf., [4]). If we study solution surfaces instead of solutions fro quasilinear equations, it looks like linear equations as the above. So our result gives a classification of singularities for solution surfaces of hyperbolic Monge-Ampère type equations whose intermediate integral are quasilinear equations.

The main result is Theorem 2.1 which is given in Section 2. For the proof of the main theorem, we apply the classification theorem for vector fields near a submanifold in [2]. Theorem 2.1 gives a generic classification of singularities of solution surfaces which are solved by the characteristics method for characteristic Cauchy data. In order to clarify the situation, we construct a geometric framework in Section 1. In Section 3 we study some geometric properties of singularities for solution surfaces by the aid of the normal forms in Theorem 2.1. For the case $n = 3$, the singularities are the trivial family of cross caps along the characteristic curve from the original point in the characteristic initial surface (cf., Figure 1 and Example 3.1 in Section 3). Moreover, we observe that the characteristic curve has second order contact with the Cauchy data.

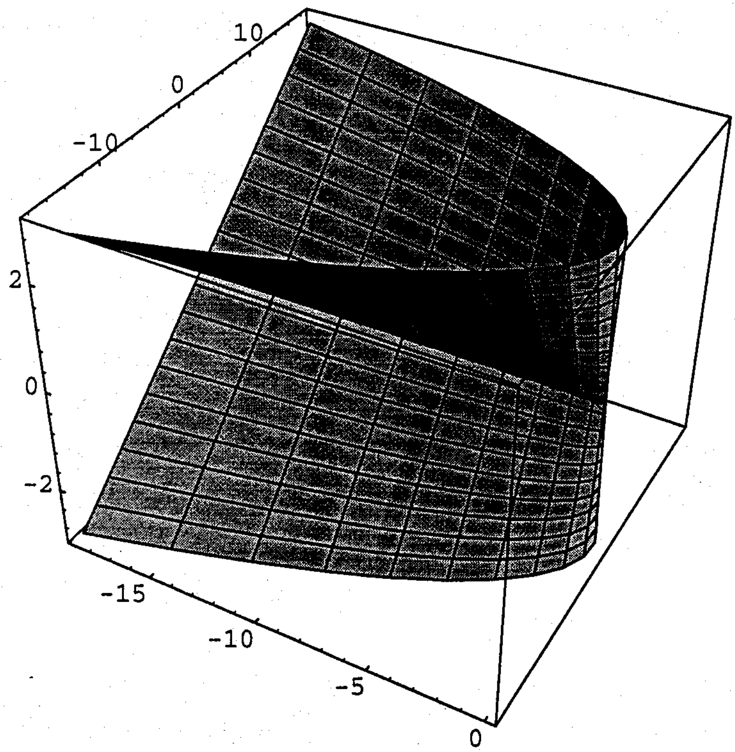

FIGURE 1

All manifolds and maps are differentiable of class C^∞ unless stated otherwise.

1. GEOMETRIC FRAMEWORK

In this section we give a geometric framework for the study of singularities for solution surfaces of $E_{(a_1,\ldots,a_{n+1})}$. Firstly we consider the non-singular case. Let S be a smooth hypersurface in $\mathbb{R}^{n+1}$. For any $x_0 \in S$, there exists a smooth submersion germ

$h : (\mathbb{R}^{n+1}, x_0) \to (\mathbb{R}, 0)$ such that $(h^{-1}(0), x_0) = (S, x_0)$. We say that S is *a smooth solution surface* of $E(a_1, \ldots, a_{n+1})$ if the function germ h satisfies

$$(\mathrm{E}_{(a_1,\ldots,a_{n+1})}) \qquad \sum_{i=1}^{n+1} a_i(x_1, \ldots, x_{n+1}) \frac{\partial h}{\partial x_i}(x_1, \ldots, x_{n+1}) = 0$$

for any $x_0 \in S$. We can easily show that the notion of smooth solution surfaces is well-defined (i.e., it does not depend on the choice of the local equation h).

We now consider the following vector field on $\mathbb{R}^{n+1}$ associated with the linear equation $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$:

$$X(a_1, \ldots, a_{n+1}) = \sum_{i=1}^{n+1} a_i(x_1 \ldots, x_{n+1}) \frac{\partial}{\partial x_i}.$$

It is called *a characteristic vector field* of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$. By definition, we have the following simple proposition.

Proposition 1.1. *Let S be a smooth hypersurface in $\mathbb{R}^{n+1}$. Then S is a solution surface of $E_{(a_1,\ldots,a_{n+1})}$ if and only if the characteristic vector field $X(a_1, \ldots, a_{n+1})$ is tangent to S.*

Proof. For any point $x_0 \in S$, we have a local representation $(h^{-1}(0), x_0) = (S, x_0)$ where $h : (\mathbb{R}^{n+1}, x_0) \to (\mathbb{R}, 0)$ is a submersion germ. It follows that S is a solution surface of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ if and only if

$$\sum_{i=1}^{n+1} a_i(x) \frac{\partial h}{\partial x_i}(x) = 0$$

for $x \in (S, x_0)$. The last condition is equivalent to the condition that $X(a_1, \ldots, a_{n+1})$ is tangent to S at x. $\square$

In classically, the Cauchy problem can be solved by the method of characteristics. *A non-characteristic Cauchy problem* (NCP) for an equation $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ is defined to be a given $(n-1)$-dimensional submanifold $i' : S' \subset \mathbb{R}^{n+1}$ such that the characteristic vector field $X(a_1, \ldots, a_{n+1})$ is not tangent to S'. The following result is well-known (cf. [1]), however it clarifies the geometric situation in this paper.

Theorem 1.2 (Local existence theorem for NCP). *A NCP $i' : S' \subset \mathbb{R}^{n+1}$ has a unique smooth solution surface, that is there exists a solution surface S of the equation $E(a_1, \ldots, a_{n+1})$ with $S' \subset S$ such that any two such smooth hypersurfaces coincide in a neighborhood of S'.*

Proof. Consider the embedding $i : S \subset \mathbb{R}^n \times \mathbb{R}$, where $S \subset (-\varepsilon, \varepsilon) \times S'$ is a neighborhood of $0 \times S'$, $\varepsilon > 0$; here $i(t, q) = T_t(q)$, $q \in S'$, $(t, q) \in S$, and T_t is an one-parameter group of translations along $X(a_1, \ldots, a_{n+1})$. By the construction, we have $X(a_1, \ldots, a_{n+1}) \in TS$, so that S is a smooth solution surface of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ by Proposition 1.1.

On the other hand, any smooth solution surface S of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ must be invariant under $X(a_1, \ldots, a_{n+1})$, so that it is coincide with S in some neighborhood. $\square$

On the other hand, if we do not assume the non-characteristic condition of the Cauchy problem (i.e., the characteristic vector field may be tangent to the initial submanifold S'), the singularities appear on the solution surface. *A generalized Cauchy problem* (GCP) for an equation $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ is defined to be a given $(n-1)$-dimensional submanifold $i : S' \subset \mathbb{R}^{n+1}$. Here we have no assumption about the characteristic vector field $X(a_1,\cdots,a_{n+1})$. Our problem in this paper is summarized as follows:

Problem. Classify the generic singularities of solution surface of GCP for a linear equation which is solved by the characteristics method.

For the purpose, we now introduce the notion of solution surfaces with singularities. Since we study local properties of singularities of solution surfaces, we consider parameterized surfaces.

Let S be an n dimensional manifold and $f : S \to \mathbb{R}^{n+1}$ be a map. We say that f is *a solution surface* (*with singularities*) of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ if there exists a non-vanishing vector field Y on S such that

$$df \circ Y = X(a_1,\cdots,a_{n+1}) \circ f.$$

By Proposition 1.1, if $f : S \to \mathbb{R}^{n+1}$ is a solution surface of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$, then $f(S-\Sigma(S))$ is a smooth solution surface of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$, where $\Sigma(S) = \{q \in S | \operatorname{rank} df_q < n-1\}$. Let $i : S' \subset \mathbb{R}^{n+1}$ be an $(n-1)$-dimensional submanifold. We say that $f : S \to \mathbb{R}^{n+1}$ is *a solution surface* of the GCP $i : S' \subset \mathbb{R}^{n+1}$ for $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ if $S' \subset S$, $f|S' = i$ and there exists a non-vanishing vector field Y on S such that $d \circ Y = X(a_1,\cdots,a_{n+1}) \circ f$ and Y is transverse to S' in S.

We now introduce a natural equivalence relation among solution surfaces of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ as follows: Let $i : S' \subset \mathbb{R}^{n+1}$ be a GCP and $f_i : S_i \to \mathbb{R}^{n+1}$ $(i = 1, 2)$ be solution surface of $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$. We say that f_0 and f_1 are $\mathcal{R}_{S'}$*-equivalent* if there exists a diffeomorphism $\Phi : S_1 \to S_2$ with $\Phi|S' = 1_{S'}$ such that $f_1 = f_2 \circ \Phi$.

Our existence and uniqueness theorem can be formulated as follows.

Theorem 1.3 (Local existence theorem for GCP). *A GCP $i : S' \subset \mathbb{R}^{n+1}$ has unique solution surface, that is, there is a solution surface $f : S \to \mathbb{R}^{n+1}$ of the GCP for $E_{(a_1,\ldots,a_{n+1})}$ such that any two such solution surfaces are $\mathcal{R}_{S'}$-equivalent in a neighbourhood of S'.*

Proof. Consider the mapping $f : S \to \mathbb{R}^{n+1}$, where $S \subset (-\varepsilon, \varepsilon) \times S'$ is a neighbourhood of $0 \times S'$, $\varepsilon > 0$; here $f(t,q) = T_t(q)$, $q \in S'$, $(t,q) \in S$ and T_t is an one-parameter group go translations along $X(a_1,\cdots,a_{n+1})$. By the construction, we have $X(a_1,\cdots,a_{n+1}) \circ f = df(\frac{\partial}{\partial t})$, so that $f : S \to \mathbb{R}^{n+1}$ is a solution surface of the GCP for $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$.

Let $g : P \to \mathbb{R}^{n+1}$ be another solution surface of the GCP for $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$, so that we have a non-vanishing vector field Z on P such that $dg \circ Z = X(a_1,\cdots,a_{n+1}) \circ g$ and Z is transverse to S' in P. Let ϕ_t be an one parameter group of translation along Z, then $g \circ \phi_t(q)$ is a flow of $X(a_1,\cdots,a_{n+1})$ through $g(q) = i(q)$ for $q \in S'$. Any point in a neighbourhood of S' in S is written in the form (t,q). Since Z is transverse to S' in p, the map $\Phi : S \to P$ defined by $\Phi(t,p) = \phi_t(p)$ is well-defined diffeomorphism on a neighbourhood of S' in S onto a neighbourhood of S' in P with $\Phi|S' = 1_{S'}$. We remark that $g \circ \phi_0(q) = g(q) = i(q) = f(q)$ for $q \in S'$. It follows from the uniqueness of the flow of $X(a_1,\cdots,a_{n+1})$ through $i(q)$ that we have $g \circ \phi_t(q) = T_t(q) = f(t,q)$. This means that g and f are $\mathcal{R}_{S'}$-equivalent on neighbourhoods of S' by the diffeomorphism Φ. □

2. Classifications

In this section we shall give normal forms of solution surfaces with singularities of the GCP for $E_{(a_1,\dots,a_{n+1})}$ under a natural equivalence relation.

Firstly we consider what is the natural equivalence relation among $E_{(a_1,\dots,a_{n+1})}$. Here, we adopt point transformations. Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a diffeomorphism which is usually called *a point transformation* in the geometric theory of partial differential equations. Returning to the classical notation of the solution surface of $E_{(a_1,\dots,a_{n+1})}$, we consider a function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ and the relation

$$\sum_{i=1}^{n+1} a_i(x_1,\dots,x_{n+1})\frac{\partial h}{\partial x_i}(x_1,\dots,x_{n+1}) = 0$$

on $u^{-1}(0)$. We can easily show that the above relation is equivalent to the following relation:

$$\sum_{j=1}^{n+1}\sum_{i=1}^{n+1} \frac{\partial \psi_j}{\partial x_i}(\psi^{-1}(x))a_i \circ \psi^{-1}(x)\frac{\partial h \circ \psi^{-1}}{\partial x_j}(x) = 0$$

on $\psi(h^{-1}(0))$. We say that $E_{(a_1,\dots,a_{n+1})}$ and $E_{(b_1,\dots,b_{n+1})}$ are *equivalent under the group of point transformations* if there exists a point transformation $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $b_j(x) = \frac{\partial \psi_j}{\partial x_i}(\psi^{-1}(x))a_i \circ \psi^{-1}(x)$.

In this case we especially say that these equations are *equivalent by* ψ.

We remark that $E_{(a_1,\dots,a_{n+1})}$ and $E_{(b_1,\dots,b_{n+1})}$ are equivalent by a point transformation ψ if and only if the corresponding characteristic vector fields are ψ-related in the usual sense.

We now define an equivalence relation among solution surfaces of GCP for quasilinear equations. Let $i : (S'_a, q_a) \subset (\mathbb{R}^{n+1}, q_a)$ (respectively, $i : (S'_b, q_b) \subset (\mathbb{R}^{n+1}, q_b)$) be a GCP germ for $E_{(a_1,\dots,a_{n+1})}$ (respectively, $E_{(b_1,\dots,b_{n+1})}$). By Theorem 1.3 we have a unique solution surface germ $f_a : (S_a, q_a) \to (\mathbb{R}^{n+1}, q_a)$ (respectively, $f_a : (S_a, q_a) \to (\mathbb{R}^{n+1}, q_a)$). We say that $(S'_a, f_a, E_{(a_1,\dots,a_{n+1})})$ and $(S'_b, f_b, E_{(b_1,\dots,b_{n+1})})$ are *equivalent* if there exist a diffeomorphism germ $\Phi : (S_a, q_a) \to (S_b, q_b)$ with $\Phi(S'_a) = S'_b$ and a point transformation germ $\psi : (\mathbb{R}^{n+1}, q_a) \to (\mathbb{R}^{n+1}, q_b)$ such that

(1) $\psi \circ f_a = f_b \circ \Phi$,

(2) $E_{(a_1,\dots,a_{n+1})}$ and $E_{(b_1,\dots,b_{n+1})}$ are equivalent by ψ.

We can state the main result in this note. We say that the equation $E_{(a_1,\dots,a_{n+1})}$ is *regular* at $q_0 \in \mathbb{R}^{n+1}$ if $a_i(q_0) \neq 0$ for some $i = 1,\dots,n+1$. We simply say that the equation is *regular* if it is regular at any point. Let S' be an closed $(n-1)$-manifold and $\mathrm{Emb}(S', \mathbb{R}^{n+1})$ be the space of embeddings of S' in $\mathbb{R}^{n+1}$ endowed with the Whitney topology.

Theorem 2.1. *Let $E_{(a_1,\dots,a_{n+1})}$ be a regular equation. Then there exists a residual subset $\mathcal{O} \subset Emb(S', \mathbb{R}^{n+1})$ with the following properties: For any $i \in \mathcal{O}$ and $q_0 \in S'$, we have the unique solution surface*

$$f : (S, i(q_0)) \to \mathbb{R}^{n+1} \text{ of } GCP: \ (S', i(q_0)) \subset \mathbb{R}^{n+1} \text{ for } E_{(a_1,\dots,a_{n+1})},$$

then there exists an integer $m = m(q_0)$, $0 \le m \le \frac{n-1}{2}$ *such that* $((S', i(q_0)), f, E_{(a_1,\dots,a_{n+1})})$ *is equivalent to* $((Q', 0), {}_mf, E(b_1, \dots, b_{n+1}))$ *which is given as follows:*

(1) $(Q', 0) = (\{(0, 0, x_3, \dots, x_{n+1}) | x_i \in \mathbb{R}\}, 0)$

(2) ${}_mf : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+1}, 0)$ *where*

$$ {}_mf(u_1, \dots, u_n) = (f_1(u_1, \dots, u_n), \dots, f_{n+1}(u_1, \dots, u_n)); $$

$$
\begin{aligned}
f_1(u_1, \dots, u_n) &= \frac{1}{(m+1)!} u_1^{m+1} + \sum_{i=1}^{m} \frac{1}{(m-i+1)!} u_1^{m+1-i} u_{2(m-i+1)} \\
f_2(u_1, \dots, u_n) &= \sum_{i=0}^{m-1} \frac{1}{(m-i)!} u_1^{m-i} u_{2(m-i)+1} \\
f_{2j-1}(u_1, \dots, u_n) &= \frac{1}{(m-j+2)!} u_1^{m-j+2} \\
&\quad + \sum_{i=1}^{m-j+2} \frac{1}{(m-j-i+2)!} u_1^{m-j+2-i} u_{2(m-i+1)},\ j = 2, \dots, m+1, \\
f_{2j}(u_1, \dots, u_n) &= \sum_{i=0}^{m-j+1} \frac{1}{(m-j+1-i)!} u_1^{m-j-i+1} u_{2(m-i)+1},\ j = 2, \dots, m+1, \\
f_k(u_1, \dots, u_n) &= u_{k-1}\ k = 2m+3, \dots, n+1.
\end{aligned}
$$

(3) $b_{2j-1} = x_{2j+1}$, $b_{2j} = x_{2j+2}$ *for* $(j = 1, \dots, m)$, $b_{2m+1} = 1$ *and* $b_k = 0$ *for* $k = 2m+2, \dots n+1$.

The proof is based on the following result.

Theorem 2.2 (Ishikawa-Izumiya-Watanabe [2]). *Let* X *be a non-vanishing vector field on* $\mathbb{R}^{n+1}$. *Then there exists a residual subset* $\mathcal{O} \subset \mathrm{Emb}(S', \mathbb{R}^{n+1})$ *with the following property: For any* $i \in \mathcal{O}$ *and* $q_0 \in S'$, *there exist a coordinate neighborhood* $(U, (x_1, \dots, x_{n+1}))$ *around* $i(q_0)$ *in* $\mathbb{R}^{n+1}$ *and an integer* $m = m(q_0)$, $(0 \le m \le \frac{n-1}{2})$, *such that*

(1) $X|U = \sum_{j=1}^{m} (x_{2j+1} \frac{\partial}{\partial x_{2j-1}} + x_{2j+2} \frac{\partial}{\partial x_{2j}}) + \frac{\partial}{\partial x_{2m+1}}$,

(2) $i(Q) \cap U = \{(x_1, \dots, x_{n+1}) | x_1 = x_2 = 0\}$.

Proof of Theorem 2.1. We consider the characteristic vector field $X(a_1, \dots, a_{n+1})$. Since the equation $E_{(a_1,\dots,a_{n+1})}$ is regular at any point of $\mathbb{R}^{n+1}$, $X(a_1, \dots, a_{n+1})$ does not vanish on $\mathbb{R}^{n+1}$. We can apply the result of Theorem 2.2, so that, there exists a residual subset $\mathcal{O} \subset \mathrm{Emb}(S', \mathbb{R}^{n+1})$ with the properties in Theorem 2.2.

For any $i \in \mathcal{O}$ and $q_0 \in S'$, there exists a diffeomorphism germ $\psi : (\mathbb{R}^{n+1}, i(q_0)) \to (\mathbb{R}^{n+1}, 0)$ with $\psi(S') = \{(x_1, \dots, x_{n+1}) | x_1 = x_2 = 0\}$ such that

$$ d\psi \circ X(a_1, \dots, a_{n+1}) \circ \psi^{-1} = \sum_{j=1}^{m} (x_{2j+1} \frac{\partial}{\partial x_{2j-1}} + x_{2j+2} \frac{\partial}{\partial x_{2j}}) + \frac{\partial}{\partial x_{2m+1}}. $$

This means that $\mathrm{E}_{(a_1,\ldots,a_{n+1})}$ and $\mathrm{E}_{(b_1,\ldots,b_{n+1})}$ are equivalent by ψ, where $b_{2j-1} = x_{2j+1}$, $b_{2j} = x_{2j+2}$ for $(j = 1,\ldots,m)$, $b_{2m+1} = 1$ and $b_k = 0$ for $k = 2m+2,\ldots n+1$.

For the proof, it is enough to get the solution surface of GCP $\{(x_1,\ldots,x_{n+1})|x_1 = x_2 = 0\} \subset \mathbb{R}^{n+1}$ for $\mathrm{E}_{(b_1,\ldots,b_{n+1})}$.

We now solve the characteristics equations. In this case the characteristics equations is the following system of ordinary differential equation:

$$\begin{cases} \frac{dx_i}{dt} & = x_{i+2}(t) \quad i = 1,\ldots,2m \\ \frac{dx_{2m+1}}{dt} & = 1 \\ \frac{dx_i}{dt} & = 0 \quad i = 2m+2,\ldots,n+1, \end{cases}$$

for the initial condition $x_1(0) = x_2(0) = 0$ and $x_i(0) = u_{i-1}$ for $i = 3,\ldots n+1$. The solution is given as follows:

$$\phi_1(t,u_2,\ldots,u_n) = \frac{1}{(m+1)!}t^{m+1} + \sum_{i=1}^{m} \frac{1}{(m-i+1)!}t^{m+1-i}u_{2(m-i+1)}$$

$$\phi_2(t,u_2,\ldots,u_n) = \sum_{i=0}^{m-1} \frac{1}{(m-i)!}t^{m-i}u_{2(m-i)+1}$$

$$\phi_{2j-1}(t,u_2,\ldots,u_n) = \frac{1}{(m-j+2)!}t^{m-j+2} + \sum_{i=1}^{m-j+2} \frac{1}{(m-j-i+2)!}t^{m-j+2-i}u_{2(m-i+1)},\ j = 2,\ldots,m+1,$$

$$\phi_{2j}(t,u_2,\ldots,u_n) = \sum_{i=0}^{m-j+1} \frac{1}{(m-j+1-i)!}t^{m-j-i+1}u_{2(m-i)+1},\ j = 2,\ldots,m+1,$$

$$\phi_k(t,\ldots,u_n) = u_{k-1} \quad k = 2m+3,\ldots,n+1.$$

It follows from Theorem 1.3 that the map germ $\phi(t,u_2,\ldots,u_n) = (\phi_1(t,u),\ldots\phi_{n+1}(t,u))$ is the unique solution of GCP : $Q' = \{(x_1,\ldots,x_{n+1})|x_1 = x_2 = 0\} \subset \mathbb{R}^{n+1}$ for $\mathrm{E}_{(b_1,\ldots,b_{n+1})}$.

If we adopt the coordinate

$$u_1 = t, u_2 = u_2, \ldots, u_n = u_n$$

of $(\mathbb{R}^n, 0)$ and denote $f_i(u_1,\ldots,u_n) = \phi_i(u_1,u_2,\ldots,u_n)$, then the map germ ${}_mf(u) = (f_1(u),\ldots,f_{n+1}(u))$ is the solution surface such that $((S', i(q_0)), f, \mathrm{E}_{(a_1,\ldots,a_{n+1})})$ is equivalent to $((Q',0),{}_mf, E(b_1,\ldots,b_{n+1}))$. $\square$

3. Geometric study of normal forms

In this section we consider geometric properties of singularities for solution surfaces by the aid of the normal forms in Theorem 2.1.

Example 3.1. We consider the case $n = 3$. In this case the normal forms are given as follows:

$$\begin{aligned} {}_0f(u_1, u_2, u_3) &= (u_1, 0, u_2, u_3) \\ {}_1f(u_1, u_2, u_3) &= (\frac{1}{2}u_1^2 + u_2u_1, u_3u_1, u_1 + u_2, u_3). \end{aligned}$$

The Jacobian for ${}_1f$is given as

$$J = \begin{pmatrix} u_1 + u_2 & u_3 & 1 & 0 \\ u_1 & 0 & 1 & 0 \\ 0 & u_1 & 0 & 1 \end{pmatrix},$$

so that rank $J = 2$ if and only if $u_2 = u_3 = 0$. It follows that ${}_1f$ has singularities along the line $u_2 = u_3 = 0$. The critical value set is $\{(\frac{1}{2}u_1^2, 0, u_1, 0) | u_2 \in (\mathbb{R}, 0)\}$. The tangent vector of the critical value curve is given by $(c, 0, 1, 0)$ at $(\frac{1}{2}c^2, 0, c, 0)$. If we consider the transversal hyperplane $\{(x_1, x_2, x_3, x_4) | x_3 = c\}$ to the critical value curve, then the restriction of the image of the solution surface into the hyperplane is given by

$$(-\frac{1}{2}u_1^2 - cu_1, u_3u_1, c, u_3).$$

This is the cross cap (cf., Figure 1 in Section 0).

In accordance with the proof of Theorem 2.1, the u_1-axis is corresponding to the parameter of the characteristic curve, so that the cross caps appear along the characteristic curve from the origin. Moreover, we can observe that the characteristic curve has second order contract with the Cauchy data $\{x_1 = x_2 = 0\}$.

In general we can state as follows: Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^{n+1}, 0)$ be a regular curve germ (i.e, an embedding germ). We say that a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+1}, 0)$ is *trivial along* γ if there exist a map germ $f_0 : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^n, 0)$ and coordinates $(t, u_1, \ldots, u_{n-1})$, $(y, x_1, \ldots, x_n)$ around the origins such that $f(t, u_1, \ldots, u_{n-1}) = (t, f_0(u_1, \ldots, u_{n-1}))$ and $\gamma(t) = (t, 0, \ldots, 0)$.

We can prove the following proposition.

Proposition 3.2. *Let $E_{(a_1,\ldots,a_{n+1})}$ be a regular equation. For any GCP germ $(S', q_0) \subset \mathbb{R}^{n+1}$, the unique solution surface germ $f : (S, i(q_0)) \to \mathbb{R}^{n+1}$ is trivial along the characteristic curve from q_0.*

Proof. Since the characteristic vector field $X(a_1, \ldots a_{n+!})$ does not vanish at q_0, there exists a coordinate $(x, v_1, \ldots, v_n)$ around $q_0 \in \mathbb{R}^{n+1}$ such that the characteristic vector field can be represented as $\frac{\partial}{\partial x}$. We now consider an embedding germ $i : (R^{n-1}, 0) \to (\mathbb{R}^{n+1}, q_0)$ which is considered as a parameterization of (S', q_0). In this case the unique solution surface is given as the map germ

$$f(t, u_1, \ldots, u_{n-1}) = (i_1(u_1, \ldots, u_{n-1}) + t, i_2(u_1, \ldots, u_{n-1}), \ldots, i_{n+1}(u_1, \ldots, u_{n-1})).$$

It follows that f is trivial along the characteristic curve from starting q_0. $\square$

If we fix the local coordinate $(x, v_1, \ldots, v_n)$ around $q_0 \in \mathbb{R}^{n+1}$ in the proof of Proposition 3.2, it is known that the generic singularities of $\pi|S' : (S', q_0) \to \mathbb{R}^n$ is the A_m-type singularity (cf., Proposition 2.4 in [2]), where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is $\pi(x, v_1, \ldots, v_n) = (v_1, \ldots, v_n)$. It follows from Proposition 3.2 that the singularities of the unique solution surface are the trivial family of the A_m-type singularity along the characteristic curve starting from q_0. It follows from the normal forms in Theorem 2.1 that the characteristic curve from the origin is

$$(\frac{1}{(m+1)!}u_1^{m+1}, 0, \frac{1}{m!}u_1^m, 0, \frac{1}{(m-1)!}u_1^{m-1}, 0, \ldots, 0, u_1, 0, \ldots, 0),$$

such that the characteristic curve has $(m+1)$th-order contact with the Cauchy data at q_0. Then we have the following theorem as a corollary of Theorem 2.1 and Proposition 3.2.

Theorem 3.3. *Let $E_{(a_1,\ldots,a_{n+1})}$ be a regular equation. Then there exists a residual subset $\mathcal{O} \subset Emb(S', \mathbb{R}^{n+1})$ with the following properties: For any $i \in \mathcal{O}$ and $q_0 \in S'$, the unique solution surface $f : (S, i(q_0)) \to \mathbb{R}^{n+1}$ of GCP: $(S', i(q_0)) \subset \mathbb{R}^{n+1}$ for $E_{(a_1,\ldots,a_{n+1})}$ has a trivial family of A_m-type singularities along the characteristic curve from the point q_0 and the characteristic curve has $(m+1)$th-order of contact with the Cauchy data at q_0 for some integer $m = m(q_0)$, $0 \leq m \leq \frac{n-1}{2}$.*

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