HARMONIC MAPS AND GINZBURG-LANDAU TYPE ELLIPTIC SYSTEM

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HARMONIC MAPS AND GINZBURG-LANDAU TYPE ELLIPTIC SYSTEM

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ABSTRACT. We prove the existence of the solutions which converge in C^0 to a harmonic map for an elliptic system depending on a large parameter.

1. Introduction

We study the existence of the solutions of the Ginzburg-Landau type elliptic equations and the relations between the solutions and harmonic maps.

Let $\Omega \subset \mathbb{R}^n (n \geqq 2)$ be a bounded domain with $C^{2+\alpha}$ $(0 < \alpha < 1)$ boundary. We consider the following Ginzburg-Landau type elliptic system:

$$\Delta u - \lambda W_u(u) = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

with the first boundary condition where

$$u = (u_1, \dots, u_m) : \Omega \to \mathbb{R}^m (m \geqq 2);$$
$$W(u) = \frac{1}{4}(a(u) - 1)^2, \quad W_u(u) = \left(\frac{\partial W(u)}{\partial u_1}, \dots, \frac{\partial W(u)}{\partial u_m} \right),$$

$a(u) \in C^\infty(\mathbb{R}^m, \mathbb{R})$ satisfies some growth condition (cf. Assumption 1) and $N := \{u \in \mathbb{R}^m | a(u) - 1 = 0\}$ is a $(m-1)$-dimensional orientable compact connected Riemannian manifold without boundary and $\lambda > 0$ is a large parameter.

Our purpose is to study the existence of the solutions u_λ of (1.1) for large λ and the relation between u_λ and a harmonic map $\bar{u} : \Omega \to N$.

When $W(u) = \frac{1}{4}(|u|^2 - 1)^2$, $m = 2$, (1.1) is the well-known Ginzburg-Landau equation which comes from phase transition problems occurring in superconductivity and superfluidity.

In the case of that $n = 2$ and Ω is a smooth bounded simply connected domain, Bethuel, Brezis and Hélein in [BBH93] proved that the minimizer of the Ginzburg-Landau functional with the first boundary condition $u|_{\partial\Omega} = g$ converges in $C^{1+\alpha}(\overline{\Omega})$

to a harmonic map $\bar{u} : \Omega \to S^1$ provided that the boundary value g satisfies $\deg(g, \partial\Omega) = 0$.

In the case of Ginzburg-Landau equation with the Neumann boundary condition, it is known that if there exists a continuous map $\theta_0 : \overline{\Omega} \to S^1$ which is not homotopy equivalent to a constant value map, then there exists stable non-constant steady state solution u_λ provided that λ is large. Moreover, $\frac{u_\lambda}{|u_\lambda|}$ is homotopic to θ_0 and u_λ converges in $C^{1+\alpha}(\overline{\Omega})$ to a harmonic map (cf. [JMZ94]).

In this paper, we want to consider more general potential W which includes the case of Ginzburg-Landau equation. Precisely, we shall prove the existence of solutions u_λ of (1.1) in Hölder space $C^\alpha(\overline{\Omega})$ which converge in $C^0(\overline{\Omega})$ to a harmonic map $\bar{u} : \Omega \to N$ as $\lambda \to \infty$ provided that $\bar{u}$ satisfies a certain assumption (cf. section 2, Assumption 2-4).

Methods developed in [BBH93] and [JMZ94] seems not to be applicable to general case. In this paper, we first solve a variational inequality with an obstacle. Second, we use the maximum principle to prove that the solutions of the variational inequality with obstacle are, in fact, the solutions of the Euler-Lagrange equation of the original variational problem.

The difficulty is how to remove the obstacle. The key step is that we calculate W carefully and find that the solutions of the variational inequality with obstacle satisfy a scale elliptic differential inequality provided that λ is large enough. Using a maximum principle which is due to Stampacchia, we obtain the estimate for the bounds of the solutions under an assumption for $\bar{u}$ and Ω. Furthermore, we find that the coincidence set of the solutions of the variational inequality with obstacle is empty for large λ. It is to say that they are solutions of the original variational problem without obstacle.

Let M, N be compact Riemannian manifolds with metrics g, h respectively. In local coordinates $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_m)$ we denote

$$g = (g_{\alpha\beta})_{1 \leqq \alpha,\beta \leqq n}, \quad h = (h_{ij})_{1 \leqq i,j \leqq m}, \quad \text{and} \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}.$$

A smooth map $u : M \to N$ is harmonic iff u satisfies

$$-\Delta_M u + \Gamma_N(u)(\nabla u, \nabla u) = 0 \tag{1.2}$$

where

$$\Delta_M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta} \right)$$

is the Laplace-Beltrami operator on M and

$$(\Gamma_N(\nabla u, \nabla u))^k = g^{\alpha\beta} \Gamma_{ij}^k(u) \frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_j}{\partial x_\beta}, \quad 1 \leqq k \leqq m.$$

Eells and Sampson [ES64] proved that if the Riemannian curvature of N is non-positive then there exists a harmonic map in every homotopy class. R.Hamilton [H75] extended the above result to compact manifolds M and N with boundary. Recently, a significant progress has been made without the assumption on the Riemannian curvature of N by Struwe (cf. [S90] and references given in there).

This paper consists of five sections. In section 2 we state assumptions and main results which are proved in section 4 and section 5. Section 3 is devoted to some preliminaries which are used in section 4 and section 5.

Notation: Let $u(x) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $n \geqq 2$, $m \geqq 2$. Let B_1 denote the unit ball in $\mathbb{R}^m$. The notation

$$u(\Omega) \subset \overline{B}_1$$

means that $u(x) \in \overline{B}_1$ for a.e.$x \in \Omega$.

Let Δ denote the Laplace operator on $\mathbb{R}^n$, i.e.

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i \partial x_i}.$$

If W satisfies Assumption 1 (cf. section 2), N is an orientable $m-1$ dimensional compact connected Riemannian manifold without boundary. Let $\gamma(u)$ denote the unit outer normal vector of N at u.

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2. Assumptions and results

We consider the following equation

$$\Delta u - \lambda W_u(u) = 0 \quad \text{in} \quad \Omega \tag{2.1}$$

with the Dirichlet boundary condition

$$u = g \quad \text{on} \quad \partial\Omega \tag{2.2}$$

where Ω is a bounded domain of $\mathbb{R}^n (n \geqq 2)$ with $C^{2+\alpha}(\alpha \in (0,1))$ boundary. $u : \Omega \to \mathbb{R}^m (m \geqq 2)$. $g : \partial\Omega \to N \subset \mathbb{R}^m, g \in C^{2+\alpha}$. $W(u) : \mathbb{R}^m \to \mathbb{R}^1$ satisfies Assumption 1.

Assumption 1. Let $W(u) = \frac{1}{4}(a(u)-1)^2$, where $a(u) \in C^\infty(T_N;\mathbb{R})$ for some tubular neiborhood T_N of N in $\mathbb{R}^m$, satisfies:

(1) Growth condition: for $u \in \mathbb{R}^m$

$$|a(u)| \leqq C(1+|u|^p) \quad \text{with} \quad \begin{cases} \frac{1}{2} < p < \infty, & n = 2 \\ p = \frac{n}{n-2}, & n \geqq 3 \end{cases}$$

where C is a positive constant.

(2) $N := \{u \in \mathbb{R}^m | a(u) - 1 = 0\}$ is a $m-1$ dimensional compact connected Riemannian manifold without boundary.

In this paper, we shall reserch the existence and the behavior of solutions of (2.1) and (2.2) for a given map $\bar{u}$ which satisfies one of the following assumptions (In Remark 4.6 and Remark 5.5 we shall discuss the meaning of these assumptions).

Assumption 2. $\bar{u} : \Omega \to N$ is a harmonic map which satisfies the boundary condition (2.2) and there exists $q > \frac{n}{2}$ such that

$$|\Omega|^{\frac{2}{n}-\frac{1}{q}} \|\frac{1}{2}\sum_{i,j,k=1}^{m} |\hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}| + \frac{1}{4}\|_{L^q(\Omega)} \leqq \frac{1}{4c(q,n)}.$$

Assumption 3. $\bar{u} : \Omega \to N$ is a harmonic map which satisfies the boundary condition (2.2) and there exists $q > \frac{n}{2}$ such that

$$|\Omega|^{\frac{2}{n}-\frac{1}{q}} \|\frac{1}{2}\sum_{i,j,k=1}^{m} |\hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}| + \frac{1}{2}\sum_{i,j,k=1}^{m} |\hat{u}_k \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}| + |\Delta \hat{u}|\|_{L^q(\Omega)}$$
$$\leqq \frac{1}{4c(q,n)}.$$

Here $c(q,n)$ is a positive constant which only depends on q and n (cf. section 3.4); $\hat{u} = \frac{2}{\sum_{i=1}^m (\frac{\partial a(\bar{u})}{\partial u_i})^2}\Gamma_N(\bar{u})(\nabla\bar{u}, \nabla\bar{u})$.

The main result of this paper is

Theorem 2.1. *Assume that W satisfies Assumption 1, $\bar{u}$ and Ω satisfy Assumption 2 or Assumption 3. Then, there exists a solution $v_\lambda \in C^\alpha(\overline{\Omega})$ of (2.1) and (2.2) for some $\alpha \in (0,1)$ provided that λ is large enough. Moreover,*

$$\|v_\lambda - \bar{u}\|_{C^0(\overline{\Omega})} \to 0, \quad \text{as} \quad \lambda \to \infty. \tag{2.3}$$

The proof of Theorem 2.1 is given in section 5.

It is easy to see that, for Ginzburg-Landau equation, $W(u) = \frac{1}{4}(|u|^2 - 1)^2$ and $N = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ which satisfies Assumption 1. Here, we give two another examples which satisfy Assumption 1.

Example 2.2. N is an ellipse.

$$\begin{cases} W(u) = \dfrac{1}{4}(a(u) - 1)^2 \\ a(u) = u_1^2 + 2u_2^2. \end{cases} \tag{2.4}$$

Example 2.3. N is a torus.

$$\begin{cases} W(u) = \dfrac{1}{4}(a(u) - 1)^2 \\ a(u) = u_1^2 + u_2^2 + u_3^2 - 4\sqrt{u_1^2 + u_2^2} + 4. \end{cases} \tag{2.5}$$

In the case of that N is an ellipse, Assumption 3 can be written as

Assumption 4. $\bar{u} : \Omega \to N \subset \mathbb{R}^m$ is a smooth harmonic map which satisfies the boundary condition (2.2) and there exists $q > \frac{n}{2}$ such that

$$|\Omega|^{\frac{2}{n} - \frac{1}{q}} \|3|\Delta \bar{u}| + |\Delta \hat{u}|\|_{L^q(\Omega)} \leqq \frac{1}{4c(q,n)}.$$

Here $c(q,n)$ is the same constant as Assumption 2 and Assumption 3.

Theorem 2.4. *Assume that W is given in (2.4). Let $\bar{u}$ satisfy Assumption 4. Then, there exists a solution $v_\lambda \in C^\alpha(\overline{\Omega})$ of (2.1) and (2.2) for some $\alpha \in (0,1)$ provided that λ is large enough. Moreover,*

$$\|v_\lambda - \bar{u}\|_{C^0(\overline{\Omega})} \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{2.6}$$

The proof of Theorem 2.4 is given in section 4.

3. Preliminaries

3.1. Function space

Let $B_r \subset \mathbb{R}^m$ $(r > 0)$ denote the ball in $\mathbb{R}^m$ with the center at 0:

$$B_r = \{y \in \mathbb{R}^m : |y| < r\}.$$

Definition 3.1.1. *Let $v \in H^1(\Omega, \mathbb{R}^m)$. We define*

$$H_0^1(\Omega, \overline{B}_r) := \{u \in H_0^1(\Omega, \mathbb{R}^m) : u(\Omega) \subset \overline{B}_r\};$$

and

$$\mathcal{K}_r := \{u : u - v \in H_0^1(\Omega, \mathbb{R}^m), \quad u(\Omega) \subset \overline{B}_r\}.$$

Here, $u(\Omega) \subset \overline{B}_r$ means that $u(x) \in \overline{B}_r$ for a.e. $x \in \Omega$.

Proposition 3.1.2. *$H_0^1(\Omega, \overline{B}_r)$ is a closed, convex and weakly closed subset of $H_0^1(\Omega, \mathbb{R}^m)$.*

Similarly, $\mathcal{K}_r$ is a closed, convex and weakly closed subset of $H^1(\Omega, \mathbb{R}^m)$.

Proof. Claim 1. $H_0^1(\Omega, \bar{B}_r)$ is a convex subset.

For $u, v \in H_0^1(\Omega, \bar{B}_r)$, we have

$$\alpha u + (1 - \alpha)v \in H_0^1(\Omega, \overline{B}_r) \quad (\forall \alpha \in (0, 1)).$$

Claim 2. If $\{u^{(j)}\}_j \subset H_0^1(\Omega, \overline{B}_r)$ and $u^{(j)} \to u$ in $H_0^1(\Omega, \mathbb{R}^m)$, then $|u| \leqq r$ a.e. in Ω.

Step 1. If not, then there exists a positive constant t_0 such that the subset

$$E_{t_0} := \{x \in \Omega : |u(x)| \geqq r + t_0\}$$

satisfies $|E_{t_0}| \neq 0$. In fact,

$$E_{t_1} \subset E_{t_2} \quad (0 \leqq t_2 \leqq t_1)$$

and for $t \in (0, \infty)$, E_t is a bounded measurable set with $|E_t| \leqq |\Omega|$. Moreover, we have

$$\cup_{t>0} E_t = \{x \in \Omega : |u(x)| > r\}$$

and

$$\lim_{t\downarrow 0}|E_t| = |\{x \in \Omega : |u(x)| > r\}|.$$

If u does not satisfy $|u| \leqq r$ a.e. in Ω, then there exist t_0 and E_{t_0} with $|E_{t_0}| \neq 0$.

Step 2. There exists subsequence $\{u^{(j)}\}$ such that

$$u^{(j)}(x) \to u(x), \quad \text{for a.e.} \quad x \in \Omega.$$

Because

$$\begin{aligned} \|u^{(j)} - u\|_{L^1(\Omega)} &\leqq |\Omega|^{\frac{1}{2}}\|u^{(j)} - u\|_{L^2(\Omega)} \\ &\to 0 \quad \text{as} \quad j \to \infty, \end{aligned}$$

we can choose a subsequence which converges to u almost everywhere in Ω.

Step 3. Note that $|u^{(j)}(x)| \leqq r$, for a.e. $x \in \Omega$, from Step 1 and Step 2, we get a contradiction.

Claim 3. $H_0^1(\Omega, \overline{B}_r)$ is weakly closed in $H_0^1(\Omega, \mathbb{R}^m)$.

Because $H_0^1(\Omega, \mathbb{R}^m)$ is a Hilbert space, we get the conclusion.

Similarly, for $\mathcal{K}_r$, we can prove the same results. □

3.2. The energy functional

Let $\bar{u} : \Omega \to N$ be a smooth harmonic map which satisfies the first boundary condition (2.2). Let $\hat{u} \in C^{2+\alpha}(\Omega, \mathbb{R}^m)$ ($\alpha \in (0,1)$). We consider the following functional

$$E_\lambda(u) = \int_\Omega \{\frac{1}{2}|\nabla(\bar{u} + \frac{\hat{u}+u}{\lambda})|^2 + \lambda W(\bar{u} + \frac{\hat{u}+u}{\lambda})\}dx \tag{3.2.1}$$

where $\lambda > 0$ and W satisfies Assumption 1.

We consider the following variational problems:

$$\min\{E_\lambda(u) |\, u \in H_0^1(\Omega, \overline{B}_r)\} \tag{3.2.2}$$

and

$$\min\{E_\lambda(u) |\, u \in \mathcal{K}_r\}. \tag{3.2.2'}$$

The functional $E_\lambda(u)$ in $H_0^1(\Omega, \overline{B}_r)$ (or $\mathcal{K}_r$) is well defined. Moreover, we have the following existence theorem.

Theorem 3.2.1. *The functional $E_\lambda(u)$ in $H_0^1(\Omega, \overline{B}_r)$ (or $\mathcal{K}_r$) satisfies coercive condition and is weakly sequentially lower semi-continuous.*

Moreover, there exist $u_\lambda \in H_0^1(\Omega, \overline{B}_r)$ and $v_\lambda \in \mathcal{K}_r$ such that

$$E_\lambda(u_\lambda) = \min\{E_\lambda(u) |\, u \in H_0^1(\Omega, \overline{B}_r)\}$$

and

$$E_\lambda(v_\lambda) = \min\{E_\lambda(u) |\, u \in \mathcal{K}_r\}.$$

Proof. For $u \in H_0^1(\Omega, \bar{B}_r)$, we estimate $E_\lambda(u)$ from below.

$$\begin{aligned}
E_\lambda(u) &\geqq \frac{1}{2}\int_\Omega |\nabla(\bar{u} + \frac{\hat{u}+u}{\lambda})|^2 dx \\
&= \frac{1}{2}\int_\Omega |\nabla \frac{u}{\lambda}|^2 + 2\nabla(\frac{u}{\lambda}) \cdot \nabla(\bar{u} + \frac{\hat{u}}{\lambda}) + |\nabla(\bar{u} + \frac{\hat{u}}{\lambda})|^2 dx \\
&\geqq \frac{3}{8\lambda}\int_\Omega |\nabla u|^2 dx - \frac{3}{2}\int_\Omega |\nabla(\bar{u} + \frac{\hat{u}}{\lambda})|^2 dx
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla u\|_{L^2(\Omega,\mathbb{R}^m)} &\geqq \frac{\sqrt{2}}{2}(\|u\|_{H_0^1(\Omega,\mathbb{R}^m)} - \|u\|_{L^2(\Omega,\mathbb{R}^m)}) \\
&\geqq \|u\|_{H_0^1(\Omega,\mathbb{R}^m)} - r|\Omega|^{\frac{1}{2}} \\
&\to \infty \quad \text{as} \quad \|u\|_{H_0^1(\Omega,\mathbb{R}^m)} \to \infty.
\end{aligned}$$

We get

$$E_\lambda(u) \to \infty \quad \text{as} \quad \|u\|_{H_0^1(\Omega,\mathbb{R}^m)} \to \infty.$$

That is, $E_\lambda(u)$ on $H_0^1(\Omega, \overline{B}_r)$ is coercive.

The weakly sequentially lower semi-continuity of E_λ follows from [G 83] pp.22, Theorem 2.5. The existence of minimizers follows from [G83] pp.16, Theorem 2.2' (or c.f. [S80] pp.4, Theorem 1.2.).

Similarly, we can prove the results for (3.2.2'). □

3.3. Variational inequality with an obstacle

The minimizing problems (3.2.2), (3.2.2') are equal to solve variational inequalities with an obstacle. The regularity theory for variational inequalities with an obstacle has been well developed (c.f. [HW79]).

Proposition 3.3.1. *(Interior regularity) If u_λ be a minimizer of the variational problem (3.2.2) or (3.2.2'), then*

$$u_\lambda \in C^\alpha(\Omega) \quad \text{for some} \quad \alpha \in (0,1).$$

Proof. c.f. [HW79], Theorem 2.1. □

Theorem 3.3.2. *(Global regularity for Dirichlet boundary condition) If u_λ be a minimizer of (3.2.2) or (3.2.2'), then*

$$u_\lambda \in C^\alpha(\overline{\Omega}) \quad \text{for some} \quad \alpha \in (0,1).$$

Proof. c.f. [HW79], Theorem 2.1. □

3.4. The maximum principle

We recall a maximum principle for scalar elliptic differential inequalities which is due to Stampacchia. The following variant comes from [HW75].

Lemma 3.4.1. *Assume that $\Omega \subset \mathbb{R}^n$ is bounded with Lipschitz boundary. Let $a_{ij}(x)$ be bounded measurable functions from Ω to $\mathbb{R}$ and satisfy:*

$$\mu|\xi|^2 \leqq a_{ij}(x)\xi_i\xi_j \quad (\text{for a.e. } x \in \Omega \text{ and } \xi \in \mathbb{R}^n)$$

for some $\mu > 0$. Let $f \in L^q(\Omega,\mathbb{R})$ for some $q > \frac{n}{2}$.

If $\eta \in H^1(\Omega,\mathbb{R})$ satisfies

$$\int_\Omega a_{ij}D_i\eta D_j\varphi dx \leqq \int_\Omega f\varphi dx \quad (\forall\varphi \in H_0^1(\Omega,\mathbb{R}),\ \varphi \geqq 0)$$

then

$$\max_\Omega \eta \leqq \max_{\partial\Omega} \eta + c(q,n)\mu^{-1}|\Omega|^{\frac{2}{n}-\frac{1}{q}}||f||_{L^q}$$

where the positive constant $c(q,n)$ only depends on q and dimension n.

The proof of Lemma 3.4.1 can be found in [HW75]. Here, we only state some estimates about $c(q,n)$ (cf. [HW75] pp.70 - 71).

For $q = \infty$, $c(q,n) = (2n(\omega_n/n)^{2/n})^{-1}$ and this is best possible (ω_n = the surface area of the unit sphere in $\mathbb{R}^n$).

For $\frac{n}{2} < q < \infty$,

$$c(q,n) \leqq \left(\frac{\omega_n}{n}\right)^{-\frac{1}{n}} \frac{(1+(2/n-1/q)^{-1})}{\sqrt{n(n+2)}} A_p$$

where A_p is defined by

$$A_p^{-2} = \inf_{v \in C_c^\infty(B_1^n(0))} \frac{\left(\frac{\omega_n}{n}\right)^{\frac{2}{n}-1+\frac{2}{p}} \int |\nabla v|^2 dx}{\left(\int |v|^p dx\right)^{\frac{2}{p}}}.$$

Particularly, for $q \geqq 2$, we have

$$c(q,n) \leqq \left(\frac{\omega_n}{n}\right)^{-\frac{2}{n}} \frac{(1+(2/n-1/q)^{-1})}{\sqrt{n(n+2)}} k_{\frac{n}{2}-1,1}^{-1}$$

where $k_{\frac{n}{2}-1,1}$ is the first zero of the Bessel function of order $n/2-1$.

4. The proof of Theorem 2.4

First we remark that in the case of example 2.2, we have

$$W_u(u) = (a(u)-1)\begin{pmatrix} u_1 \\ 2u_2 \end{pmatrix}.$$

Substituting $\bar{u} + \frac{\hat{u}+u}{\lambda}$ for u in W_u, we calculate W_u as following:

$$\begin{aligned}
W_u(\bar{u} + \frac{\hat{u}+u}{\lambda}) &= (a(\bar{u} + \frac{\hat{u}+u}{\lambda}) - 1)\begin{pmatrix} \bar{u}_1 + \frac{\hat{u}_1+u_1}{\lambda} \\ 2\bar{u}_2 + \frac{2\hat{u}_2+2u_2}{\lambda} \end{pmatrix} \\
&= \frac{1}{\lambda}(2\bar{u}_1\hat{u}_1 + 4\bar{u}_2\hat{u}_2)\begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix} + \frac{1}{\lambda}(2\bar{u}_1 u_1 + 4\bar{u}_2 u_2)\begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix} \\
&+ \frac{1}{\lambda^2}(2\bar{u}_1\hat{u}_1 + 4\bar{u}_2\hat{u}_2)\begin{pmatrix} \hat{u}_1 \\ 2\hat{u}_2 \end{pmatrix} + \frac{1}{\lambda^2}(2\bar{u}_1 u_1 + 4\bar{u}_2 u_2)\begin{pmatrix} \hat{u}_1 \\ 2\hat{u}_2 \end{pmatrix} \\
&+ \frac{1}{\lambda^2}(2\bar{u}_1\hat{u}_1 + 4\bar{u}_2\hat{u}_2)\begin{pmatrix} u_1 \\ 2u_2 \end{pmatrix} + \frac{1}{\lambda^2}(2\bar{u}_1 u_1 + 4\bar{u}_2 u_2)\begin{pmatrix} u_1 \\ 2u_2 \end{pmatrix} \\
&+ \frac{1}{\lambda^2}((\hat{u}_1+u_1)^2 + 2(\hat{u}_2+u_2)^2)\begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix} \\
&+ \frac{1}{\lambda^3}((\hat{u}_1+u_1)^2 + 2(\hat{u}_2+u_2)^2)\begin{pmatrix} \hat{u}_1+u_1 \\ 2(\hat{u}_2+u_2) \end{pmatrix} \\
&= \frac{1}{\lambda}W_{uu}(\bar{u})\{\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\} + \frac{1}{\lambda^2}(V_1+V_2)\begin{pmatrix} \hat{u}_1 \\ 2\hat{u}_2 \end{pmatrix} \\
&+ \frac{1}{\lambda^2}(V_1+V_2)\begin{pmatrix} u_1 \\ 2u_2 \end{pmatrix} + \frac{1}{\lambda^2}V_3\begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix} + \frac{1}{\lambda^3}V_3\begin{pmatrix} \hat{u}_1+u_1 \\ 2\hat{u}_2+2u_2 \end{pmatrix}
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
W_{uu}(\bar{u}) &:= \begin{pmatrix} 2\bar{u}_1^2 & 4\bar{u}_1\bar{u}_2 \\ 4\bar{u}_1\bar{u}_2 & 8\bar{u}_2^2 \end{pmatrix} \\
V_1 &:= 2\bar{u}_1\hat{u}_1 + 4\bar{u}_2\hat{u}_2 \\
V_2 &:= 2\bar{u}_1 u_1 + 4\bar{u}_2 u_2 \\
V_3 &:= (\hat{u}_1+u_1)^2 + 2(\hat{u}_2+u_2)^2.
\end{aligned}$$

Let $\gamma(\bar{u})$ denote the unit normal vector of N at $\bar{u}$. That is,

$$\gamma(\bar{u}) = \frac{1}{\sqrt{\bar{u}_1^2 + 4\bar{u}_2^2}} \begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix}. \tag{4.2}$$

Let $\tau(\bar{u})$ denote the unit tangent vector of N at $\bar{u}$ defined by

$$\tau(\bar{u}) = \frac{1}{\sqrt{\bar{u}_1^2 + 4\bar{u}_2^2}} \begin{pmatrix} 2\bar{u}_2 \\ -\bar{u}_1 \end{pmatrix}. \tag{4.2'}$$

If we take $u = \bar{u} + \frac{1}{\lambda}\hat{u} + O(\frac{1}{\lambda^2})$ in (2.1), we find that $\hat{u}$ satisfies the following equation:

$$\Delta\bar{u} - W_{uu}(\bar{u})\hat{u} = 0. \tag{4.3}$$

We determine $\hat{u}$ below. Put

$$\hat{u} = c(x)\gamma(\bar{u}), \tag{4.4}$$

where $c(x)$ is a scalar valued function. Note that $\bar{u}$ is a harmonic map, we can express the equations for $\bar{u}$ in the following form:

$$\begin{cases} -\Delta\bar{u} = b(x)\gamma(\bar{u}) & \text{in } \Omega \\ \bar{u} = g & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $b(x) : \Omega \to \mathbb{R}$ is the scalar valued function determined by $b(x)\gamma(\bar{u}) = \Gamma_N(\nabla\bar{u}, \nabla\bar{u})$. Using (4.3)-(4.5), we have

$$c(x) = \frac{-b(x)}{2\bar{u}_1^2 + 8\bar{u}_2^2}. \tag{4.6}$$

From Theorem 3.2.1 and Theorem 3.3.2, we have

Lemma 4.1. *Let $r > \max_\Omega |\hat{u}|$ and define $\mathcal{K}_r$ as*

$$\mathcal{K}_r = \{u : u + \hat{u} \in H_0^1(\Omega, \mathbb{R}^2), u(\Omega) \subset \overline{B}_r\}.$$

Then the minimizing problem: $\min_{u\in\mathcal{K}_r} E_\lambda(u)$ has a solution u_λ in $\mathcal{K}_r$. Moveover, there exists $\alpha \in (0,1)$ such that $u_\lambda \in C^\alpha(\overline{\Omega})$.

Lemma 4.2. *For small $\epsilon > 0$, there exists $\lambda_0 = \lambda_0(r) > 0$ such that*

$$\int_\Omega \frac{1}{2}\nabla\varphi \cdot \nabla |u_\lambda|^2 + \varphi|\nabla u_\lambda|^2 - \varphi(h+\epsilon)dx \leqq 0 \quad (\forall \varphi \in H_0^1(\Omega,\mathbb{R}), \quad \varphi \geqq 0) \tag{4.7}$$

with

$$h(x) = 3|b(x)||u_\lambda(x)|^2 + |\Delta\hat{u}(x)||u_\lambda(x)| + |b(x)||\hat{u}(x)|^2,$$

provided that $\lambda \geqq \lambda_0$.

Proof. For $\varphi \in C_0^\infty(\Omega,\mathbb{R})$ with $\varphi \geqq 0$, let $v_t = (1-t\varphi)u_\lambda$. Clearly, there exists $t_0 > 0$ such that for all $t \in (0,t_0)$

$$|v_t| = |1-t\varphi||u_\lambda| \leqq r,$$

and $v_t + \hat{u} = (u_\lambda + \hat{u}) - t\varphi u_\lambda \in H_0^1(\Omega,\mathbb{R}^2)$. That is, $v_t \in \mathcal{K}$. Therefore,

$$E_\lambda(v_t) \geqq E_\lambda(u_\lambda) \quad (\forall t \in (0,t_0)). \tag{4.8}$$

In fact, we can calculate the first variation of E_λ(c.f. [G83] Theorem 5.1 of Chapter 1). From (4.8), we obtain

$$\frac{d}{dt}_{|t=0} E_\lambda(v_t) \geqq 0.$$

That is

$$\begin{aligned} 0 \geqq &\int_\Omega \nabla(\varphi u_\lambda)\cdot\nabla(\frac{u_\lambda}{\lambda}) + \varphi u_\lambda(-\Delta(\bar{u}+\frac{\hat{u}}{\lambda}) + \lambda W_u(\bar{u}+\frac{\hat{u}+u_\lambda}{\lambda}))dx \\ &+ \int_{\partial\Omega} \varphi u_\lambda \frac{\partial}{\partial\nu}(\bar{u}+\frac{\hat{u}}{\lambda})ds. \end{aligned}$$

Note that $\varphi = 0$ in $\partial\Omega$ and (4.1), (4.3), we have

$$\begin{aligned} 0 \geqq &\int_\Omega \nabla(\varphi u_\lambda)\cdot\nabla u_\lambda + \varphi u_\lambda(-\Delta\hat{u} + \lambda W_{uu}(\bar{u})u_\lambda \\ &+ (V_1+V_2)\begin{pmatrix}\hat{u}_1\\2\hat{u}_2\end{pmatrix} + (V_1+V_2)\begin{pmatrix}u_{\lambda 1}\\2u_{\lambda 2}\end{pmatrix} \\ &+ V_3\begin{pmatrix}\bar{u}_1\\2\bar{u}_2\end{pmatrix} + \frac{1}{\lambda}V_3\begin{pmatrix}\hat{u}_1+u_{\lambda 1}\\2\hat{u}_2+2u_{\lambda 2}\end{pmatrix})dx. \end{aligned} \tag{4.9}$$

Let

$$\begin{aligned} C_0 := 1 + \sup_{\lambda\geqq 1}\max_\Omega |u_\lambda\{-\Delta\hat{u} + (V_1+V_2)\begin{pmatrix}\hat{u}_1+u_{\lambda 1}\\2\hat{u}_2+2u_{\lambda 2}\end{pmatrix} \\ + V_3\begin{pmatrix}\bar{u}_1+\frac{\hat{u}_1+u_{\lambda 1}}{\lambda}\\2\bar{u}_2+\frac{2\hat{u}_2+2u_{\lambda 2}}{\lambda}\end{pmatrix}\}| \end{aligned} \tag{4.10}$$

and define D_λ as

$$D_\lambda := \{x \in \Omega : |u_\lambda^\gamma(x)|^2 \geqq \frac{C_0}{\lambda}\} \tag{4.11}$$

where u_λ^γ is the normal part of u_λ, i.e.

$$u_\lambda^\gamma := \langle u_\lambda, \gamma(\bar{u})\rangle_{\mathbb{R}^2}.$$

In order to estimate the right side of (4.9) from below, we divide Ω into D_λ and $\Omega \setminus D_\lambda$.

Case 1. $x \in D_\lambda$.

Then, from (4.10), we have

$$\begin{aligned} J :=\ & \lambda u_\lambda W_{uu}(\bar{u})u_\lambda + u_\lambda(-\Delta\hat{u} + (V_1 + V_2)\begin{pmatrix} \hat{u}_1 + u_{\lambda 1} \\ 2\hat{u}_2 + 2u_{\lambda 2}\end{pmatrix} \\ & + V_3\begin{pmatrix} \bar{u}_1 + \frac{\hat{u}_1 + u_{\lambda 1}}{\lambda} \\ 2\bar{u}_2 + \frac{2\hat{u}_2 + 2u_{\lambda 2}}{\lambda}\end{pmatrix}) \geqq 0. \end{aligned} \tag{4.12}$$

Case 2. $x \in \Omega \setminus D_\lambda$.

In this case, for $\epsilon > 0$, there exists $\lambda_0 > 0$ such that

$$\begin{aligned} J \geqq\ & -|V_1(u_{\lambda 1}^2 + 2u_{\lambda 2}^2) - u_\lambda \cdot \Delta\hat{u} + V_1(u_{\lambda 1}\hat{u}_1 + 2u_{\lambda 2}\hat{u}_2)| \\ & - |u_\lambda \begin{pmatrix} \hat{u}_1 + u_{\lambda 1} \\ 2\hat{u}_2 + 2u_{\lambda 2}\end{pmatrix} V_2 + V_3 u_\lambda \begin{pmatrix} \bar{u}_1 \\ 2\bar{u}_2 \end{pmatrix}| \\ & - \frac{1}{\lambda}|V_3 u_\lambda \begin{pmatrix} \hat{u}_1 + u_{\lambda 1} \\ 2\hat{u}_2 + 2u_{\lambda 2}\end{pmatrix}| \\ \geqq\ & -h(x) - \epsilon, \quad (\forall \lambda \geqq \lambda_0). \end{aligned} \tag{4.13}$$

From (4.12) and (4.13), we get (4.7). □

If harmonic map $\bar{u}$ and domain Ω satisfy Assumption 4, i.e.

$$|\Omega|^{\frac{2}{n} - \frac{1}{q}} \| 3|\Delta\bar{u}| + |\Delta\hat{u}| \|_{L^q(\Omega,\mathbb{R})} \leqq \frac{1}{4c(q,n)}$$

for some $q > \frac{n}{2}$, where $c(q,n)$ is given in Lemma 3.4.1, then we can obtain a estimate for $\max_\Omega |u_\lambda|^2$. That is

Lemma 4.3. *If Assumption 4 is satisfied, then*

$$\begin{aligned} \max_{\Omega} |u_\lambda|^2 &\leqq \max\{1,\ 2\max_{\partial\Omega} |\hat{u}|^2 + 4c(q,n)|\Omega|^{\frac{2}{n} - \frac{1}{q}}(\| b|\hat{u}|^2 \|_{L^q(\Omega,\mathbb{R})} + |\Omega|^{\frac{1}{q}})\} \\ &=: r_0 \end{aligned}$$

for $\lambda \geqq \lambda_0(r)$.

Proof. Suppose $\max_\Omega |u_\lambda|^2 > 1$. Using Lemma 3.4.1 and Lemma 4.2, we have

$$\max_\Omega |u_\lambda|^2 \leqq 2\max_{\partial\Omega} |u_\lambda|^2 + 4c(q,n)|\Omega|^{\frac{2}{n}-\frac{1}{q}}(\||b|\hat{u}|^2\|_{L^q(\Omega,\mathbb{R})} + \epsilon|\Omega|^{\frac{1}{q}}). \quad \square$$

Lemma 4.4. *Let* $r_1 = 1+\sqrt{r_0}$. *If Assumption 4 is satisfied, then, for* $\lambda \geqq \lambda_0(\sqrt{r_0}+1)$, u_λ *is also a weak solution of the following equations:*

$$\begin{cases} \Delta(\bar{u} + \dfrac{\hat{u}+u}{\lambda}) = \lambda W_u(\bar{u} + \dfrac{\hat{u}+u}{\lambda}) \quad in \quad \Omega \\ \qquad u = -\hat{u} \quad on \quad \partial\Omega. \end{cases}$$

Proof. Note that, from Lemma 4.3, we have $\max_\Omega |u_\lambda| < r_1$ for $\lambda \geqq \lambda_0(r_1)$. Assume $v \in C_0^\infty(\Omega, \mathbb{R}^2)$. Then, for $\lambda \geqq \lambda_0(r_1)$, there exists $t_0 > 0$ such that for $t \in (0, t_0)$, $u_\lambda + tv \in \overline{B}_{r_1}$.

On the other hand, we have

$$u_\lambda + tv + \hat{u} = (u_\lambda + \hat{u}) + tv \in H_0^1(\Omega, \mathbb{R}^m).$$

Thus, for $t \in (0, t_0)$,

$$u_\lambda + tv \in \mathcal{K}_{r_1}.$$

Because u_λ is the minimizer of E_λ in $\mathcal{K}_{r_1}$, we have

$$E_\lambda(u_\lambda + tv) \geqq E_\lambda(u_\lambda).$$

By calculating the first variation of E_λ we obtain

$$\int_\Omega \nabla v \cdot (\bar{u} + \frac{\hat{u}+u_\lambda}{\lambda}) + \lambda v W_u(\bar{u} + \frac{\hat{u}+u_\lambda}{\lambda}) dx \geqq 0.$$

Replacing v by $-v$, we get the reverse inequality. Then the lemma is proved. $\square$

The proof of Theorem 2.4: Let $v_\lambda = \bar{u} + \frac{\hat{u}+u_\lambda}{\lambda}$. From Lemma 4.1 - 4.4, we get the conclusions of Theorem 2.4.

Remark 4.5. Let $H^1_{-\hat{u}}(\Omega, \mathbb{R}^2)$ denote the subset of $H^1(\Omega, \mathbb{R}^2)$:

$$H^1_{-\hat{u}}(\Omega, \mathbb{R}^2) = \{u : u + \hat{u} \in H_0^1(\Omega, \mathbb{R}^2)\}.$$

From the proof of Lemma 4.4, we only know that u_λ is a critical point of functional $E_\lambda(u)$ in $H^1_{-\hat{u}}(\Omega,\mathbb{R}^2)$ for $\lambda \geqq \lambda_0(r_1)$. Note that it is not necessary that u_λ is a minimizer of the variational problem

$$\min\{E_\lambda(u)| \quad u \in H^1_{-\hat{u}}(\Omega,\mathbb{R}^2)\}.$$

Remark 4.6. Assume that $\bar{u}$ in Ω satisfies Assumption 4. Let $k \in (0,1)$ and

$$x = ky, \quad \bar{u}_k(y) = \bar{u}(ky),$$
$$\Omega_1 = \{k^{-1}x : x \in \Omega\}.$$

Then, $\bar{u}_k$ is also a harmonic map from Ω_1 into N and satisfies Assumption 4 in Ω_1. We can prove it directly:

$$\begin{aligned}\Delta_y \bar{u}_k(y) &= k^2 \Delta_x \bar{u}(x)\\ &= k^2 \Gamma_N(\bar{u})(\nabla_x \bar{u}, \nabla_x \bar{u})\\ &= \Gamma_N(\bar{u}_k)(\nabla_y \bar{u}_k, \nabla_y \bar{u}_k),\end{aligned}$$

$$\begin{aligned}&|\Omega_1|^{\frac{2}{n}-\frac{1}{q}} \| 3|\Delta_y \bar{u}_k| + |\Delta \hat{u}| \|_{L^q(\Omega_1)}\\ &\quad = k^{-(2-\frac{n}{q})+(2-\frac{n}{q})} |\Omega|^{\frac{2}{n}-\frac{1}{q}} \| 3|\Delta_x \bar{u}| + k^2 |\Delta_x \hat{u}| \|_{L^q(\Omega)}\\ &\quad \leqq |\Omega|^{\frac{2}{n}-\frac{1}{q}} \| 3|\Delta_x \bar{u}| + |\Delta_x \hat{u}| \|_{L^q(\Omega)}.\end{aligned}$$

Note that $|\Omega_1| = k^{-n}|\Omega|$, we can say that Assumption 4 requires a relation between $|\Omega|$ and $\bar{u}$ rather than the smallness of $|\Omega|$.

5. The proof of Theorem 2.1

In this section, we assume that W satisfies Assumption 1 which includes the case that N is an ellipse or a torus (see: example 2.2 - 2.3). We shall prove Theorem 2.1 using the general form of W. For simplicity we assume that $m = 3$.

First, we calculate W_u :

$$W_u(u) = \frac{1}{2}(a(u)-1)\left(\frac{\partial a}{\partial u_1}, \frac{\partial a}{\partial u_2}, \frac{\partial a}{\partial u_3}\right). \tag{5.1}$$

As the previous section, we denote $\bar{u}$ the harmonic map, that is

$$\begin{cases} -\Delta \bar{u} = b(x)\gamma(\bar{u}) & \text{in} \quad \Omega \\ \bar{u} = g & \text{on} \quad \partial\Omega, \end{cases} \tag{5.2}$$

where $\gamma(\bar{u})$ is the unit normal vector of N at $\bar{u}$:

$$\gamma(\bar{u}) := \frac{1}{(\sum_{i=1}^{3}(\frac{\partial a(\bar{u})}{\partial u_i})^2)^{\frac{1}{2}}}\left(\frac{\partial a(\bar{u})}{\partial u_1}, \frac{\partial a(\bar{u})}{\partial u_2}, \frac{\partial a(\bar{u})}{\partial u_3}\right). \tag{5.3}$$

Let

$$\hat{u} = c(x)\gamma(\bar{u}).$$

Substituting $\bar{u} + \frac{\hat{u}}{\lambda} + O(\frac{1}{\lambda^2})$ for u in (2.1), we find that $\hat{u}$ satisfies:

$$\Delta\bar{u} - W_{uu}(\bar{u})\hat{u} = 0. \tag{5.4}$$

From (5.2) - (5.4) and

$$\begin{aligned} W_{uu}(\bar{u}) &= \frac{1}{2}\left(\frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial a(\bar{u})}{\partial u_j}\right)_{(i,j)} + \frac{1}{2}(a(\bar{u})-1)\left(\frac{\partial^2 a(\bar{u})}{\partial u_i \partial u_j}\right)_{(i,j)} \\ &= \frac{1}{2}\left(\frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial a(\bar{u})}{\partial u_j}\right)_{(i,j)}, \end{aligned}$$

we have

$$-b(x)\begin{pmatrix} \frac{\partial a(\bar{u})}{\partial u_1} \\ \frac{\partial a(\bar{u})}{\partial u_2} \\ \frac{\partial a(\bar{u})}{\partial u_3} \end{pmatrix} = \frac{c(x)}{2}\left(\frac{\partial a(\bar{u})}{\partial u_i}\frac{\partial a(\bar{u})}{\partial u_j}\right)_{(i,j)}\begin{pmatrix} \frac{\partial a(\bar{u})}{\partial u_1} \\ \frac{\partial a(\bar{u})}{\partial u_2} \\ \frac{\partial a(\bar{u})}{\partial u_3} \end{pmatrix}.$$

Then

$$c(x) = \frac{-2b(x)}{\sum_{i=1}^{3}(\frac{\partial a(\bar{u})}{\partial u_i})^2}. \tag{5.5}$$

We consider the following equations:

$$\begin{cases} \Delta(\bar{u} + \dfrac{\hat{u}+u}{\lambda}) - \lambda W_u(\bar{u} + \dfrac{\hat{u}+u}{\lambda}) = 0 & \text{in } \Omega, \\ u = -\hat{u} & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

From Theorem 3.2.1 - 3.2.2, we have

Lemma 5.1. *Let $r > \max_{\Omega} |\hat{u}|$, and*

$$\mathcal{K}_r := \{u : u + \hat{u} \in H_0^1(\Omega, \mathbb{R}^3), u(\Omega) \subset \overline{B}_r\}. \tag{5.7}$$

Then there exists $u_\lambda \in \mathcal{K}_r \cap C^\alpha(\overline{\Omega}, \mathbb{R}^3)$ for some $\alpha \in (0,1)$ such that

$$E_\lambda(u_\lambda) = \min_{u \in \mathcal{K}_r} E_\lambda(u).$$

Lemma 5.2. *For $\epsilon > 0$ there exists $\lambda_0(r) > 0$, such that*

$$\int_\Omega \frac{1}{2}\nabla|u_\lambda|^2 \cdot \nabla\varphi + \varphi|\nabla u_\lambda|^2 - \varphi(h+\epsilon)dx \leqq 0 \tag{5.8}$$
$$(\forall \varphi \in C_0^\infty(\Omega, \mathbb{R}), \quad \varphi \geqq 0)$$

where

$$h := |\frac{1}{2}(\hat{u}_1 \frac{\partial a(\bar{u})}{\partial u_1} + \hat{u}_2 \frac{\partial a(\bar{u})}{\partial u_2} + \hat{u}_3 \frac{\partial a(\bar{u})}{\partial u_3}) u_\lambda \left(\frac{\partial^2 a(\bar{u})}{\partial u_i u_j}\right)_{ij} (u_\lambda + \hat{u}) - u_\lambda \Delta \hat{u}|, \tag{5.9}$$

provided that $\lambda \geqq \lambda_0(r)$.

Proof. For $\varphi \in C_0^\infty(\Omega, \mathbb{R}), \varphi \geqq 0$, let $v_t := (1 - t\varphi)u_\lambda$. Clearly, there exists $t_0 > 0$ such that

$$v_t = (1 - t\varphi)u_\lambda \in \overline{B}_r \quad (\forall t \in (0, t_0))$$

and

$$v_t + \hat{u} = (u_\lambda + \hat{u}) - t\varphi u_\lambda \in H_0^1(\Omega, \mathbb{R}^3).$$

Then $v_t \in \mathcal{K}_r$ for $t \in (0, t_0)$.

Following [G83] (Th.5.1 Chap.1), if $a(u)$ satisfies the growth condition of Assumption 1, then we can calculate the first variation of E_λ. Note that u_λ is a minimizer, we have

$$\frac{d}{dt}_{|t=0} E_\lambda(u_\lambda - t\varphi u_\lambda) \geqq 0.$$

That is

$$\int_\Omega \nabla(\varphi u_\lambda) \cdot \nabla(\bar{u} + \frac{\hat{u} + u_\lambda}{\lambda}) + \lambda \varphi u_\lambda W_u(\bar{u} + \frac{\hat{u} + u_\lambda}{\lambda}) dx \leqq 0.$$

Note that $\varphi = 0$ on $\partial\Omega$ and (5.4), we get

$$0 \geqq \int_\Omega \nabla(\varphi u_\lambda) \cdot \nabla u_\lambda + \varphi u_\lambda H dx$$

with

$$\begin{aligned}
H := & -\Delta\hat{u} + \frac{\lambda}{2}(u_{\lambda 1}\frac{\partial a(\bar{u})}{\partial u_1} + u_{\lambda 2}\frac{\partial a(\bar{u})}{\partial u_2} + u_{\lambda 3}\frac{\partial a(\bar{u})}{\partial u_3}) \begin{pmatrix} \frac{\partial a(\bar{u})}{\partial u_1} \\ \frac{\partial a(\bar{u})}{\partial u_2} \\ \frac{\partial a(\bar{u})}{\partial u_3} \end{pmatrix} \\
& + \frac{1}{4}(\hat{u} + u_\lambda)\left(\frac{\partial^2 a(\bar{u})}{\partial u_i \partial u_j}\right)_{ij} (\hat{u} + u_\lambda) \begin{pmatrix} \frac{\partial a(\bar{u})}{\partial u_1} \\ \frac{\partial a(\bar{u})}{\partial u_2} \\ \frac{\partial a(\bar{u})}{\partial u_3} \end{pmatrix} \\
& + \frac{1}{2}(\hat{u}_1\frac{\partial a(\bar{u})}{\partial u_1} + \hat{u}_2\frac{\partial a(\bar{u})}{\partial u_2} + \hat{u}_3\frac{\partial a(\bar{u})}{\partial u_3}) \left(\frac{\partial^2 a(\bar{u})}{\partial u_i \partial u_j}\right)_{ij} (u_\lambda + \hat{u}) \\
& + \frac{1}{2}(u_{\lambda 1}\frac{\partial a(\bar{u})}{\partial u_1} + u_{\lambda 2}\frac{\partial a(\bar{u})}{\partial u_2} + u_{\lambda 3}\frac{\partial a(\bar{u})}{\partial u_3}) \left(\frac{\partial^2 a(\bar{u})}{\partial u_i \partial u_j}\right)_{ij} (u_\lambda + \hat{u}) + O(\frac{1}{\lambda}).
\end{aligned}$$

Here we have used the formula:

$$W_u(\bar u+\frac{\hat u+u_\lambda}{\lambda})=\frac{1}{2}(a(\bar u+\frac{\hat u+u_\lambda}{\lambda})-1)\begin{pmatrix}\frac{\partial a(\bar u+\frac{\hat u+u_\lambda}{\lambda})}{\partial u_1}\\ \frac{\partial a(\bar u+\frac{\hat u+u_\lambda}{\lambda})}{\partial u_2}\\ \frac{\partial a(\bar u+\frac{\hat u+u_\lambda}{\lambda})}{\partial u_3}\end{pmatrix}$$

$$=\frac{1}{2}(\frac{1}{\lambda}\sum_{i=1}^{3}\frac{\partial a(\bar u)}{\partial u_i}(\hat u_i+u_{\lambda i})+\frac{1}{2\lambda^2}\sum_{i,j=1}^{3}\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_j}(\hat u_i+u_{\lambda i})(\hat u_j+u_{\lambda j})$$

$$+\frac{1}{3!\lambda^3}\sum_{i,j,k=1}^{3}\frac{\partial^3 a(\bar u)}{\partial u_i\partial u_j\partial u_k}(\hat u_i+u_{\lambda i})(\hat u_j+u_{\lambda j})(\hat u_k+u_{\lambda k}))\cdot$$

$$\begin{pmatrix}\frac{\partial a(\bar u)}{\partial u_1}+\frac{1}{\lambda}\sum_{i=1}^3\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_1}(\hat u_i+u_{\lambda i})+\frac{1}{2\lambda^2}\sum_{i,j=1}^3\frac{\partial^3 a(\bar u)}{\partial u_i\partial u_j\partial u_1}(\hat u_i+u_{\lambda i})(\hat u_j+u_{\lambda j})\\ \frac{\partial a(\bar u)}{\partial u_2}+\frac{1}{\lambda}\sum_{i=1}^3\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_2}(\hat u_i+u_{\lambda i})+\frac{1}{2\lambda^2}\sum_{i,j=1}^3\frac{\partial^3 a(\bar u)}{\partial u_i\partial u_j\partial u_2}(\hat u_i+u_{\lambda i})(\hat u_j+u_{\lambda j})\\ \frac{\partial a(\bar u)}{\partial u_3}+\frac{1}{\lambda}\sum_{i=1}^3\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_3}(\hat u_i+u_{\lambda i})+\frac{1}{2\lambda^2}\sum_{i,j=1}^3\frac{\partial^3 a(\bar u)}{\partial u_i\partial u_j\partial u_3}(\hat u_i+u_{\lambda i})(\hat u_j+u_{\lambda j})\end{pmatrix}$$

$$+O(\frac{1}{\lambda^4}).$$

We let

$$\begin{aligned}C_0=1+&\\ &+2\sup_{\lambda\geqq 1}\max_{\Omega}\left\{\left(\sum_{i=1}^{3}(\frac{\partial a(\bar u)}{\partial u_i})^2\right)^{-1}\Bigg|-u_\lambda\Delta\hat u+\right.\\ &+\frac{1}{4}(\hat u+u_\lambda)\left(\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_j}\right)_{(i,j)}(\hat u+u_\lambda)\sum_{i=1}^{3}u_{\lambda i}\frac{\partial a(\bar u)}{\partial u_i}+\\ &\left.+\frac{1}{2}(\sum_{i=1}^{3}\frac{\partial a(\bar u)}{\partial u_i}(\hat u_i+u_{\lambda i}))u_\lambda\left(\frac{\partial^2 a(\bar u)}{\partial u_i\partial u_j}\right)_{(i,j)}(\hat u+u_\lambda)\Bigg|\right\}\end{aligned}$$

and define $u_\lambda^\gamma:=\langle u_\lambda,\gamma(\bar u)\rangle_{\mathbb{R}^m}$. Let

$$D_\lambda:=\{x\in\Omega:|u_\lambda^\gamma(x)|^2\geqq\frac{C_0}{\lambda}\}. \tag{5.10}$$

It is clear that

$$u_\lambda(x)H(x)\geqq 0\quad(\forall x\in D_\lambda), \tag{5.11}$$

and

$$u_\lambda(x)H(x)\geqq -h(x)-\epsilon\quad(\forall x\in\Omega\setminus D_\lambda), \tag{5.12}$$

provided that λ is large enough. Then we proved the lemma. □

Assume that $\bar u$ and Ω satisfy Assumption 2 or Assumption 3. As Lemma 4.3, we have

Lemma 5.3. *(1) If Assumption 2 is satisfied, then, for $\lambda \geqq \lambda_0(r)$,*

$$\max_{\Omega} |u_\lambda|^2 \leqq r_0$$

with

$$r_0 := 2\max_{\partial\Omega} |\hat{u}|^2 + 4c(q,n)|\Omega|^{\frac{2}{n}-\frac{1}{q}} \left(\|\{\frac{1}{2}\sum_{i,j,k=1}^{3} |\hat{u}_k \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}| + |\Delta \hat{u}|\}^2\|_{L^q(\Omega)} + |\Omega|^{\frac{1}{q}} \right).$$

(2) If Assumption 3 is satisfied, then, for $\lambda \geqq \lambda_0(r)$,

$$\max_{\Omega} |u_\lambda|^2 \leqq r_0'$$

with

$$r_0' := \max\{1, 2\max_{\partial\Omega} |\hat{u}|^2 + 4c(q,n)|\Omega|^{\frac{2}{n}}\}.$$

Proof. From Lemma 3.4.1, we have

$$\max_{\Omega} |u_\lambda|^2 \leqq \max_{\partial\Omega} |u_\lambda|^2 + 2c(q,n)|\Omega|^{\frac{2}{n}-\frac{1}{q}} \|h+\epsilon\|_{L^q(\Omega)}.$$

$$\begin{aligned}
h &= |\frac{1}{2}(\sum_{i=1}^{3} \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}) \sum_{j,k=1}^{3} u_{\lambda j} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k} (\hat{u}_k + u_{\lambda k}) - u_\lambda \Delta \hat{u}| \\
&\leqq \frac{1}{2}\sum_{j,k=1}^{3} |(\sum_{i=1}^{3} \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}) \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k} u_{\lambda j} u_{\lambda k}| \\
&+ \frac{1}{2}\sum_{j,k=1}^{3} |(\sum_{i=1}^{3} \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i}) \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k} u_{\lambda j} \hat{u}_k| + |u_\lambda \Delta \hat{u}| \\
&\leqq (\frac{1}{2}\sum_{i,j,k=1}^{3} |\hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}|)|u_\lambda|^2 \\
&+ (\frac{1}{2}\sum_{i,j,k=1}^{3} |\hat{u}_i \hat{u}_k \frac{\partial a(\bar{u})}{\partial u_i} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}|)|u_\lambda| + |\Delta \hat{u}||u_\lambda| \\
&= h_1|u_\lambda|^2 + h_2|u_\lambda| \\
&\leqq (h_1 + \frac{1}{4})|u_\lambda|^2 + h_2^2.
\end{aligned}$$

Here we put

$$h_1 := \frac{1}{2}\sum_{i,j,k=1}^{3} |\hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}|$$

$$h_2 := \frac{1}{2} \sum_{i,j,k=1}^{3} |\hat{u}_k \hat{u}_i \frac{\partial a(\bar{u})}{\partial u_i} \frac{\partial^2 a(\bar{u})}{\partial u_j \partial u_k}| + |\Delta \hat{u}|$$

The calculations and Assumption 2 yields (1).

If $\max_\Omega |u_\lambda|^2 > 1$, then

$$h(x) \leqq (h_1(x) + h_2(x)) \max_\Omega |u_\lambda|^2.$$

Using Assumption 3, we obtain (2). □

Lemma 5.4. *Let* $r_1 = \sqrt{r_0}+1$ *(or* $r_1 = \sqrt{r_0'}+1$*). If Assumption 2 (or Assumption 3) is satisfied, then* u_λ *is also the weak solution of (5.6) for* $\lambda \geqq \lambda_0(r_1)$.

The proof of Lemma 5.4 is similar to the proof of Lemma 4.4.

The proof of Theorem 2.1: If we let

$$v_\lambda = \bar{u} + \frac{\hat{u} + u_\lambda}{\lambda},$$

then Theorem 2.1 is proved.

Remark 5.5. As Remark 4.6, we can scale the inequality given in Assumption 3 and conclude that if $\bar{u}$ and Ω satisfy Assumption 3, then, for k $(0 < k < 1)$, $\bar{u}_k$ and Ω_1 also satisfy Assumption 3.

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