GENERIC AFFINE DIFFERENTIAL GEOMETRY OF PLANE CURVES

S. Izumiya and T. Sano

Series #310. October 1995

HOKKAIDO UNIVERSITY

PREPRINT SERIES IN MATHEMATICS

♯ 284 T. Nakazi, M. Yamada, Riesz's Functions In Weighted Hardy And Bergman Spaces, 20 pages. 1995.

♯ 285 K. Hidano, K. Tsutaya, Scattering theory for nonlinear wave equations in the invariant Sobolev space, 32 pages. 1995.

♯ 286 A. Arai, Strong coupling limit of the zero-energy-state density of the Dirac-Weyl operator with a singular vector potential, 8 pages. 1995.

♯ 287 T. Nakazi, Factorizations of outer functions and extremal problems, 15 pages. 1995.

♯ 288 A. Kishimoto, The Rohlin property for automorphisms of UHF algebras, 15 pages. 1995.

♯ 289 K. Goto, A. Yamaguchi and I. Tsuda, Nine-bit states cellular automata are capable of simulating the pattern dynamics of coupled map lattice, 24 pages. 1995.

♯ 290 Y. Giga, Interior derivative blow-up for quasilinear parabolic equations, 16 pages. 1995.

♯ 291 F. Hiroshima, Functional Integral Representation of a Model in QED, 48 pages. 1995.

♯ 292 N. Kawazumi, A Generalization of the Morita-Mumford Classes to Extended Mapping Class Groups for Surfaces, 11 pages. 1995.

♯ 293 P. Aviles and Y. Giga, The distance function and defect energy, 23 pages. 1995.

♯ 294 S. Izumiya and A. Takiyama, A time-like surface in Minkowski 3-space which contains pseudocircles, 11 pages. 1995.

♯ 295 S. Izumiya, Local classifications of multi-valued solutions of quasilinear first order partial differential equations, 12 pages. 1995.

♯ 296 A. Kishimoto, A Rohlin property for one-parameter automorphism groups, 27 pages. 1995.

♯ 297 F. Hiroshima, Diamagnetic Inequalities for Systems of Nonrelativistic Particles with a Quantized Field, 23 pages. 1995.

♯ 298 A. Higuchi, Lattices of closure operators, 6 pages. 1995.

♯ 299 S. Izumiya and W-Z. Sun, Singularities of solution surfaces for quasilinear 1st order partial differential equations, 9 pages. 1995.

♯ 300 D. Lehmann, M. Soares and T. Suwa, On the index of a holomorphic vector field tangent to a singular variety, 14 pages. 1995.

♯ 301 J. Zhai, Harmonic maps and Ginzburg-Landau type elliptic system, 20 pages. 1995.

♯ 302 M.-H. Giga and Y. Giga, Geometric evolution by nonsmooth interfacial energy, 15 pages. 1995.

♯ 303 S. Jimbo and J. Zhai, Ginzburg-Landau equation with magnetic effect: non-simply-connected domains, 21 pages. 1995.

♯ 304 T. Ozawa, On the nonlinear Schrödinger equations of derivative type, 27 pages. 1995.

♯ 305 N.H. Bingham and A. Inoue, Jordan's theorem for fourier and hankel transforms, 30 pages. 1995.

♯ 306 T. Honda and T. Suwa, Residue formulas for singular foliations defined by meromorphic functions on surfaces, 19 pages. 1995.

♯ 307 J. Yoshizaki, On the structure of the singular set of a complex analytic foliation, 25 pages. 1995.

♯ 308 A. Arai, Representation of canonical commutation relations in a gauge theory, the Aharonov-Bohm effect, and Dirac-Weyl operator, 17 pages. 1995.

♯ 309 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of the spin-boson Hamiltonian, 20 pages. 1995.

GENERIC AFFINE DIFFERENTIAL GEOMETRY OF PLANE CURVES

SHYUICHI IZUMIYA AND TAKASI SANO

ABSTRACT. We study affine invariants of plane curves from the view point of the singularity theory of smooth functions

1. INTRODUCTION

There are several articles which study about "generic differential geometry" in the Euclidian space ([2,3,4,5,6,7,etc.]). The main tools in these articles are the distance-squared function and the height function. The classical invariants of the extrinsic differential geometry can be treated as "singularities" of these functions, however, as Fidal[7] pointed out, the geometric interpretation of sextactic points of a convex curve is quite complicated from this point of view. We say that a point p of a convex curve C is *a sextactic point* if there exists a unique conic touching C at p with at least six-point contact.

On the other hand, it has been classically known that the sextactic point of a convex curve is corresponding to the stationary point of the affine curvature (i.e., so called "the affine vertex") in the affine differential geometry (cf., [1,8,9]). We also say that a point p is *a parabolic point* if there exists a unique parabola touching C at p with five-point contact which is known as the zero point of the affine curvature (i.e., so called "the affine inflexion"). In this paper we introduce the new notions of affine distance-cubed functions and affine height functions of a convex curve. These functions are quite useful for the study of generic properties of invariants of the extrinsic affine differential geometry on convex plane curves.

As a consequence, we can apply ordinary techniques of the singularity theory for these functions and get information of sextactic points and parabolic points of a convex plane curve. It is incredibly easy comparing with Fidal's arguments about sextactic points. Not only to simplify the arguments about sextactic points but also we give new interpretation about parabolic points. In Section 2 we shall introduce the notion of *the affine normal curve* of a convex plane curve. We shall show that the cusp singular point of the affine normal curve corresponds to a parabolic points of the original curve.

Let $\gamma : S^1 \to \mathbb{R}^2$ be a smooth curve without inflexions. We assume throughout that γ has the following properties, both of which are satisfied generically. (We do not prove the fact here (cf., [6,7]):

(A 1) There is no conic having greater than six-point contact with $\gamma(S^1)$.

(A 2) The number of points p of $\gamma(S^1)$ where the unique non-singular conic touching $\gamma(S^1)$ at p with at least five-point contact is a parabola is finite.

1991 *Mathematics Subject Classification.* 53A15, 58C27.

Key words and phrases. generic properties, affine differential geometry, plane curves.

(A 3) There is no parabola having six-point contact with $\gamma(S^1)$.

Under these assumptions, we shall give new interpretations of affine invariants of convex plane curves from the view point of singularity theory. The main result in this paper is Theorem 2.2 which will be given in Section 2. The basic techniques we used in this paper depend heavily on those in the attractive book of Bruce and Giblin [5], so that the authors are grateful to both of them.

All curves and maps considered here are of class C^∞ unless stated otherwise.

2. Basic notions

Let $\mathbb{R}^2$ be an affine plane which adopt the coordinate such that the area of the parallelogram spanned by two vectors $a = (a_1, a_2)$, $b = (b_1, b_2)$ is given by $|a, b| = a_1 b_2 - a_2 b_1$. We fix the above coordinate in this section. Let $\gamma : I \to \mathbb{R}^2$ be a curve with $|\dot{\gamma}(t), \ddot{\gamma}(t)| \neq 0$ (i.e., without inflexion points), where $\dot{\gamma}(t) = \frac{d\gamma}{dt}(t)$. *The affine arc-length* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is $s(t) = \int_{t_0}^{t} |\dot{\gamma}(t), \ddot{\gamma}(t)|^{\frac{1}{3}} dt$. Then a parameter s is determined such that $|\gamma'(s), \gamma''(s)| = 1$, where $\gamma'(s) = \frac{d\gamma}{ds}(s)$. So we say that a curve γ is *parameterized by the affine arc-length* if it satisfies that $|\gamma'(s), \gamma''(s)| = 1$.

By the similar arguments as those of in the Euclidian differential geometry, we have the following Frenet-Serret type formula:

$$\gamma'''(s) = -k_a(s)\gamma'(s), \tag{2.1}$$

where $k_a(s) = |\gamma''(s), \gamma'''(s)|$ which is called *an affine curvature* of γ. We also call $\gamma''(s)$ *an affine normal vector* of γ. Suppose that $k_a(s) \neq 0$, then the point $\gamma(s) + \frac{1}{k_a(s)}\gamma''(s)$ is called *an affine center of curvature* of γ at s. *The affine evolute* is define to be the locus of affine centers of curvature. The curve $\gamma'' : I \to \mathbb{R}^2$ contains important geometric information (cf., Theorem 2.2 and Section 5). We call the curve $\gamma'' : I \to \mathbb{R}^2$ *an affine normal curve* of γ. It is classically known the following fact (cf., [1,8,9]).

Proposition 2.1. *Under the generic assumption* (A 1) *and* (A 2), *we have the following assertions:*
(1) $p = \gamma(s_0)$ *is a parabolic point of* $\gamma(I)$ *if and only if* $k_a(s_0) = 0$.
(2) $p = \gamma(s_0)$ *is a sextactic point of* $\gamma(I)$ *if and only if* $k'_a(s_0) = 0$.

By Proposition 2.1, there is a parabola having six-point contact with $\gamma(I)$ at $p = \gamma(s_0)$ if and only if $k_a(s_0) = k'_a(s_0) = 0$. It is known that the affine evolute is the center locus of the unique conics touching the curve with five-point contact (cf., [1,8,9]).

The main result in this paper is the following theorem.

Theorem 2.2. *Let* $\gamma : I \to \mathbb{R}^2$ *be a curve without inflexions satisfying* (A 1)-(A 3). *Then:*
(1) *Let p be a point of the affine evolute of γ at s_0, then, locally at p, the affine evolute is*
 (i) *diffeomorphic to a line in $\mathbb{R}^2$ if the point $\gamma(s_0)$ is not a sextactic point;*
 (ii) *diffeomorphic to an ordinary cusp in $\mathbb{R}^2$ if the point $\gamma(s_0)$ is a sextactic point.*
(2) *Let $p = \gamma''(s_0)$ be a point of the affine normal of γ, then, locally at p, the affine normal curve is*
 (i) *diffeomorphic to a line in $\mathbb{R}^2$ if the point $\gamma(s_0)$ is not a parabolic point of γ;*

(ii) *diffeomorphic to an ordinary cusp in* $\mathbb{R}^2$ *if the point* $\gamma(s_0)$ *is a parabolic point of* γ. *The ordinary cusp is a curve which is defined by* $C = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 = x_2^3\}$.

Remark. We cannot find any arguments about the affine normal curve in classical textbook on the affine differential geometry ([1,8,9]). Therefore the above theorem gives a new information about the affine curvature.

3. Sextactic points

In this section we briefly review some properties of sextactic points of a plane curve without inflexions from the view point of the Euclidian differential geometry. We will not use any results in this section, however, we can understand that it is very hard to treat the sextactic points from this point of view. We now fix an Euclidian structure on $\mathbb{R}^2$ which may be considered as an affine structure.

For any convex regular curve γ, we have two kinds of parameter with respect to the Euclidian structure. We adopt t as the arclength and s as the affine arclength. Let $k(t)$ be the usual curvature of $\gamma(t)$ which is given by $k(t) = |\dot{\gamma}, \ddot{\gamma}|$, where $\dot{\gamma} = \frac{d\gamma}{dt}$. It follows that $\frac{dt}{ds} = |\dot{\gamma}, \ddot{\gamma}|^{-\frac{1}{3}} = k(t)^{-\frac{1}{3}}$. In [7] Fidal has given a characterization of a sextactic point in terms of the curvature $k(t)$ as follows: $\gamma(t_0)$ is a sextactic point if and only if

$$36k(t_0)^4\dot{k}(t_0) + 40\dot{k}(t_0)^3 - 45k(t_0)\dot{k}(t_0)\ddot{k}(t_0) + 9k(t_0)^2\dddot{k}(t_0) = 0.$$

Therefore it is very hard to study sextactic points from the view point of the Euclidian differential geometry, however, it is known that sextactic points are corresponding to affine vertices. Since $\frac{dt}{ds} = k(t)^{-\frac{1}{3}}$, we have $\frac{d^2t}{ds^2} = -\frac{1}{3}k(t)^{-\frac{5}{3}}\dot{k}(t)$ and $\frac{d^3t}{ds^3} = \frac{5}{9}k(t)^{-3}\dot{k}^2 - \frac{1}{3}k(t)^{-2}\ddot{k}(t)$. By the Frenet-Serret formula, we have $|\ddot{\gamma}, \dddot{\gamma}| = k^3$. It follows from that we have

$$k_a(s) = k(t)^{\frac{4}{3}} - \frac{5}{9}k(t)^{-\frac{8}{3}}\dot{k}(t)^2 + \frac{1}{3}k(t)^{-\frac{5}{3}}\ddot{k}(t).$$

Differentiating the both side of the above equation with respect to s, we can show that $k_a'(s_0) = 0$ if and only if $\gamma(t_0)$ is a sextactic point, where $s_0 = s(t_0)$.

We can also represent the affine normal curve in terms of the unit tangent vector $T(t)$ and the unit normal vector $N(t)$ obtained from $T(t)$ by rotating anticlockwise through $\frac{\pi}{2}$. By a simple calculation, we have

$$\begin{aligned}\gamma''(s) =& k(t)^{-\frac{1}{3}}\ddot{\gamma}(t) - \frac{1}{3}k(t)^{-\frac{4}{3}}\dot{k}(t)\dot{\gamma}(t)\\ =& k(t)^{-\frac{1}{3}}\dot{T}(t) - \frac{1}{3}k(t)^{-\frac{4}{3}}\dot{k}(t)T(t).\end{aligned}$$

Substituing the Euclidian Frenet-Serret formula $\dot{T}(t) = k(t)N(t)$ to the above equation, we have

$$\gamma''(s) = -\frac{1}{3}k(t)^{-\frac{4}{3}}\dot{k}(t)T(t) + k(t)^{\frac{2}{3}}N(t).$$

We cannot directly guess the shape of the affine normal curve by this formula, so that we need some new techniques.

4. Affine support functions

In this section we introduced two kind of families of functions on a plane curve without inflexions which are useful for the study of the affine differential geometric properties of the curve. Let $\gamma : I \to \mathbb{R}^2$ be a plane curve with $|\gamma(s)', \gamma(s)''| = 1$.

4-1) Affine distance-cubed functions. We now define a two parameter family of smooth functions on I

$$F : I \times \mathbb{R}^2 \to \mathbb{R}$$

by

$$F(s, x) = |\gamma'(s), \gamma(s) - x|.$$

We call F *an affine distance cubed function on* γ.

Differentiating $F(s, x)$ with respect to s, we have

$$f_x'(s) = |\gamma''(s), \gamma(s) - x|, \tag{4.1}$$

$$f_x''(s) = |\gamma'''(s), \gamma(s) - x| - 1, \tag{4.2}$$

$$f_x'''(s) = |\gamma^{IV}(s), \gamma(s) - x|, \tag{4.3}$$

$$f_x^{IV}(s) = |\gamma^{V}(s), \gamma(s) - x| + |\gamma^{IV}(s), \gamma'(s)|, \tag{4.4}$$

where $f_x(s) = F(s, x)$ for any $x \in \mathbb{R}^2$.

It follows from these formulae that we have the following proposition.

Proposition 4.1. *Let $\gamma : I \to \mathbb{R}^2$ be a convex plane curve with $|\gamma'(s), \gamma''(s)| = 1$. Then*
(a) *$f_x'(s_0) = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\gamma(s_0) - x = \lambda\gamma''(s_0)$.*
(b) *$f_x'(s_0) = f_x''(s_0) = 0$ if and only if $k_a(s_0) \neq 0$ and $x = \gamma(s_0) + \frac{1}{k_a(s_0)}\gamma''(s_0)$.*
(c) *$f_x'(s_0) = f_x''(s_0) = f_x'''(s_0) = 0$ if and only if $k_a(s_0) \neq 0$, $x = \gamma(s_0) + \frac{1}{k_a(s_0)}\gamma''(s_0)$ and $k_a'(s_0) = 0$.*
(d) *$f_x'(s_0) = f_x''(s_0) = f_x'''(s_0) = 0$ and $f^{IV}(s_0) \neq 0$ if and only if $k_a(s_0) \neq 0$, $x = \gamma(s_0) + \frac{1}{k_a(s_0)}\gamma''(s_0)$ and $k_a'(s_0) = 0$ and $k_a''(s_0) \neq 0$.*

Proof. (a) By the formula (4.1), $f_x'(s_0) = 0$ if and only if $\gamma''(s_0)$ and $\gamma(s_0) - x$ are parallel.
(b) It follows from the formula (4.2) that $f_x''(s_0) = 0$ if and only if $|\gamma'''(s_0), \gamma(s_0) - x| = 1$, so that $f_x'(s_0) = f_x''(s_0) = 0$ if and only if $-\lambda k_a(s_0) = |\gamma'''(s_0), \lambda\gamma''(s_0)| = 1$ because of (a). The last condition is equivalent to the condition that $k_a(s_0) \neq 0$ and $\lambda = -\frac{1}{k_a(s_0)}$.
(c) We also assert that $f_x'(s_0) = f_x''(s_0) = f_x'''(s_0) = 0$ if and only if $k_a(s_0) \neq 0$, $x = \gamma(s_0) + \frac{1}{k_a(s_0)}\gamma''(s_0)$ and $|\gamma^{IV}(s_0), -\frac{1}{k_a(s_0)}\gamma''(s_0)| = 0$. On the other hand, differentiating the both side of the equality $k_a(s) = |\gamma''(s), \gamma'''(s)|$, we have $k_a'(s) = |\gamma''(s), \gamma^{IV}(s)|$. This implies that $k_a'(s_0) = 0$ if and only if $|\gamma^{IV}(s_0), -\frac{1}{k_a(s_0)}\gamma''(s_0)| = 0$.
(d) We have $k_a''(s) = |\gamma'''(s), \gamma^{IV}(s)| + |\gamma''(s), \gamma^{V}(s)|$. Differentiating the both side of the relation $\gamma'(s), \gamma''(s)| = 1$, we have $|\gamma'(s), \gamma'''(s)| = 0$. We differentiate again, so that we have an equation $|\gamma''(s), \gamma'''(s)| + |\gamma'(s), \gamma^{IV}(s)| = 0$. It follows that we have $|\gamma'(s), \gamma^{IV}(s)| = -k_a(s)$. By the formula (2.?) we have $|\gamma'''(s), \gamma^{IV}(s)| = |-k_a(s)\gamma'(s), \gamma^{IV}(s)| = k_a(s)^2$. Therefore we have $k_a''(s) = k_a(s)^2 + |\gamma''(s), \gamma^{IV}(s)|$, so that $k_a''(s_0) = 0$ if and only if $|\gamma''(s_0), \gamma^{IV}(s_0)| = -k_a(s_0)^2$. Under the condition (c), $|\gamma''(s_0), \gamma^{IV}(s_0)| = -k_a(s_0)^2$ if and only if $|\gamma^{V}(s_0), \gamma(s_0) -$

$x| = -k_a(s_0)$. By the formula (4.5), it is equivalent to the condition that $f_x^{IV}(s_0) = 0$. This completes the proof. $\square$

4-2) Affine height functions. Let S^1 be the "unit" circle in $\mathbb{R}^2$ given by $S^1 = \{(x_1, x_2) | x_1^2 + x_2^2 = 1\}$. We also define a family of smooth functions on γ parameterized by S^1

$$H : I \times S^1 \to \mathbb{R}$$

by

$$H(s, u) = |\gamma'(s), u|.$$

We call H *an affine height function on* γ.

Differentiating $H(s, u)$ with respect to s, we have the followings:

$$h'_u(s) = |\gamma''(s), u|, \tag{4.5}$$

$$h''_u(s) = |\gamma'''(s), u|, \tag{4.6}$$

$$h'''_u(s) = |\gamma^{IV}(s), u|, \tag{4.7}$$

where $h_u(s) = H(s, u)$ for any $u \in S^1$.

It follows from these formulae that we have the following proposition.

Proposition 4.2. *Let* $\gamma : I \to \mathbb{R}^2$ *be a plane curve with* $|\gamma'(s), \gamma''(s)| = 1$. *Then*
(a) $h'_u(s_0) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $u = \lambda\gamma''(s_0)$.
(b) $h'_u(s_0) = h''_u(s_0) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $u = \lambda\gamma''(s_0)$ *and* $k_a(s_0) = 0$.
(c) $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ *if and only if there exists* $\lambda \in \mathbb{R}$ *such that* $u = \lambda\gamma''(s_0)$ *and* $k_a(s_0) = k'_a(s_0) = 0$.

Proof. (a) By the formula (4.5), $h'_u(s_0) = 0$ if and only if $\gamma''(s_0)$ and u are parallel.
(b) It follows from the formula (a) and (4.6) that $h'_u(s_0) = h''_u(s_0) = 0$ if and only if $-k_a(s_0)\lambda = \lambda|\gamma'''(s_0), \gamma''(s_0)| = |\gamma'''(s_0), u| = 0$. If $\lambda = 0$, then $u = 0$. This contradicts the fact that $u \in S^1$, so that the above condition is equivalent to the condition $k_a(s_0) = 0$.
(c) Differentiating the both side of the formula $\gamma'''(s) = -k_a(s)\gamma'(s)$, we have $\gamma^{IV}(s) = -k'(s)\gamma'(s) - k_a(s)\gamma''(s)$. By (b) and (4.7), $h'_u(s_0) = h''_u(s_0) = h'''_u(s_0) = 0$ if and only if there exists $\lambda \in \mathbb{R}$ such that $u = \lambda\gamma''(s_0)$, $k_a(s_0) = 0$ and $|\gamma'''(s_0), u| = -k'_a(s_0)|\gamma'(s_0), u| = -k'_a(s_0)\lambda$. Since $\lambda \neq 0$, we have $k'_a(s_0) = 0$. This completes the proof. $\square$

We remark that $\lambda(s) = \pm\frac{1}{\sqrt{x_1''(s)^2 + x_2''(s)^2}}$, where $\gamma(s) = (x_1(s), x_2(s))$.

5. Unfoldings of functions of one-variable

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [5]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (t_0, x_0)) \to \mathbb{R}$ be a function germ. We call f *an* r*-parameter unfolding* of $f(t) = F(t, x_0)$. The crucial notion is universal unfoldings, however, we only use some features of it, so we do not need to give the original definition of universal unfoldings (cf., [5]). We say that $f(t)$ has *an* A_k *singularity at* t_0 if $f^{(p)}(t_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(t_0) \neq 0$. In this case we adopt the following definition. Let $j^{k-1}(\frac{\partial F}{\partial x_i}(t,x_0))(t_0) = \sum_{j=1}^{k-1} \alpha_{j,i}t^j$ for $i = 1,\dots,r$, where j^{k-1} denotes the $(k-1)$-jet. We say that F is *infinitesimally (p)versal* (respectively, *infinitesimally versal*) if the $(k-1)\times r$ (respectively, $k \times r$) matrix of coefficients (α_{ji}) (respectively, $(\alpha_{0i}, \alpha_{ji})$) has rank $k-1$ (respectively, k). There are two important sets of unfoldings relative to the above two notions. *The bifurcation set* of F is the set

$$\mathcal{B}_F = \{x \in \mathbb{R}^r | \text{there exists } t \text{ with } \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0 \text{ at } (t,x)\}$$

and *the discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{there exists } t \text{ with } F = \frac{\partial F}{\partial t} = 0 \text{ at } (t,x)\}.$$

The proof of Theorem 2.2 is based on the following result.

Proposition 5.1 (cf., [5]). *Let $F : (\mathbb{R}\times\mathbb{R}^r, (t_0,x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(t)$ which has the A_k singularity at t_0.*
(1) Suppose that F is an infinitesimally (p)versal unfolding.
(a) If $k=2$, then $\mathcal{B}_F$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$.
(b) If $k=3$, then $\mathcal{B}_F$ is diffeomorphic to $C\times\mathbb{R}^{r-2}$.
(2) Suppose that F is an infinitesimally versal unfolding.
(a) If $k=1$, then $\mathcal{D}_F$ is diffeomorphic to $\{0\}\times\mathbb{R}^{r-1}$.
(b) If $k=2$, then $\mathcal{D}_F$ is diffeomorphic to $C\times\mathbb{R}^{r-2}$.
Here, $C = \{(x_1,x_2) | x_1^2 = x_2^3\}$ is the ordinary cusp.

Proof of Theorem 2.2, (1). We consider the distance-cubed function $F(s,x) = |\gamma'(s), \gamma(s)-x|$. Using the results of Proposition 4.1, the bifurcation set of F is precisely the affine evolute of γ.

Next, the condition for $f = F_{x_0}$ to have exactly an A_2 at s_0 is for x_0 is on the affine evolute of γ at s_0 but $k_a'(s_0) \neq 0$. Likewise the condition for f to have exactly A_3 at s_0 is for x_0 is on the affine evolute of γ at s_0 and $k_a'(s_0)=0$, $k_a''(s_0)\neq 0$. By the assumption (A 1)-(A 3), f has no higher A_k.

Now $F(s,x) = x_1'(s)(x_2(s)-x_2) - x_2'(s)(x_1(s)-x_1)$, so that we should like to show that $F(s,x)$ is an infinitesimally (p)versal unfolding of $f(s)$ for the above two cases. Here,

$$\begin{aligned}
\frac{\partial F}{\partial x_1} =& x_2'(s);\ \text{2-jet at } s_0 = x_2''(s_0)s + \frac{1}{2}x_2''(s_0)s^2,\\
\frac{\partial F}{\partial x_2} =& -x_1'(s);\ \text{2-jet at } s_0 = -x_1''(s_0)s - \frac{1}{2}x_1''(s_0)s^2.
\end{aligned}$$

The condition for infinitesimally (p)versal unfolding is as follows:

(i) When f has A_2 at s_0, we require the 1×2 matrix $(x_2''(s_0), -x_1''(s_0))$ to have rank 1, which is always does since $|\gamma'(s), \gamma''(s)| = 1$.

(ii) When f has A_3 at s_0, we require 2×2 matrix

$$\begin{pmatrix} x_2''(s_0) & -x_1''(s_0) \\ \frac{1}{2}x_2''(s_0) & -\frac{1}{2}x_1''(s_0) \end{pmatrix}$$

to be nonsingular. But, this just says $k_a(s_0) \neq 0$ which is true since $f'_x(s_0) = f''_x(s_0) = 0$ (cf., Section 4).

Hence the infinitesimally (p)versal conditions are automatically satisfied and we have the required result by Proposition 5.1. $\square$

Proof of Theorem 2.2, (2). We consider

$$\widetilde{H} : I \times S^1 \times \mathbb{R} \to \mathbb{R}$$

given by

$$\widetilde{H}(s, u, y) = H(s, u) - y = x'_1(s)u_2 - x'_2(s)u_1 - y.$$

The discriminant set of $\widetilde{H}$ is

$$\mathcal{D}^{\pm}_{\widetilde{H}} = \{(\lambda(s)\gamma''(s), \lambda(s)) | \lambda(s) = \pm\frac{1}{\sqrt{x''_1(s)^2 + x''_2(s)^2}},\ s \in I\}.$$

We only consider $\mathcal{D}^{+}_{\widetilde{H}}$. Define a map

$$\Phi : \mathbb{R}^2 - \{0\} \to S^1 \times \mathbb{R}_+$$

by

$$\Phi(x_1, x_2) = ((\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}), \frac{1}{\sqrt{x_1^2 + x_2^2}}),$$

where $\mathbb{R}_+$ is the set of all positive real numbers. It is clear that Φ is a diffeomorphism and $\Phi(\text{Image}\,(\gamma'')) = \mathcal{D}^{+}_{\widetilde{H}}$.

Using the results of Proposition 4.2 and the assumption (A 1)-(A 3), we only need to consider A_1 and A_2 singularities.

We now define a family of functions

$$F : I \times \mathbb{R}^2 \to \mathbb{R}$$

by

$$F(s, x_1, x_2) = x'_1(s) \sin x_1 - x'_2(s) \cos x_1 - x_2.$$

This is considered as a local representation of $\widetilde{H}$. We may use F instead of $\widetilde{H}$. Thus

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= x'_1(s) \cos x_1 + x'_2(s) \sin x_1, \\ \frac{\partial F}{\partial x_2} &= -1. \end{aligned}$$

The condition for infinitesimally versal unfolding is also as follows:

(i) When f has A_1 at s_0, we require the 1×2 matrix $(x'_1(s) \cos x_1 + x'_2(s) \sin x_1, -1)$ to have rank 1, which is always does.

(ii) When f has A_2 at s_0, we require 2×2 matrix

$$\begin{pmatrix} x_1'(s_0)\cos x_1 + x_2'(s_0)\sin x_1 & -1 \\ x_1''(s_0)\cos x_1 + x_2''(s_0)\sin x_1 & 0 \end{pmatrix}$$

to be nonsingular.

On the other hand,

$$(\cos x_1, \sin x_1) = \pm\frac{1}{\sqrt{x_1''(s_0)^2 + x_2''(s_0)^2}}(x_1''(s_0), x_2''(s_0))$$

when s_0 is corresponding to $((\cos x_1, \sin x_1), x_2) \in \mathcal{D}_F^{\pm}$, so that $x_1''(s_0)\cos x_1 + x_2''(s_0)\sin x_1 = \pm\sqrt{x_1''(s_0)^2 + x_2''(s_0)^2} \neq 0$. This means that the above matrix is nonsingular. So the infinitesimally versal conditions are automatically satisfied and we have the required result by Proposition 5.1. $\square$

References

1. W. Blaschke, *Vorlesungen uber Differentialgeometrie II, Affine Differentialgeometrie*, Springer, 1923.
2. J. W. Bruce, *On singulariteis, envelopes and elementary differential geometry*, Math.Proc.Camb.Phil.Soc. **89** (1981), 43–48.
3. J. W. Bruce and P. J. Giblin, *Generic curves and surfaces*, J. London Math. Soc. **24** (1981), 555–561.
4. J. W. Bruce and P. J. Giblin, *Generic Geometry*, Amer. Math. Monthly **90** (1983), 529–545.
5. J. W. Bruce and P. J. Giblin, *Curves and singularities*, Cambridge University press, 1984.
6. D. L. Fidal and P. J. Giblin, *Generic 1-parameter families of caustics by reflexion in the plane*, Math. Proc. Camb. Phil. Soc. **96** (1984), 425–432.
7. D. L. Fidal, *The existence of sextactic points*, Math. Proc. Camb. Phil. Soc. **96** (1984), 433–436.
8. P. A. Schirokow and A. P. Schirokow, *Affine Differentialgeometrie*, Teubner, 1962.
9. B. Su, *Affine Differential Geometry*, Gordon and Breach, 1983.

Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan