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H. M. Ito and T. Mikami

Series #312. October 1995

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Poissonian Asymptotics of a Randomly Perturbed Dynamical System: Flip-flop of the Stochastic Disk Dynamo

H. M. Ito[1] and T. Mikami[2]

[1] Seismology Department, Meteorological Research Institute, Tsukuba, Ibaraki 305, Japan

[2] Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Abstract A dynamical system with two stable equilibrium points will show a flip-flop motion between the neighborhoods of the two points when it is perturbed by small random noises. A typical example is the stochastic disk dynamo model where the two equilibrium points correspond to the two polarities of the earth's magnetic field. We will prove what has been suggested by a computer simulation, that is, the counting process of the flips or the reversals of the earth's field converges to a standard Poisson process if the time is suitably scaled.

KEY WORDS: Poisson, asymptotics, small random perturbation, reversal of the earth's magnetic field

1. INTRODUCTION

Geophysical phenomena recorded on the earth sometimes offer novel stochastic features of nature which are difficult to find in laboratory experiments of limited time span. Analyses of the paleomagnetic data revealed that the earth's magnetic field has reversed its polarities many times(1). One of the most interesting statistical properties of the reversals is that their counting process is a Poisson process.

Simple models mimicking the dynamo action of the earth have been analyzed numerically to account for the statistical properties. In one approach the statistical properties are attributed to chaos of deterministic systems composed of a couple of disk dynamos. See Ref.2 for present status of this approach.

The other approach regards the reversals as a result of perturbations acted on the single disk dynamo.(3,4) In particular, Honkura and Hoshi(4) emphasized the importance of the randomness of the perturbations. One of the authors proposed independently a stochastic model(5), following the recipe by van Kampen(6) and Kubo et al.(7) This model, called the stochastic disk dynamo model, describes a system of mutually interacting single disk dynamos, and is expressed by a two-dimensional dynamical system subject to small random perturbations. The unperturbed non-gradient dynamical system has two stable equilibrium points corresponding to the two polarities of the earth's magnetic field. The perturbations showing the effects of the eliminated degrees of freedom other than the dipole field, are of order of the inverse of the number of the interacting disks.

A computer simulation suggested[5] that the stochastic disk dynamo has the Poisson property above mentioned. Later, its mathematical justification was given to a one-dimensional simplified version with a double well potential.[8] The purpose of the present paper is to give a justification to the original two-dimensional model.

Efforts have been devoted to the asymptotics of dynamical systems subject to small random perturbations from practical[9,10] as well as mathematical[11,24] viewpoints. The problem discussed here will be new in two respects. Firstly, instead of relaxation from a metastable state to a stable state, we are interested in flip-flop motion between the two states equally stable. Secondly unlike most of the works which are concerned with the ensemble averaged properties, we deal with pathwise properties, importance of which was emphasized by Cassandro et al.[12]

We constitute our paper as follows. In section 2, we give a definition of the stochastic disk dynamo model together with its physical background as well as its mathematical properties. The main result is Theorem 3.1 which assures that the counting process of the reversals, Y^ϵ of (3.3), converges in the limit of small perturbation to a standard Poisson process if the time is scaled suitably. It is proved by preparing two theorems in Section 3, proofs of which are given in the following subsections.

2. STOCHASTIC DISK DYNAMO MODEL

Stochastic disk dynamo model is described by the following stochastic differential equation: for $t \geq 0, x = (x_1, x_2) \in \mathbf{R}^2, \varepsilon > 0$,

$$\begin{aligned} dX^\varepsilon(t,x) &= b(X^\varepsilon(t,x))dt + \varepsilon^{1/2} dW(t), \\ X^\varepsilon(0,x) &= x, \end{aligned} \tag{2.1}$$

where $W(\cdot)$ is a 2-dimensional Wiener process[13], and $b(x) = (b_1(x), b_2(x))$ is given by

$$\begin{aligned} b_1(x_1,x_2) &= -\mu x_1 + x_1 x_2, \\ b_2(x_1,x_2) &= -\nu x_2 + 1 - x_1^2. \end{aligned} \tag{2.2}$$

Here μ and ν are positive constants such that $\nu\mu < 1$.

This model describes a system of several disks interacting mutually. The variables x_1 and x_2 physically stand for the current running through the coil and the angular velocity of the disks, respectively, and the $1/\epsilon$ has a meaning of the number of the disks. See Ref. 5 for further physical background and some related mathematical arguments.

The deterministic equation (2.1) with $\epsilon = 0$ is called the single disk dynamo model, which does not show the reversal of the polarity, i.e., the change of the signs of X_1^0. In fact, since $\nu\mu < 1$, $\{X^0(t,x)\}$ has three equilibrium points[14,15]: a hyperbolic equilibrium point

$$H(0, 1/\nu) \tag{2.3}$$

and exponentially stable equilibrium points

$$F_+(\lambda, \mu), F_-(-\lambda, \mu), \tag{2.4}$$

where

$$\lambda = (1 - \mu\nu)^{1/2}. \tag{2.5}$$

On the other hand, the stochastic disk dynamo exhibits reversals. More precisely, the solution to (2.1) exists, is positively recurrent, and has a unique invariant probability density function[15,16] since

$$\begin{aligned} < x, b(x) > &= x_1(-\mu x_1 + x_1 x_2) + x_2(-\nu x_2 + 1 - x_1^2) \\ &= -\mu x_1^2 - \nu x_2^2 + x_2 \\ &(= -\mu x_1^2 - \nu(x_2 - 1/(2\nu))^2 + 1/(4\nu)) \\ &\leq -\mu x_1^2 - (\nu/2)x_2^2 + 1/(2\nu) \\ &\leq -\min(\mu, \nu/2)|x|^2 + 1/(2\nu). \end{aligned} \tag{2.6}$$

Here we used

$$x_2 = \nu^{-1/2}\nu^{1/2}x_2 \leq \{1/\nu + \nu x_2^2\}/2.$$

The regions $\{x \in \mathbf{R}^2 : x_1 > \lambda - \delta\}$ and $\{x \in \mathbf{R}^2 : x_1 < -\lambda + \delta\}$ are understood as the normal polarity and the reversed polarity of the earth's field, respectively, where $\delta \in (0, \lambda)$ is a parameter to be determined by paleomagnetic observation. We will discuss the asymptotics of the flip-flop motion of X^ϵ between the two polarities in the limit of $\epsilon \to 0$.

3. MATHEMATICAL RESULTS

For the stochastic process $\{X^\varepsilon(t,x)\}_{0\geq t<\infty}$ determined by (2.1), we consider a sequence of stopping times $\{\tau_n^\varepsilon(x;\delta)\}_{n\geq 0}$ defined by the following: for $\delta\in(0,\lambda)$ and $x=(x_1,x_2)\in\mathbf{R}^2$ for which $|x_1|\geq\lambda-\delta$, put

$$\tau_0^\varepsilon(x;\delta)=0,$$
$$\tau_{n+1}^\varepsilon(x;\delta)=\begin{cases}\inf\{t>\tau_n^\varepsilon(x;\delta);\lambda-\delta\leq X_1^\varepsilon(t,x)\} & \text{if } X_1^\varepsilon(\tau_n^\varepsilon(x;\delta),x)\leq-\lambda+\delta,\\ \inf\{t>\tau_n^\varepsilon(x;\delta);X_1^\varepsilon(t,x)\leq-\lambda+\delta\} & \text{if } \lambda-\delta\leq X_1^\varepsilon(\tau_n^\varepsilon(x;\delta),x),\end{cases}\tag{3.1}$$

for $n\geq 0$.

Take $\beta^\varepsilon(\delta)>0$ so that

$$P(\tau_1^\varepsilon(F_+;\delta)>\beta^\varepsilon(\delta))=P(\tau_1^\varepsilon(F_-;\delta)>\beta^\varepsilon(\delta))=e^{-1},\tag{3.2}$$

which is possible since $(X_1^\varepsilon(\cdot,(x_1,x_2)),X_2^\varepsilon(\cdot,(x_1,x_2)))$ has the same probability law as that of $(-X_1^\varepsilon(\cdot,(-x_1,x_2)),X_2^\varepsilon(\cdot,(-x_1,x_2)))$ (see (2.2)). Here $F_\pm$ and λ are defined in (2.4) and (2.5), respectively.

Put $\beta^\varepsilon=\beta^\varepsilon(\lambda/2)>0$, and consider the following stochastic processes:

$$Y^\varepsilon(t;x,\delta)=n\quad\text{if } \tau_n^\varepsilon(x;\delta)\leq\beta^\varepsilon t<\tau_{n+1}^\varepsilon(x;\delta).\tag{3.3}$$

Denote by $D_{\mathbf{N}}[0,\infty)$ the space of right continuous functions $f:[0,\infty)\mapsto N$ with left limits. $D_{\mathbf{N}}[0,\infty)$ is given the Skorohod topology[21].

The following is our main result.

Theorem 3.1. *For any $\delta\in(0,\lambda)$ and any x for which $|x_1|\geq\lambda-\delta$, $\{Y^\varepsilon(t;x,\delta)\}_{0\leq t<\infty}$ converges, as $\varepsilon\to 0$, in distribution in $D_{\mathbf{N}}[0,\infty)$[21] to a Poisson process with parameter 1*[17].

Theorem 3.1 can be proved by the following Theorems 3.2 and 3.3, proof of which will be given in sections 3.2 and 3.3. Theorem 3.2 assures that any finite dimensional marginal distribution of $\{Y^\varepsilon(t;x,\delta)\}_{0\leq t<\infty}$ converges, as $\varepsilon\to 0$, to that of a Poisson process with parameter 1. Theorem 3.3 and Corollary 7.4 (p. 129, Ref. 21) guarantee the tightness of $\{Y^\varepsilon(t;x,\delta)\}_{0\leq t<\infty}$.

Theorem 3.2. *For any $\delta\in(0,\lambda)$ and $r>0$, the following holds: for any $\ell\in N$ and any $0=n_0\leq n_1\leq\cdots\leq n_\ell$,*

$$\begin{aligned}&\lim_{\varepsilon\to 0}P(Y^\varepsilon(t_1;x,\delta)=n_1,\cdots,Y^\varepsilon(t_\ell;x,\delta)=n_\ell)\\&=\Pi_{i=1}^\ell(t_i-t_{i-1})^{n_i-n_{i-1}}\exp(-(t_i-t_{i-1}))/(n_i-n_{i-1})!,\end{aligned}\tag{3.4}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ *and* $0 = t_0 < t_1 < \cdots < t_\ell < \infty$.

Here and in the following, $U_r(x) \equiv \{y \in \mathbf{R}^2; |x-y| < r\}$ for $x \in \mathbf{R}^2$ and $r > 0$, and the symbol o stands for the origin.

Remark 3.1. Roughly speaking, Theorem 3.2 means that $[\tau_{n+1}^\varepsilon(x;\delta) - \tau_n^\varepsilon(x;\delta)]/\beta^\varepsilon$ $(n \geq 0)$ are, for sufficiently small $\varepsilon > 0$, independent of each other (from the strong Markov property of $X^\varepsilon(t,x)$), and have exponential distributions, from which we can believe that Theorem 3.1 is true[17].

Before stating Theorem 3.3, we give a notation.

For $r > 0$, $T > 0$ and $f \in D_{\mathbf{N}}[0,\infty)$, put

$$\omega'(f,r,T) \equiv \inf\{\max_i \sup_{s,t \in [t_{i-1},t_i)} |f(t) - f(s)|\}, \tag{3.5}$$

where the infimum is taken over all partitions $\{t_i\}$ for which $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > r$ and over all $n \geq 1$.

Theorem 3.3. *For any* $\delta \in (0,\lambda)$ *and* $r > 0$, *the following holds.*

(I).*For any* $\gamma > 0$ *and* $t > 0$, *there exists* $N_{\gamma,t} \in \mathbf{N}$ *such that*

$$\limsup_{\varepsilon \to 0}(\sup\{P(Y^\varepsilon(t;x,\delta) \geq N_{\gamma,t}); |x| < r, |x_1| \geq \lambda - \delta\}) < \gamma. \tag{3.6}$$

(II). *For any* $\gamma > 0$ *and* $T > 0$, *there exists* $\tilde{r} > 0$ *such that*

$$\limsup_{\varepsilon \to 0}(\sup\{P(\omega'(Y^\varepsilon(\cdot;x,\delta),\tilde{r},T) \geq \gamma); |x| < r, |x_1| \geq \lambda - \delta\}) < \gamma. \tag{3.7}$$

3.1. Special case of Theorem 3.2.

In this subsection, we prove Theorem 3.4, a special case of Theorem 3.2 with $\ell = 1$ and $n_1 = 0$, dealing with asymptotics of exit probabilities. Closely related problems have been discussed in Refs. 18,19,22 and 23.

Theorem 3.4. *For any* $\delta \in (0,\lambda)$ *and* $r > 0$

$$\lim_{\varepsilon \to 0} P(\tau_1^\varepsilon(x;\delta) > \beta^\varepsilon(\lambda/2)t) = e^{-t}, \tag{3.8}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ *and* $t \geq 0$.

Theorem 3.4 can be proved from Theorem 3.5 below, which gives an exponential law for a stopping time

$$\sigma^\varepsilon(x;\delta) = \begin{cases} \inf\{t > 0; |X^\varepsilon(t,x) - F_+| \leq \delta\} & \text{if } |x - F_-| \leq \delta, \\ \inf\{t > 0; |X^\varepsilon(t,x) - F_-| \leq \delta\} & \text{if } |x - F_+| \leq \delta, \end{cases} \tag{3.9}$$

where $\delta \in (0,\lambda)$ and $x \in \overline{U_\delta(F_+) \cup U_\delta(F_-)}$.

Theorem 3.5. *There exists $\delta_1 > 0$ such that for any $\delta \in (0, \delta_1)$,*

$$\lim_{\varepsilon \to 0} P(\sigma^\varepsilon(x;\delta) > \beta^\varepsilon(\lambda/2)t) = e^{-t}, \tag{3.10}$$

uniformly in $x \in \overline{U_\delta(F_+) \cup U_\delta(F_-)}$, and $t \geq 0$.

Before proving Theorem 3.5 we note that the following (R.1)-(R.4) hold for $R > R_0$ if $R_0 > 0$ is sufficiently large.

(R.1). $U_\lambda(F_+) \cup U_\lambda(F_-) \subset U_R(o)$.

(R.2).

$$< x, b(x) >< 0$$

for all $x \in \partial U_R(o)$ (see (2.6)).

(R.3). For any $x \in \partial U_R(o) \cap \{y; y_1 \leq 0\}$,

$$V_R(x) > V_R(H).$$

Here we put

$$V_R(y) = \inf\{\int_0^t |d\varphi(s)/ds - b(\varphi(s))|^2 ds/2; \varphi(0) = F_-, \varphi(t) = y,$$
$$\{\varphi(s)\}_{0 \leq s < t} \subset U_R(o) \cap \{y; y_1 < 0\}, t > 0\}.$$

(R.4). There exists $\delta_0 \in (0, \lambda)$ such that $V_R(x)$ is smooth in $U_{\delta_0}(F_-)$.

Remark 3.2. For any $r > 0$, there exists $R(r) > 0$ such that

$$V_R(x) = V_{R'}(x)$$

for $x \in U_r(o) \cap \{y; y_1 < 0\}$ and $R, R' \geq R(r)$ (see the proof of Proposition 3.1).

It is easy to see that the conditions R.1 and R.2 are satisfied. As for R.3 and R.4, see Proposition 3.1 at the end of this subsection and Theorem 2 in Ref. 20.

Let us prepare some lemmas. Fix $R > R_0$ and put

$$C_\eta = \{x \in \overline{U_R(o)}; dist(x, \{y \in U_R(o); y_1 = 0\}) < \eta\}, \tag{3.11}$$

$$\tau_A^\varepsilon(x) \equiv \inf\{t > 0; X^\varepsilon(t,x) \notin A\} \quad \text{for } x \in \mathbf{R}^2, A \subset \mathbf{R}^2. \tag{3.12}$$

Then the following lemma can be proved in the same way as in Lemma 1[22], and we omit the proof.

Lemma 3.1. *For any $\alpha > 0$, there exists $\eta > 0$ so that*

$$\sup_{x \in C_\eta} E[\tau^\varepsilon_{C_\eta}(x)] < \exp(\alpha/\varepsilon) \tag{3.13}$$

for sufficiently small $\varepsilon > 0$.

Put

$$M(\eta, \delta) = \sup\{\tau^0_{R^2 \setminus (U_{\delta/2}(F_+) \cup U_{\delta/2}(F_-))}(x); x \in \overline{U_R(o)}, |x_1| \geq \eta\} \tag{3.14}$$

which is finite from the upper semicontinuity of $\tau^0_{R^2 \setminus (U_{\delta/2}(F_+) \cup U_{\delta/2}(F_-))}(x)$, and put

$$S^\varepsilon_\eta(x) = \begin{cases} \inf\{t > 0; X^\varepsilon(t,x) \notin U_R(o) \cap \{y; y_1 < \eta\}\} & \text{if } x_1 \leq 0, \\ \inf\{t > 0; X^\varepsilon(t,x) \notin U_R(o) \cap \{y; y_1 > -\eta\}\} & \text{if } x_1 > 0. \end{cases} \tag{3.15}$$

From the condition (R.3), the following can be proved in the same way as in Lemma 2[22], and the proof is omitted.

Lemma 3.2. *For any $\delta \in (0, \lambda)$ and any sufficiently small $\eta \in (0, \lambda - \delta)$,*

$$\lim_{\varepsilon \to 0} P(\sigma^\varepsilon(x; \delta) < S^\varepsilon_\eta(x) + M(\eta, \delta)) = 1, \tag{3.16}$$

uniformly in $x \in \overline{U_\delta(F_+) \cup U_\delta(F_-)}$.

Take $\gamma^\varepsilon(\eta) > 0$ so that

$$P(S^\varepsilon_\eta(F_-) > \gamma^\varepsilon(\eta)) = e^{-1}. \tag{3.17}$$

Lemma 3.3. *There exists $\eta^\varepsilon > 0$ for which $\eta^\varepsilon/\gamma^\varepsilon(\eta) \to 0$ (as $\varepsilon \to 0$) for any $\eta \in (0, \lambda)$ and the following holds: there exists $\delta_1 \in (0, \lambda)$ such that for any $\delta \in (0, \delta_1)$ and $\eta \in (0, \lambda)$,*

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \{P(S^\varepsilon_\eta(y) > u) - P(S^\varepsilon_\eta(F_-) > u - \eta^\varepsilon)\} \leq 0, \\ &\liminf_{\varepsilon \to 0} \{P(S^\varepsilon_\eta(y) > u) - P(S^\varepsilon_\eta(F_-) > u + \eta^\varepsilon)\} \geq 0, \end{aligned} \tag{3.18}$$

uniformly in $u \geq 0$ and $y \in \overline{U_\delta(F_-)}$.

(Proof). Take $\tilde{\alpha} > 0$ sufficiently small so that $\{y; V_R(y) \leq 2\tilde{\alpha}\} \subset U_{\delta_0}(F_-)$ and that for any $\eta \in (0, \lambda)$

$$\exp(2\tilde{\alpha}/\varepsilon)/\gamma^\varepsilon(\eta) \to 0,$$

as $\varepsilon \to 0$[11]. Put $\eta^\varepsilon = \exp(2\tilde{\alpha}/\varepsilon)$ and $K = \{y; V_R(y) \le \tilde{\alpha}\}$. Then in K, $V_R(y)$ is smooth from the condition (R.4). Take $\delta_1 > 0$ so that $U_{\delta_1}(F_-) \subset Int(K)$. Then the rest of the proof is the same as that of Lemma 3[22].

Q. E. D.

Put, for $t > 0$,

$$f_\varepsilon(t;\eta) = P(S^\varepsilon_\eta(F_-) > \gamma^\varepsilon(\eta)t). \tag{3.19}$$

From the condition (R.3), the following can be proved in the same way as in Lemma 4[22], and the proof is omitted.

Lemma 3.4. *For sufficiently small $\eta \in (0, \lambda)$, there exists a sequence of positive numbers $\{\theta_\varepsilon\}$ for which $\theta_\varepsilon \to 0$, as $\varepsilon \to 0$, such that the following holds: for any $t_0 > 0$,*

$$\begin{aligned} \limsup_{\varepsilon\to 0}\{f_\varepsilon(s+\theta_\varepsilon;\eta)f_\varepsilon(t+\theta_\varepsilon;\eta) - f_\varepsilon(s+t;\eta)\} \le 0, \\ \liminf_{\varepsilon\to 0}\{f_\varepsilon(s;\eta)f_\varepsilon(t-\theta_\varepsilon;\eta) - f_\varepsilon(s+t;\eta)\} \ge 0, \end{aligned} \tag{3.20}$$

uniformly in $(s,t) \in [0,\infty) \times [t_0,\infty)$.

Now we prove Theorem 3.5.

(Proof of Theorem 3.5). In the same way as in Theorem 1[22], we get the following from Lemmas 3.1-3.4: for any $\delta \in (0,\delta_1)$ (see Lemma 3.3) and sufficiently small $\eta \in (0, \lambda - \delta_1)$,

$$\lim_{\varepsilon\to 0} P(\sigma^\varepsilon(x;\delta) > \gamma^\varepsilon(\eta)t) = e^{-t}, \tag{3.21}$$

uniformly in $x \in \overline{U_\delta(F_+) \cup U_\delta(F_-)}$ and $t \ge 0$.

Hence the following relation (3.22) together with Lemma 3.5 below, completes the proof (see (3.2) and (3.17)): for any $\delta \in (0,\delta_1)$ (see Lemma 3.3) and sufficiently small $\eta \in (0, \lambda - \delta_1)$,

$$\lim_{\varepsilon\to 0} \gamma^\varepsilon(\eta)/\beta^\varepsilon(\delta) = 1. \tag{3.22}$$

Let us prove (3.22).

We first show that for any $\delta \in (0,\delta_1)$ and $\eta \in (0, \lambda - \delta_1)$,

$$\gamma^\varepsilon(\eta) < \beta^\varepsilon(\delta). \tag{3.23}$$

Since $S^\varepsilon_\eta(F_-) < \tau^\varepsilon_1(F_-;\delta)$ by (3.1) and (3.15), the relation (3.17) leads to

$$\begin{aligned}\exp(-1) &= P(S^\varepsilon_\eta(F_-) > \gamma^\varepsilon(\eta)) \\ &< P(\tau^\varepsilon_1(F_-;\delta) > [\gamma^\varepsilon(\eta)/\beta^\varepsilon(\delta)]\beta^\varepsilon(\delta)),\end{aligned}$$

from which we get (3.23) by (3.2).

Next we show that for any $\delta \in (0, \delta_1)$ and sufficiently small $\eta \in (0, \lambda - \delta_1)$,

$$\limsup_{\varepsilon\to 0} \beta^\varepsilon(\delta)/\gamma^\varepsilon(\eta) \leq 1. \tag{3.24}$$

Since $\tau^\varepsilon_1(F_-;\delta) \leq \sigma^\varepsilon(F_-;\delta)$ by (3.1) and (3.9), (3.2) leads to

$$\begin{aligned}\exp(-1) &= P(\tau^\varepsilon_1(F_-;\delta) > \beta^\varepsilon(\delta)) \\ &\leq P(\sigma^\varepsilon(F_-;\delta) > [\beta^\varepsilon(\delta)/\gamma^\varepsilon(\eta)]\gamma^\varepsilon(\eta)),\end{aligned}$$

from which we get (3.24) by (3.21).

Q. E. D.

Lemma 3.5. *For any δ and $\delta' \in (0, \lambda)$,*

$$\lim_{\varepsilon\to 0} \beta^\varepsilon(\delta)/\beta^\varepsilon(\delta') = 1. \tag{3.25}$$

(Proof). For any $\tilde{\delta} \in (0, \lambda)$, take $\delta \in (0, \min(\delta_1, \tilde{\delta}))$ and $\eta \in (0, \lambda - \delta_1)$ for which (3.22) holds. Then inequalities

$$\gamma^\varepsilon(\eta) < \beta^\varepsilon(\tilde{\delta}) < \beta^\varepsilon(\delta) \tag{3.26}$$

are shown to hold in the same way as in (3.23) since

$$S^\varepsilon_\eta(F_-) < \tau^\varepsilon_1(F_-;\tilde{\delta}) < \tau^\varepsilon_1(F_-;\delta).$$

The relation (3.22) and the inequalities (3.26) complete the proof.

Q. E. D.

Next we prove Theorem 3.4 from Theorem 3.5.

(Proof of Theorem 3.4). For $r > 0$, take $R > \max(r, R_0)$. For $\delta \in (0, \lambda)$, take $\delta' \in (0, \min(\delta, \lambda - \delta, \delta_1))$ (see Theorem 3.5).

We first show that

$$\liminf_{\varepsilon\to 0} P(\tau^\varepsilon_1(x;\delta) > \beta^\varepsilon t) \geq e^{-t}, \tag{3.27}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ and $t \geq 0$.

For $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$, $\eta \in (0, \lambda - \delta)$ and $t > 0$,

$$
\begin{aligned}
&P(\tau_1^\varepsilon(x;\delta) > \beta^\varepsilon t) \\
&\geq P\Big(\sup_{0 \leq s \leq \tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x)} |X^\varepsilon(s,x) - X^0(s,x)| < \delta'/2, \\
&\quad S^\varepsilon_\eta(X^\varepsilon(\tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x), x)) > \beta^\varepsilon t\Big) \\
&\geq \inf_{|y|<r, |y_1| \geq \lambda - \delta'} P\Big(\sup_{0 \leq s \leq C(r,\delta')} |X^\varepsilon(s,y) - X^0(s,y)| < \delta'/2\Big) \\
&\quad \times \inf_{y \in \overline{U_{\delta'}(F_+) \cup U_{\delta'}(F_-)}} P(S^\varepsilon_\eta(y) > \beta^\varepsilon t)
\end{aligned} \tag{3.28}
$$

by the strong Markov property of $X^\varepsilon(t,x)$, since $\tau_1^\varepsilon(y;\delta) \geq S^\varepsilon_\eta(y)$ for $y \in \overline{U_\delta(F_+) \cup U_\delta(F_-)}$. Here we put

$$
C(r,\delta) \equiv \sup_{|y|<r, |y_1| \geq \lambda - \delta} \tau^0_{R^2 \setminus (U_{\delta/2}(F_+) \cup U_{\delta/2}(F_-))}(y), \tag{3.29}
$$

which is finite because of the upper semicontinuity of $\tau^0_{R^2 \setminus (U_{\delta/2}(F_+) \cup U_{\delta/2}(F_-))}(\cdot)$.

The first probability in the last part of (3.28) converges, as $\varepsilon \to 0$, to one[(11)]. The second probability in the last part of (3.28) converges, as $\varepsilon \to 0$, to $\exp(-t)$, uniformly in $t \geq 0$ for sufficiently small $\eta \in (0, \lambda - \delta_1)$ from Theorem 3.5, Lemma 3.2, $S^\varepsilon_\eta(y) < \sigma^\varepsilon(y;\delta')$ and $\beta^\varepsilon \to \infty$ as $\varepsilon \to 0$[(11)].

Next let us show that

$$
\limsup_{\varepsilon \to 0} P(\tau_1^\varepsilon(x;\delta) > \beta^\varepsilon t) \leq e^{-t}, \tag{3.30}
$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ and $t \geq 0$.

For $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ and $t \geq 0$,

$$
\begin{aligned}
&P(\tau_1^\varepsilon(x;\delta) > \beta^\varepsilon t) \\
&\leq P\Big(\sup_{0 \leq s \leq \tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x)} |X^\varepsilon(s,x) - X^0(s,x)| \geq \delta'/2\Big) \\
&\quad + P\Big(\sup_{0 \leq s \leq \tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x)} |X^\varepsilon(s,x) - X^0(s,x)| < \delta'/2, \\
&\tau_1^\varepsilon(X^\varepsilon(\tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x), x); \delta) > \beta^\varepsilon t - \tau^0_{R^2 \setminus (U_{\delta'/2}(F_+) \cup U_{\delta'/2}(F_-))}(x)\Big).
\end{aligned} \tag{3.31}
$$

The first probability on the right hand side of (3.31) converges, as $\varepsilon \to 0$, to zero[11], uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ (see (3.29)). The second probability on the right hand side of (3.31) is dominated by

$$\sup_{y \in \overline{U_{\delta'}(F_+) \cup U_{\delta'}(F_-)}} P(\sigma^\varepsilon(y;\delta') > \beta^\varepsilon t - C(r,\delta')) \tag{3.32}$$

(see (3.29) for notation), by the strong Markov property of $X^\varepsilon(t,x)$, since $\tau_1^\varepsilon(y;\delta) \leq \sigma^\varepsilon(y;\delta')$ for $y \in \overline{U_{\delta'}(F_+) \cup U_{\delta'}(F_-)}$.

The probability in (3.32) converges, as $\varepsilon \to 0$, to $\exp(-t)$, uniformly in $t \geq 0$ and in $y \in \overline{U_{\delta'}(F_+) \cup U_{\delta'}(F_-)}$ from Theorem 3.5 and from $\beta^\varepsilon \to \infty$ as $\varepsilon \to 0$[11].

Q.E.D.

Finally we show that the condition (R.3) is satisfied.

Proposition 3.1. *For sufficiently large $R > 0$,*

$$\inf\{V_R(x); x \in \partial U_R(o) \cap \{y; y_1 \leq 0\}\} > V_R(H). \tag{3.33}$$

(Proof). Put $\kappa = \min(\mu, \nu/2)/2$. Take $R_1 > 0$ such that for $x \notin U_{R_1}(o)$,

$$< x, b(x) >< -\kappa|x|^2, \tag{3.34}$$

which is possible from (2.6).

For $R > R_1$ and $\{\varphi(s)\}_{0 \leq s \leq t}$ for which $\varphi(0) = F_-$, $\varphi(t) \in \partial U_R(o) \cap \{y; y_1 < 0\}$ and $\{\varphi(s)\}_{0 \leq s < t} \subset U_R(o) \cap \{y; y_1 < 0\}$, put

$$T(\varphi) \equiv \sup\{s < t; \varphi(s) \in U_{R_1}(o) \cap \{y; y_1 < 0\}\}. \tag{3.35}$$

Then

$$\int_0^t |d\varphi(s)/ds - b(\varphi(s))|^2 ds/2 \geq \kappa(|\varphi(t)|^2 - |\varphi(T(\varphi))|^2)/2 = \kappa(R^2 - R_1^2)/2 \to \infty \tag{3.36}$$

as $R \to \infty$. This is true, since for $u \in \mathbf{R}^2$ and $x \notin U_{R_1}(o)$,

$$\begin{aligned} |u - b(x)|^2/2 &= \sup_{z \in R^2} [< z, u > - < b(x), z > -|z|^2/2] \\ &\geq < \kappa x, u > - < b(x), \kappa x > -|\kappa x|^2/2 \\ &\geq < \kappa x, u > +|\kappa x|^2/2 \geq \kappa < x, u > \end{aligned} \tag{3.37}$$

from (3.34), and since

$$\begin{aligned}\int_0^t |d\varphi(s)/ds - b(\varphi(s))|^2 ds/2 &\geq \int_{T(\varphi)}^t |d\varphi(s)/ds - b(\varphi(s))|^2 ds/2 \\ &\geq \int_{T(\varphi)}^t \kappa < \varphi(s), d\varphi(s)/ds > ds \quad \text{(from (3.37))} \\ &= \kappa(|\varphi(t)|^2 - |\varphi(T(\varphi))|^2)/2.\end{aligned}$$

Q. E. D.

3.2. Proof of Theorem 3.2.

In this subsection we prove Theorem 3.2 by induction. First we prove Theorem 3.6 which is a special case of Theorem 3.2 with $\ell = 1$.

Theorem 3.6. *For any $\delta \in (0, \lambda)$ and $r > 0$, the following holds: for any $n \in N$,*

$$\lim_{\varepsilon \to 0} P(Y^\varepsilon(t; x, \delta) = n) = t^n \exp(-t)/n!, \tag{3.38}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ and $t \geq 0$.

We need the following lemma to prove Theorem 3.6, proof of which is given at the end of this subsection.

Lemma 3.6. *There exists $r_0 > 0$ such that for any $\delta \in (0, \lambda)$, $r > 0$, and $n \in N$,*

$$\lim_{\varepsilon \to 0} P(|X^\varepsilon(\tau_n^\varepsilon(x; \delta), x)| \geq r_0) = 0, \tag{3.39}$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$.

(Proof of Theorem 3.6). We prove it by induction on n. When $n = 0$, it has been proved in Theorem 3.4. Suppose that Theorem 3.6 is true for $n \leq k$. Put

$$A_{r,\delta} \equiv U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}. \tag{3.40}$$

Then

$$\begin{aligned}&P(Y^\varepsilon(t; x, \delta) = k + 1) \\ &= P(Y^\varepsilon(t; x, \delta) = k + 1, X^\varepsilon(\tau_{k+1}^\varepsilon(x; \delta), x) \notin A_{r_0,\delta}) \\ &\quad + P(Y^\varepsilon(t; x, \delta) = k + 1, X^\varepsilon(\tau_{k+1}^\varepsilon(x; \delta), x) \in A_{r_0,\delta}).\end{aligned} \tag{3.41}$$

The first probability on the right hand side of (3.41) converges, as $\varepsilon \to 0$, to zero from Lemma 3.6, uniformly in $x \in A_{r,\delta}$ and $t \geq 0$.

The second probability on the right hand side of (3.41) is estimated as follows:

$$
\begin{aligned}
&P(Y^\varepsilon(t;x,\delta) = k+1, X^\varepsilon(\tau^\varepsilon_{k+1}(x;\delta),x) \in A_{r_0,\delta}) \\
&= \int_{0\leq s\leq t, y\in A_{r_0,\delta}} P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \in ds, X^\varepsilon(\tau^\varepsilon_{k+1}(x;\delta),x) \in dy) P(\beta^\varepsilon(t-s) < \tau^\varepsilon_1(y;\delta)) \\
&= \int_{0\leq s\leq t, y\in A_{r_0,\delta}} P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \in ds, X^\varepsilon(\tau^\varepsilon_{k+1}(x;\delta),x) \in dy) \exp(-(t-s)) + o(1) \\
&= \int_0^t P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \in ds) \exp(-(t-s)) + o(1),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $t \geq 0$, from Theorem 3.4 and Lemma 3.6; and

$$
\begin{aligned}
&\int_0^t P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \in ds) \exp(-(t-s)) \\
&= P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \leq t) - \int_0^t P(\tau^\varepsilon_{k+1}(x;\delta)/\beta^\varepsilon \leq s) \exp(-(t-s)) ds \\
&= P(Y^\varepsilon(t;x,\delta) \geq k+1) - \int_0^t P(Y^\varepsilon(s;x,\delta) \geq k+1) \exp(-(t-s)) ds \\
&= \sum_{j=k+1}^{\infty} t^j \exp(-t)/j! - \int_0^t \sum_{j=k+1}^{\infty} [s^j \exp(-s)/j!] \exp(-(t-s)) ds + o(1) \\
&= t^{k+1} \exp(-t)/(k+1)! + o(1),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $t \geq 0$, by the assumption on induction.

Q.E.D.

Next we prove Theorem 3.2 from Theorem 3.6.

(Proof of Theorem 3.2). We prove it by induction. When $\ell = 1$, it is done by Theorem 3.6. Suppose that Theorem 3.2 is true when $\ell = k \geq 2$. Then we only have to show the following to complete the proof: for $m = n_1 (\geq 1)$, $n_1 + 1$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} P(m \leq Y^\varepsilon(t_1;x,\delta), Y^\varepsilon(t_2;x,\delta) = n_2, \cdots, Y^\varepsilon(t_{k+1};x,\delta) = n_{k+1}) \qquad (3.42) \\
&= \left(\sum_{j=m}^{n_2} [(t_1)^j \exp(-t_1)/j!](t_2-t_1)^{n_2-j} \exp(-(t_2-t_1))/(n_2-j)!\right) \\
&\quad \times \Pi_{i=3}^{\ell} (t_i - t_{i-1})^{n_i - n_{i-1}} \exp(-(t_i - t_{i-1}))/(n_i - n_{i-1})!,
\end{aligned}
$$

uniformly in $x \in U_r(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ and $0 = t_0 < t_1 < \cdots < t_\ell < \infty$ since

$$
\begin{aligned}
&P(Y^\varepsilon(t_1; x, \delta) = n_1, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}) \\
&= P(n_1 \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}) \\
&\quad - P(n_1 + 1 \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}).
\end{aligned}
$$

Let us prove (3.42). For $m \geq 1$,

$$
\begin{aligned}
&P(m \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}) \\
&= P(m \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}, X^\varepsilon(\tau_m^\varepsilon(x; \delta), x) \notin A_{r_0,\delta}) \\
&+ P(m \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1}, X^\varepsilon(\tau_m^\varepsilon(x; \delta), x) \in A_{r_0,\delta}).
\end{aligned}
\tag{3.43}
$$

The first probability on the right hand side of (3.43) converges, as $\varepsilon \to 0$, to zero from Lemma 3.6, uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < \cdots < t_\ell < \infty$.

The second probability on the right hand side of (3.43) is estimated as follows:

$$
\begin{aligned}
&P(m \leq Y^\varepsilon(t_1; x, \delta), Y^\varepsilon(t_2; x, \delta) = n_2, \cdots, Y^\varepsilon(t_{k+1}; x, \delta) = n_{k+1} \\
&\quad , X^\varepsilon(\tau_m^\varepsilon(x; \delta), x) \in A_{r_0,\delta}) \\
&= \int_{0 \leq s \leq t_1, y \in A_{r_0,\delta}} P(\tau_m^\varepsilon(x; \delta)/\beta^\varepsilon \in ds, X^\varepsilon(\tau_m^\varepsilon(x; \delta), x) \in dy) \\
&\quad \times P(Y^\varepsilon(t_2 - s; y, \delta) = n_2 - m, \cdots, Y^\varepsilon(t_{k+1} - s; y, \delta) = n_{k+1} - m) \\
&\quad \text{(by the strong Markov property of } X^\varepsilon(t, x)) \\
&= \int_0^{t_1} P(\tau_m^\varepsilon(x; \delta)/\beta^\varepsilon \in ds)(t_2 - s)^{n_2 - m} \exp(-(t_2 - s))/(n_2 - m)! \\
&\quad \times \Pi_{i=3}^{k+1}(t_i - t_{i-1})^{n_i - n_{i-1}} \exp(-(t_i - t_{i-1}))/(n_i - n_{i-1})! + o(1),
\end{aligned}
\tag{3.44}
$$

as $\varepsilon \to 0$,uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < \cdots < t_\ell < \infty$, again by Lemma 3.6 and by the assumption on induction.

From (3.44) we only have to prove that

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^{t_1} P(\tau_m^\varepsilon(x; \delta)/\beta^\varepsilon \in ds)(t_2 - s)^{n_2 - m} \exp(-(t_2 - s))/(n_2 - m)! \\
&= \sum_{j=m}^{n_2} [(t_1)^j \exp(-t_1)/j!](t_2 - t_1)^{n_2 - j} \exp(-(t_2 - t_1))/(n_2 - j)!,
\end{aligned}
\tag{3.45}
$$

uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < t_2 < \infty$, which is done as

$$
\begin{aligned}
&\int_0^{t_1} P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \in ds)(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)! \qquad (3.46)\\
&= P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \le t_1)(t_2-t_1)^{n_2-m}\exp(-(t_2-t_1))/(n_2-m)!\\
&\quad -\int_0^{t_1} P(\tau_m^\varepsilon(x;\delta)/\beta^\varepsilon \le s)\{d[(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!]/ds\}ds\\
&= (\sum_{j=m}^{\infty}(t_1)^j\exp(-t_1)/j!)(t_2-t_1)^{n_2-m}\exp(-(t_2-t_1))/(n_2-m)!\\
&\quad -\int_0^{t_1}(\sum_{j=m}^{\infty}s^j\exp(-s)/j!)(d[(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!]/ds)ds+o(1),
\end{aligned}
$$

as $\varepsilon \to 0$, uniformly in $x \in A_{r,\delta}$ and $0 < t_1 < t_2 < \infty$ by Theorem 3.6 and

$$
\begin{aligned}
&\int_0^{t_1} |d[(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!]/ds|ds\\
&\le \int_0^{t_1}\{(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!\\
&\qquad +(t_2-s)^{n_2-m-1}\exp(-(t_2-s))/(n_2-m-1)!\}ds\\
&< 2\exp(-(t_2-t_1))\sum_{i=0}^{m}(t_2-t_1)^{n_2-i}/(n_2-i)! < 2.
\end{aligned}
$$

The last part of (3.46) is transformed inductively as follows:

$$
\begin{aligned}
&(\sum_{j=m}^{\infty}(t_1)^j\exp(-t_1)/j!)(t_2-t_1)^{n_2-m}\exp(-(t_2-t_1))/(n_2-m)!\\
&\qquad -\int_0^{t_1}(\sum_{j=m}^{\infty}s^j\exp(-s)/j!)(d[(t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!]/ds)ds\\
&= \int_0^{t_1}[s^{m-1}\exp(-s)/(m-1)!](t_2-s)^{n_2-m}\exp(-(t_2-s))/(n_2-m)!ds\\
&= [(t_1)^m\exp(-t_2)/m!](t_2-t_1)^{n_2-m}/(n_2-m)!\\
&\qquad +\exp(-t_2)\int_0^{t_1}[s^m/m!](t_2-s)^{n_2-m-1}/(n_2-m-1)!ds\\
&= \exp(-t_2)\sum_{j=m}^{n_2}[(t_1)^j/j!](t_2-t_1)^{n_2-j}/(n_2-j)!.
\end{aligned}
$$

Q.E.D.

Finally we prove Lemma 3.6.

(Proof of Lemma 3.6). We first consider the case $n = 1$. Fix $R > r + \lambda + R_0$ sufficiently large so that the conditions (R.1)-(R.3) hold, and that the following holds: there exists $r_1 \in (R_0, R - \lambda)$ such that for sufficiently small $\eta \in (0, (\lambda - \delta)/2)$,

$$\lim_{\varepsilon \to 0} P(|X_1^\varepsilon(S_\eta^\varepsilon(x), x)| = \eta, |X^\varepsilon(S_\eta^\varepsilon(x), x)| < r_1) = 1, \tag{3.47}$$

uniformly in $x \in U_{R-\lambda/2}(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ (see Theorem 5.2[11]).

In this proof, $S_\eta^\varepsilon(x)$ and $M(\eta, \delta)$ are defined, for $R(> r + \lambda)$ fixed at the beginning of this proof, by (3.15) and (3.14), respectively.

We only have to prove (3.39) for $x \in U_{R-\lambda/2}(o) \cap \{y; |y_1| \geq \lambda - \delta\}$ since $R - \lambda/2 > r$.

Put, for $\eta \in (0, (\lambda - \delta)/2)$ chosen in (3.47),

$$r_2 = \sup\{|X^0(s, y)|; |X_1^0(s, y)| \geq (\lambda - \delta)/2, 0 \leq s \leq M(\eta, \delta), |y_1| = \eta, |y| < r_1\}. \tag{3.48}$$

Then $r_2 < r_1$, from the condition (R.3) with $R = r_1$. For $x \in U_{R-\lambda/2}(o) \cap \{y; |y_1| \geq \lambda - \delta\}$,

$$\begin{aligned}
&P(|X^\varepsilon(\tau_1^\varepsilon(x; \delta), x)| \geq \min(\lambda - \delta, \delta)/2 + r_2) \\
&\leq P(|X_1^\varepsilon(S_\eta^\varepsilon(x), x)| \neq \eta, \text{ or } |X^\varepsilon(S_\eta^\varepsilon(x), x)| \geq r_1) \\
&\quad + P(|X_1^\varepsilon(S_\eta^\varepsilon(x), x)| = \eta, |X^\varepsilon(S_\eta^\varepsilon(x), x)| < r_1, \\
&\quad \sup_{0 \leq t \leq M(\eta, \delta)} |X^\varepsilon(t + S_\eta^\varepsilon(x), x) - X^0(t, X^\varepsilon(S_\eta^\varepsilon(x), x))| \geq \min(\lambda - \delta, \delta)/2).
\end{aligned} \tag{3.49}$$

This is true. In fact, if $|X_1^\varepsilon(S_\eta^\varepsilon(x), x)| = \eta$, $|X^\varepsilon(S_\eta^\varepsilon(x), x)| < r_1 (< R)$, and if

$$\sup_{0 \leq t \leq M(\eta, \delta)} |X^\varepsilon(t + S_\eta^\varepsilon(x), x) - X^0(t, X^\varepsilon(S_\eta^\varepsilon(x), x))| < \min(\lambda - \delta, \delta)/2,$$

then $\tau_1^\varepsilon(x; \delta) \leq S_\eta^\varepsilon(x) + M(\eta, \delta)$, and

$$\begin{aligned}
&|X_1^0(\tau_1^\varepsilon(x; \delta) - S_\eta^\varepsilon(x), X^\varepsilon(S_\eta^\varepsilon(x), x))| \\
&\geq |X_1^\varepsilon(\tau_1^\varepsilon(x; \delta), x)| - |X_1^\varepsilon(\tau_1^\varepsilon(x; \delta), x) - X_1^0(\tau_1^\varepsilon(x; \delta) - S_\eta^\varepsilon(x), X^\varepsilon(S_\eta^\varepsilon(x), x))| \\
&> (\lambda - \delta)/2
\end{aligned}$$

(see (3.1)), from which we get

$$
\begin{aligned}
&|X^\varepsilon(\tau_1^\varepsilon(x;\delta),x)| \\
&\le |X^\varepsilon(\tau_1^\varepsilon(x;\delta),x) - X^0(\tau_1^\varepsilon(x;\delta) - S_\eta^\varepsilon(x), X^\varepsilon(S_\eta^\varepsilon(x),x))| \\
&\qquad + |X^0(\tau_1^\varepsilon(x;\delta) - S_\eta^\varepsilon(x), X^\varepsilon(S_\eta^\varepsilon(x),x))| < \min(\lambda-\delta,\delta)/2 + r_2.
\end{aligned}
$$

The first probability on the right hand side of (3.49) converges, as $\varepsilon \to 0$, to zero, uniformly in $x \in U_{R-\lambda/2}(o) \cap \{y; |y_1| \ge \lambda - \delta\}$ from (3.47). The second probability on the right hand side of (3.49) converges, as $\varepsilon \to 0$, to zero[11], uniformly in $x \in U_{R-\lambda/2}(o) \cap \{y; |y_1| \ge \lambda - \delta\}$ since

$$
\begin{aligned}
&P(|X_1^\varepsilon(S_\eta^\varepsilon(x),x)| = \eta, |X^\varepsilon(S_\eta^\varepsilon(x),x)| < r_1, \\
&\qquad \sup_{0\le t\le M(\eta,\delta)} |X^\varepsilon(t+S_\eta^\varepsilon(x),x) - X^0(t, X^\varepsilon(S_\eta^\varepsilon(x),x))| \ge \min(\lambda-\delta,\delta)/2) \\
&\le \sup_{|y_1|=\eta, |y|<r_1} P(\sup_{0\le t\le M(\eta,\delta)} |X^\varepsilon(t,y) - X^0(t,y)| \ge \min(\lambda-\delta,\delta)/2) \\
&\to 0,
\end{aligned}
$$

as $\varepsilon \to 0$[11].

Since $\min(\lambda-\delta,\delta)/2 + r_2 < R - \lambda/2$, we can prove this lemma inductively by the strong Markov property of $X^\varepsilon(t,x)$ (see (3.49)).

Q. E. D.

3. Proof of Theorem 3.3.

In this subsection we prove Theorem 3.3.

(Proof of (I)). For $\gamma > 0$ and $t > 0$ take $N_{\gamma,t} \in \mathbf{N}$ sufficiently large so that

$$
\sum_{j=N_{\gamma,t}}^\infty t^j \exp(-t)/j! \le \gamma/4. \tag{3.50}
$$

Then from Theorem 3.6,

$$
P(Y^\varepsilon(t;x,\delta) \ge N_{\gamma,t}) = 1 - P(Y^\varepsilon(t;x,\delta) \le N_{\gamma,t} - 1) \le 2 \sum_{j=N_{\gamma,t}}^\infty t^j \exp(-t)/j! < \gamma/2,
$$

for sufficiently small $\varepsilon > 0$, uniformly in $x \in A_{r,\delta}$.

Q. E. D.

Next we prove (II).

(Proof of (II) (see p. 134, Lemma 8.2[21])).

For $\gamma > 0$ and $T > 0$, take $N_{\gamma,T}$ which satisfies (3.50) with $t = T$, and take $\tilde{r} \in (0, \gamma/(4N_{\gamma,T}))$. Then

$$
\begin{aligned}
&P(\omega'(Y^\varepsilon(\cdot; x, \delta), \tilde{r}, T) \geq \gamma) \qquad (3.51)\\
&\leq P(\min_{0 \leq k \leq Y^\varepsilon(T; x,\delta)} (\tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta)) < \beta^\varepsilon \tilde{r})\\
&\leq P(Y^\varepsilon(T; x, \delta) \geq N_{\gamma,T})\\
&\quad + P(Y^\varepsilon(T; x, \delta) < N_{\gamma,T}, \min_{0 \leq k \leq Y^\varepsilon(T; x,\delta)} (\tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta)) < \beta^\varepsilon \tilde{r})\\
&\leq P(Y^\varepsilon(T; x, \delta) \geq N_{\gamma,T}) + N_{\gamma,T} \max_{k=0}^{N_{\gamma,T}-1} P(\tau^\varepsilon_k(x;\delta) \leq \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta) < \beta^\varepsilon \tilde{r}).
\end{aligned}
$$

The first probability in the last part of (3.51) is less than $\gamma/2$, for sufficiently small $\varepsilon > 0$, uniformly in $x \in A_{r,\delta}$, in the same way as in the proof of (I).

The second probability in the last part of (3.51) is transformed as follows: for $k = 0, \cdots, N_{\gamma,T} - 1$,

$$
\begin{aligned}
&P(\tau^\varepsilon_k(x;\delta) \leq \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta) < \beta^\varepsilon \tilde{r}) \qquad (3.52)\\
&= P(\tau^\varepsilon_k(x;\delta) \leq \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta) < \beta^\varepsilon \tilde{r}, X^\varepsilon(\tau^\varepsilon_k(x;\delta), x) \notin A_{r_0,\delta})\\
&\quad + P(\tau^\varepsilon_k(x;\delta) \leq \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta) < \beta^\varepsilon \tilde{r}, X^\varepsilon(\tau^\varepsilon_k(x;\delta), x) \in A_{r_0,\delta})
\end{aligned}
$$

The first probability on the right hand side of (3.52) converges, as $\varepsilon \to 0$, to zero, uniformly in $x \in A_{r,\delta}$ from Lemma 3.6.

The second probability on the right hand side of (3.52) is estimated as follows:

$$
\begin{aligned}
&P(\tau^\varepsilon_k(x;\delta) \leq \beta^\varepsilon T, \tau^\varepsilon_{k+1}(x;\delta) - \tau^\varepsilon_k(x;\delta) < \beta^\varepsilon \tilde{r}, X^\varepsilon(\tau^\varepsilon_k(x;\delta), x) \in A_{r_0,\delta})\\
&\leq \sup_{y \in A_{r_0,\delta}} P(Y^\varepsilon(\tilde{r}; y, \delta) \geq 1) \quad \text{(by the strong Markov property of } X^\varepsilon(t,x))\\
&\leq 2(1 - \exp(-\tilde{r})) \leq 2\tilde{r} < \gamma/(2N_{\gamma,T}),
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$ from Theorem 3.6.

Q. E. D.

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