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Slice Maps And Multipliers Of Invariant Subspaces

by

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Abstract. Let $\overline{D^2}$ be the closed bidisc and T^2 be its distinguished boundary. For $(\alpha, \beta) \in \overline{D^2}$, let $\Phi_{\alpha\beta}$ be a slice map, that is, $(\Phi_{\alpha\beta} f)(\lambda) = f(\alpha\lambda, \beta\lambda)$ for $\lambda \in D$ and $f \in H^2(D^2)$. Then $ker\Phi_{\alpha\beta}$ is an invariant subspace, and it is not difficult to describe $ker\Phi_{\alpha\beta}$ and $\mathcal{M}(ker\Phi_{\alpha\beta}) = \{\phi \in L^\infty(T^2) : \phi ker\Phi_{\alpha\beta} \subset H^2(D^2)\}$. In this paper, we study the set $\mathcal{M}(M)$ of all multipliers for an invariant subspace M such that the common zero set of M contains that of $ker\Phi_{\alpha\beta}$.

§1 Introduction

Let D^2 be the open unit disc in $\mathbf{C}^2$ and T^2 be its distinguished boundary. Normalized Lebesgue measure on T^2 is denoted by dm. For $1 \le p < \infty$, $H^p(D^2)$ is the Hardy space and $L^p(T^2)$ is the Lebesgue space on T^2. Let $N(D^2)$ denote the Nevanlinna class. Each f in $N(D^2)$ has radial limits f^* defined on T^2 *a.e.* Moreover, there is a singular measure $d\sigma_f$ on T^2 determined by f such that the least harmonic majorant $u(log \mid f \mid)$ of $log \mid f \mid$ is given by $u(log \mid f \mid)(\zeta) = P_\zeta(log \mid f^* \mid + d\sigma_f)$ where P_ζ denotes Poisson integration and $\zeta = (z, w) \in D^2$. Put $N_*(D^2) = \{f \in N(D^2)\ ;\ d\sigma_f \le 0\}$, then $H^p(D^2) \subset N_*(D^2) \subset N(D^2)$ and $H^p(D^2) = N_*(D^2) \cap L^p(T^2) \subset N(D^2) \cap L^p(T^2)$. These facts are shown in [6, Theorem 3.3.5].

A closed subspace M of $H^2(D^2)$ is said to be invariant if $zM \subset M$ and $wM \subset M$. For an invariant subspace M of $H^2(D^2)$, set

$$\mathcal{M}(M) = \{\phi \in L^\infty(T^2)\ ;\ \phi M \subseteq H^2(D^2)\}.$$

If $M = qH^2(D^2)$ for some inner function q, it is trivial to see $\mathcal{M}(M) = \bar{q}H^\infty(D^2)$. In the case of one variable, an arbitrary invariant subspace M has the form $qH^2(D)$ for some inner function q by the famous Beurling theorem [1]. Hence $\mathcal{M}(M) = \bar{q}H^\infty(D)$. Hence the map : $M \to \mathcal{M}(M)$ is one-to-one. However this result for invariant subspaces of $H^2(D^2)$ is not true. The author [4] studied the relation between M and $\mathcal{M}(M)$. To study $\mathcal{M}(M)$, D.Douglas and K.Yan [2] introduced the common zero set $\mathcal{Z}(M)$ and the singular measure $\mathcal{Z}_\partial(M)$, that is,

$$\mathcal{Z}(M) = \{\zeta \in D^2\ ;\ f(\zeta) = 0 \text{ for } f \in M\}$$

and

$$\mathcal{Z}_\partial(M) = inf\{-d\sigma_f\ ;\ f \in M,\ f \ne 0\}.$$

They showed that if the real 2-dimensional Hausdorff measure of $\mathcal{Z}(M)$ is zero and $\mathcal{Z}_\partial(M) \doteq 0$, then $\mathcal{M}(M) = H^\infty(D^2)$. In this paper, we are interested in invariant subspaces M of $H^2(D^2)$ such that the real 2-dimensional Hausdorff measure of $\mathcal{Z}(M)$ is positive and $\mathcal{Z}_\partial(M) = 0$.

Fix $(\alpha, \beta) \in \overline{D^2}$. For f in $H^p(D^2)$,

$$(\Phi^p_{\alpha\beta} f)(\lambda) = f(\alpha\lambda, \beta\lambda) \qquad (\lambda \in D).$$

$\Phi^p_{\alpha\beta}$ is called a slice map. $\Phi^2_{\alpha\beta}$ maps $H^2(D^2)$ into $L^2_a(D)$, where $L^2_a(D)$ is the Bergman space (*cf.*[p53]). In this paper, we study the kernel $ker\Phi^p_{\alpha\beta}$ and the range $ran\Phi^p_{\alpha\beta}$ for $p = 2, \infty$. $ker\Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$ and the closure of $ran\Phi^2_{\alpha\beta}$ is an invariant subspace of $L^2_a(D)$. Put

$$\mathcal{D}_{\alpha\beta} = \{(\alpha\lambda, \beta\lambda) \in D^2\ ;\ \lambda \in D\},$$

then $\mathcal{Z}(ker\Phi^2_{\alpha\beta}) = \mathcal{D}_{\alpha\beta}$ if $(\alpha,\beta) \in T^2 \cup T \times D \cup D \times T$. The 2-dimensional Hausdorff measure of $\mathcal{Z}(ker\Phi^2_{\alpha\beta})$ is positive and $\mathcal{Z}_\partial(ker\Phi^2_{\alpha\beta}) = 0$. In this paper, we show $\mathcal{M}(M) = H^\infty(D^2)$ when $\mathcal{Z}(M) = \mathcal{D}_{\alpha\beta}$ for some $(\alpha,\beta) \in T^2$ and $\mathcal{Z}_\partial(M) = 0$ and M satisfies some additional natural condition. The main result in this paper is Theorem 4 in Section 3. Theorem 1 of [2] has a lot of corollaries on the rigidity of invariant subspaces. Theorem 3 in this paper has similarly such corollaries. Hence our results can be seen as the generalizations of results of R.G.Douglas and K.Yan.

For f in $N(D^2), f(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f and F_j is a polynomial which is homogeneous of degree j. The smallest $j = j(f)$ such that F_j is not the zero-polynomial is called the order of the zero which f has at $\zeta = (0,0)$. For $p \in D^2$, the order of the zero of f at p is simply the order of the zero of $f(p+\zeta)$ at $\zeta = (0,0)$. We will write $f_p(\zeta) = f(p+\zeta)$.

§2. Slice maps

In this section, we study the slice map $\Phi_{\alpha\beta} = \Phi^p_{\alpha\beta}$ for $(\alpha,\beta) \in \overline{D^2}$.

Proposition 1. Let $(\alpha,\beta) \in \overline{D^2}$.

(1) $\Phi^2_{\alpha\beta}$ is a contractive map from $H^2(D^2)$ to $L^2_a(D)$.

(2) If $(\alpha,\beta) \in D^2$, then $ran\Phi^2_{\alpha\beta}$ is a subset of analytic functions on $\bar{D}$.

(3) If $(\alpha,\beta) \in T^2$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $L^2_a(D)$ with $\|\Phi^2_{\alpha\beta}\| = 1$.

(4) If $(\alpha,\beta) \in T \times D \cup D \times T$, then $\Phi^2_{\alpha\beta}$ is an onto map from $H^2(D^2)$ to $H^2(D)$ with $\|\Phi^2_{\alpha\beta}\| \leq (1-|\beta|^2)^{-1}$.

Proof. (1) For $f \in H^2(D^2)$, let $f(z,w) = \sum_{j=0}^\infty F_j(z,w)$ be a homogeneous expansion of f. Then $F_j(z,w) = \sum_{\ell=0}^j a_\ell z^{j-\ell}w^\ell$ and $\int | F_j |^2 \, dm = \sum_{\ell=0}^j | a_\ell |^2$. Moreover

$$\int | f |^2 \, dm = \sum_{j=0}^\infty \int | F_j |^2 \, dm = \sum_{j=0}^\infty \sum_{\ell=0}^j | a_\ell |^2 < \infty.$$

$(\Phi_{\alpha\beta}f)(\lambda) = \sum_{j=0}^\infty F_j(\alpha,\beta)\lambda^j$ and

$$| F_j(\alpha,\beta) |^2 \leq (\sum_{\ell=0}^j | a_\ell |^2)(\sum_{\ell=0}^j | \beta |^{2\ell}) \leq (j+1)(\sum_{\ell=0}^j | a_\ell |^2).$$

Hence

$$\int_0^1 \int_0^{2\pi} | \Phi_{\alpha\beta} f |^2 \, (re^{i\theta}) r d\theta dr/\pi$$

$$= \int_0^1 \sum_{j=0}^{\infty} | F_j(\alpha,\beta) |^2 r^{2j+1} 2dr = \sum_{j=0}^{\infty} | F_j(\alpha,\beta) |^2 \frac{1}{j+1}$$

$$\leq \sum_{j=0}^{\infty}\sum_{\ell=0}^{j} | a_\ell |^2 = \int | f |^2 dm.$$

Thus $\Phi_{\alpha\beta} f \in L_a^2(D)$ and $\|\Phi_{\alpha\beta}\| \leq 1$.

(2) is clear. (3). For $g \in L_a^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} \frac{b_j}{j+1} (\bar{\beta} w)^{j-\ell} (\bar{\alpha} z)^\ell$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (3). (4). We may assume $(\alpha,\beta) \in T \times D$. Then

$$| F_j(\alpha,\beta) |^2 \leq (1 - | \beta |^2)^{-1} \textstyle\sum_{\ell=0}^{j} | a_\ell |^2$$

and hence

$$\int_0^{2\pi} | \Phi_{\alpha\beta} f |^2 (re^{i\theta}) d\theta / 2\pi$$

$$\leq \sum_{j=0}^{\infty} \frac{1}{1 - | \beta |^2} \sum_{\ell=0}^{j} | a_\ell |^2 \leq \frac{1}{1 - | \beta |^2} \int | f |^2 dm.$$

For $g \in H^2(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$, put $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j$. Then $f \in H^2(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This implies (4).

(3) of Proposition 1 is essentially known (see [6, p53]). Now we study the slice map $\Phi_{\alpha\beta}^{\infty}$ on $H^\infty(D^2)$. Let L be the norm closed linear span of $\overline{H^\infty(D^2)}\, H^\infty(D^2)$ in $L^\infty(T^2)$. Then $L \neq L^\infty(T^2)$ (see[5]).

Proposition 2. Let $(\alpha,\beta) \in \overline{D^2}$.

(1) $\Phi_{\alpha\beta}^{\infty}$ is a contractive homomorphism from $H^\infty(D^2)$ to $H^\infty(D)$.

(2) If $(\alpha,\beta) \in T^2 \cup T \times D \cup D \times T$, then $\Phi_{\alpha\beta}^{\infty}$ is a contractive homomorphism from $H^\infty(D^2)$ onto $H^\infty(D)$.

(3) If $(\alpha,\beta) \in T^2$, there exists a contractive $*$- homomorphism $\tilde{\Phi}_{\alpha\beta}^{\infty}$ from L onto $L^\infty(T)$ such that $\tilde{\Phi}_{\alpha\beta}^{\infty} \mid H^\infty(T^2) = \Phi_{\alpha\beta}^{\infty} \mid H^\infty(T^2)$.

Proof. (1) is clear. (2). If $g \in H^\infty(D)$ with $g(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^j$ and $| \alpha | = 1$, then $f(z,w) = \sum_{j=0}^{\infty} b_j (\bar{\alpha} z)^j \in H^\infty(D^2)$ and $(\Phi_{\alpha\beta} f)(\lambda) = g(\lambda)$. This and (1) imply (2).

(3) For $f_j, g_j \in H^\infty(D^2)$ and $j = 1, \cdots, n$, put

$$\{\tilde{\Phi}_{\alpha\beta}(\sum_{j=1}^{n} f_j \bar{g}_j)\}(\lambda) = \sum_{j=1}^{n} f_j(\alpha\lambda, \beta\lambda) \overline{g_j(\alpha\lambda, \beta\lambda)}$$

for $\lambda \in D$, then $\tilde{\Phi}_{\alpha\beta}(\sum_{j=1}^{n} f_j \bar{g}_j)$ can be seen as an element in $L^\infty(T)$ by its radial limits. Hence for $a.e.\ \lambda \in T$

$$
\begin{aligned}
&| \Phi_{\alpha\beta}(\sum_{j=1}^{n} f_j\bar{g}_j)(\lambda) | \\
&\leq ess \sup_{\lambda\in T} | \sum_{j=1}^{n} f_j(\alpha\lambda,\beta\lambda)\overline{g_j(\alpha\lambda,\beta\lambda)} | \\
&\leq ess \sup_{(z,w)\in T^2} | \sum_{j=1}^{n} f_j(\alpha z,\beta w)\overline{g_j(\alpha z,\beta w)} | = \|\sum_{j=1}^{n} f_j\bar{g}_j\|_\infty
\end{aligned}
$$

because $(\alpha,\beta) \in T^2$. Then $\tilde{\Phi}_{\alpha\beta}$ is the extension of $\Phi_{\alpha\beta}$ from $H^\infty(D^2)$ to L, then $\tilde{\Phi}_{\alpha\beta}$ is a contractive $*$- homomorphism from L to $L^\infty(T)$. If $U(\lambda) = \sum_{j=1}^{n} F_j(\lambda)\bar{G}_j(\lambda)$ *a.e.* on T where $F_j, G_j \in H^\infty(D)$, then $u(z,w) = \sum_{j=1}^{n} F_j(\bar{\alpha}z)\bar{G}_j(\bar{\beta}w)$ belongs to L and $(\tilde{\Phi}_{\alpha\beta}u)(\lambda) = U(\lambda)$ *a.e.* on T. Since arbitrary function U in $L^\infty(T)$ can be approximated by such functions, $\tilde{\Phi}_{\alpha\beta}$ is onto.

The following lemma will be used in the proofs in the following proposition and the main theorem. We can prove it by an approximation method as in [4] but we prove it using Proposition 2.

Lemma. If $\phi \in L^\infty(T^2), (\alpha,\beta) \in T^2$ and $\phi(z,w)(\beta z - \alpha w) \in H^\infty(D^2)$, then $\phi \in H^\infty(D^2)$.

Proof. Note that $\beta z - \alpha w \in ker\Phi_{\alpha\beta}$. If $\phi(\beta z - \alpha w) = g$ for some $g \in H^\infty(D^2)$, then g belongs to $ker\Phi_{\alpha\beta}$. In fact, $\hat{\phi}(\beta z - \alpha w)^\wedge = \hat{g}$ on $Spec\ L^\infty(T^2)$ which is the maximal ideal space of $L^\infty(T^2)$ and $(\beta z - \alpha w)^\wedge = 0$ on $hull(ker\tilde{\Phi}_{\alpha\beta})$. Hence $\hat{g} = 0$ on $hull(\ker\tilde{\Phi}_{\alpha\beta}) \cap SpecL^\infty(T^2)$. Since L is a commutative C^*-algebra, every element of $SpecL$ extends to an element of $SpecL^\infty(T^2)$. Therefore $\hat{g} = 0$ on $hull(\ker\tilde{\Phi}_{\alpha\beta})$. Thus $g \in (\ker\tilde{\Phi}_{\alpha\beta}) \cap H^\infty(D^2) = \ker\Phi_{\alpha\beta}$. Hence if $g = \sum_{j=0}^{\infty} G_j$ and $G_j(z,w) = \sum_{\ell=0}^{j} b_\ell z^{j-\ell}w^\ell$, then

$$
G_j(z,w) = z^j\sum_{\ell=0}^{j} b_\ell(\bar{z}w)^\ell = k\prod_{\ell=1}^{j}(w - k_\ell z)
$$

where $k \in \mathbf{C}$ and $k_\ell \in \mathbf{C}$ for $1 \leq \ell \leq j$ and $G_j(\alpha\lambda,\beta\lambda) \equiv 0$ for $\lambda \in D$ because $g \in \ker\Phi^2_{\alpha\beta}$. Thus $G_j(z,w) = m(\beta z - \alpha w)\,\Pi_{\ell=2}^{j}(w - m_\ell z)$ where $m \in \mathbf{C}$ and $m_\ell \in \mathbf{C}$ for $2 \leq \ell \leq j$ and hence $g/(\beta z - \alpha w)$ is analytic on D^2. Since $d\sigma_{\beta z-\alpha w} = 0$, $g/(\beta z - \alpha w) \in N_*(D^2) \cap L^\infty(T^2) = H^\infty(D^2)$ and hence ϕ belongs to $H^\infty(D^2)$.

Proposition 3. Let $(\alpha,\beta) \in \overline{D^2}$.
(1) For any $r \in (0,1], ker\Phi^2_{\alpha\beta} = ker\Phi^2_{r\alpha,r\beta}$.
(2) $ker\Phi^2_{\alpha\beta}$ is an invariant subspace of $H^2(D^2)$,

$$
\mathcal{Z}(ker\Phi^2_{\alpha\beta}) = \mathcal{D}_{\alpha\beta} \text{ and } \mathcal{Z}_\partial(ker\Phi^2_{\alpha\beta}) = 0.
$$

For any $p \in \mathcal{D}_{\alpha\beta}$, $\beta z - \alpha w \in ker\Phi^2_{\alpha\beta}$ has a zero of order 1 at p.

(3) If $(\alpha, \beta) \in T^2$, then $(\beta z - \alpha w)H^2(D^2)$ is dense in $ker\Phi^2_{\alpha\beta}$ but $ker\Phi^2_{\alpha\beta} \neq (\beta z - \alpha w)H^2(D^2)$. If $(\alpha, \beta) \in T \times D \cup D \times T$, then $ker\Phi^2_{\alpha\beta} = (\beta z - \alpha w)H^2(D^2)$.

(4) If $(\alpha, \beta) \in T^2$, then $\mathcal{M}(ker\Phi^2_{\alpha\beta}) = H^\infty(D^2)$ and if $(\alpha, \beta) \in T \times D \cup D \times T$, then $\mathcal{M}(ker\Phi^2_{\alpha\beta}) = (\beta z - \alpha w)^{-1}H^\infty(D^2)$.

(5) If $(\alpha, 0) \in \bar{D} \times D$ and $\alpha \neq 0$, then $ker\Phi^2_{\alpha 0} = wH^2(D^2)$ and hence $\mathcal{M}(ker\Phi^2_{\alpha 0}) = w^{-1}H^\infty(D^2)$.

(6) Let M be an invariant subspace of $H^2(D^2)$ with $ker\Phi^2_{\alpha\beta} \subsetneq M, \mathcal{M}(M) = H^\infty(D^2)$. If $(\alpha, \beta) \in T^2$, then $\mathcal{Z}(M) = \{(\alpha a_j, \beta a_j) \in D^2 \ ; \ \sum_{j=1}^\infty (1- \mid a_j \mid) \times [-log(1- \mid a_j \mid)]^{1-\varepsilon} < \infty$ for all $\varepsilon > 0\}$. If $(\alpha, \beta) \in T \times D \cup D \times T$, then $\mathcal{Z}(M) = \{(\alpha a_j, \beta a_j) \in D^2 \ ; \ \sum_{j=1}^\infty (1- \mid a_j \mid) < \infty\}$. If $(\alpha, 0) \in \bar{D} \times D$ and $\alpha \neq 0$, then $M = qH^2(D) \oplus wH^2(D^2)$ where q is a one variable inner function with $q = q(z)$ and hence $\mathcal{Z}(M) = \{(s, 0) \in D^2 \ ; \ q(s) = 0$ and $s \in D\}$.

Proof. (1) and (2) are clear. (3). Let $(\alpha, \beta) \in T^2$.If $f \in ker\Phi_{\alpha\beta}, f = \sum_{j=0}^\infty F_j$ and $F_j(z, w) = \sum_{\ell=0}^j a_\ell z^{j-\ell}w^\ell$, then $F_j(z, w) = c(\beta z - \alpha w)\prod_{\ell=2}^j (w - c_\ell z)$ and hence f can be approximated by the functions in $(\beta z - \alpha w)H^2(D^2)$. This implies that $(\beta z - \alpha w)H^2(D^2)$ is dense in $ker\Phi^2_{\alpha\beta}$. Suppose $ker\Phi_{\alpha\beta} = (\beta z - \alpha w)H^2(D^2)$, then the multiplication operator by $\beta z - \alpha w$ is a left invertible operator from $H^2(D^2)$ to $ker\Phi_{\alpha\beta}$. Hence there exists a positive constant ε such that

$$\int_{T^2} \mid g \mid^2 \mid \beta z - \alpha w \mid^2 dm \geq \varepsilon \int_{T^2} \mid g \mid^2 dm$$

for all $g \in H^\infty(D^2)$ and so

$$\int_{T^2} u \mid \beta z - \alpha w \mid^2 dm \geq \varepsilon \int_{T^2} u dm$$

for all nonnegative continuous functions u on T^2. Thus $\mid \beta z - \alpha w \mid^2 \geq \varepsilon > 0$ *a.e.* on T^2 and this contradiction implies that $ker\Phi_{\alpha\beta} \neq (\beta z - \alpha w)H^2(D^2)$. Let $(\alpha, \beta) \in T \times D \cup D \times T$. Since $\beta z - \alpha w$ is invertible in L^∞, $ker\Phi_{\alpha\beta} = (\beta z - \alpha w)H^2(D^2)$ because $(\beta z - \alpha w)H^2(D^2)$ is dense in $ker\Phi_{\alpha\beta}$.

(4) Let $(\alpha, \beta) \in T^2$. If $\phi \in \mathcal{M}(ker\Phi_{\alpha\beta})$, then $\phi(\beta z - \alpha w) = g$ for some $g \in H^\infty(D^2)$. By Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(ker\Phi_{\alpha\beta}) = H^\infty(D^2)$. Let $(\alpha, \beta) \in T \times D \cup D \times T$ and $\phi \in \mathcal{M}(ker\Phi_{\alpha\beta})$. By (3), $ker\Phi_{\alpha\beta} = (\beta z - \alpha w)H^2(D^2)$ and hence

$$\phi(\beta z - \alpha w)H^2(D^2) \subset H^2(D^2).$$

This implies that $\phi(\beta z - \alpha w) \in H^\infty(D^2)$ and so $\mathcal{M}(ker\Phi_{\alpha\beta}) = (\beta z - \alpha w)^{-1}H^\infty(D^2)$. (5) is easy to see. (6) If $(\alpha, \beta) \in T^2$ and $ker\Phi_{\alpha\beta} \subsetneq M$, then by (3) of Proposition 1 and [3, Corollary 3.6], $\mathcal{Z}([\Phi_{\alpha\beta}M]_2) = \{a_j \in D \ : \ \sum_{j=1}^\infty (1- \mid a_j \mid)[-log(1- \mid a_j \mid)]^{1-\varepsilon} < \infty$ for all $\varepsilon > 0\}$. This implies the first part. For $(\alpha, \beta) \in T \times D \cup D \times T$, we can show similarly by (4) of Proposition 1.

§3. Multipliers

By (4) of Proposition 3, we know the set of all multipliers $\mathcal{M}(M)$ of an invariant subspace such that $ker\Phi^2_{\alpha\beta} \subseteq M \subseteq H^2(D^2)$ when $(\alpha, \beta) \in T^2 \cup T \times D \cup D \times T$ or $(\alpha, 0) \in D \times D \setminus (0,0)$. When $(\alpha, \beta) \in D \times D$ and $|\alpha| = |\beta| > 0$, there exists $(\alpha_0, \beta_0) \in T \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. When $(\alpha, \beta) \in D \times D$ and $0 \leq |\alpha| < |\beta|$, there exists $(\alpha_0, \beta_0) \in D \times T$ such that $\alpha = r\alpha_0$ and $\beta = r\beta_0$ for some $r \in (0,1)$. (1) of Proposition 3 implies $ker\Phi^2_{\alpha\beta} = ker\Phi^2_{\alpha_0\beta_0}$. Hence for arbitrary $(\alpha, \beta) \in \bar{D} \times \bar{D} \setminus (0,0)$, we can describe $\mathcal{M}(M)$ by Proposetion 3. In this section, we study $\mathcal{M}(M)$ without such a condition. In this section, for example, we study $\mathcal{M}(M)$ of an invariant subspace such that $M \subseteq ker\Phi^2_{\alpha\beta}$. In fact, we study such a problem more generally, that is, when the 2-dimensional Hausdorff measure of $\mathcal{Z}(M) \cap \mathcal{D}^c_{\alpha\beta}$ is zero. For $\Lambda \subset T^2 \cup T \times D \cup D \times T$, put

$$\mathcal{D}_\Lambda = \{\cup\, \mathcal{D}_{\alpha\beta}\ ;\ (\alpha, \beta) \in \Lambda\} \setminus \{(0,0)\}.$$

Note that if $\mathcal{Z}(M) \supseteq \mathcal{D}_\Lambda$ and Λ is an infinite set such that $\mathcal{D}_{\alpha\beta} \cap \mathcal{D}_{\gamma\delta} = \{(0,0)\}$ when $(\alpha, \beta) \neq (\gamma, \delta)$, then $M = \{0\}$.

Theorem 4. Let Λ be a finite set of T^2. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1) $\sim$ (3), then $\mathcal{M}(M) = H^\infty(D^2)$.

(1) For any $p \in \mathcal{Z}(M) \cap \mathcal{D}_\Lambda$, there exists a function f in M such that f has a zero of order 1 at p.

(2) The 2-dimensional Hausdorff measure of $\mathcal{Z}(M) \cap \mathcal{D}^c_\Lambda$ is zero.

(3) $\mathcal{Z}_\partial(M) = 0$.

Proof. Suppose $\phi \in \mathcal{M}(M)$. Fix $p \in \mathcal{Z}(M) \cap \mathcal{D}_\Lambda$. By (1), let f be a function in M such that f has a zero of order 1 at p. Let $(\alpha, \beta) \in \Lambda$ with $p \in \mathcal{D}_{\alpha\beta}$. By definition of $\mathcal{M}(M), \phi f = g$ for some $g \in H^2(D^2)$. Put $k(z,w) = \beta z - \alpha w$, then $k_p(\zeta) = k(\zeta + p) = k(\zeta)$ and $k_p(\zeta)\phi_p(\zeta)f_p(\zeta) = k(\zeta)g_p(\zeta)$. Suppose $f_p(\zeta) = \sum_{j=0}^\infty F_j(\zeta)$ is a homogeneous expansion of f_p. Since $1 = s(f_p), F_1(0,w) = cw$ for $c \neq 0$. By the Weierstrass preparation theorem (*cf* [6, Theorem 1.2.1]), there exists a polydisc Δ in $\mathbf{C}^2$, centered at $(0,0)$, such that

$$f_p(z,w) = W(z,w)h(z,w)$$

for $(z,w) \in \Delta$ where h is analytic in Δ, h has no zero in $\Delta, W(z,w) = w + b_0(z)$ and b_0 is analytic in Δ with $b_0(0) = 0$. Since $f_p(\alpha\lambda, \beta\lambda) \equiv 0$ on $D, \beta\lambda + b_0(\alpha\lambda) = 0$ if $(\alpha\lambda, \beta\lambda) \in \Delta$ and hence $b_0(\alpha\lambda) = -\beta\lambda$. Thus $b_0(z) = -\frac{\beta}{\alpha}z$ and $W(z,w) = -\frac{1}{\alpha}(\beta z - \alpha w)$. Therefore $k_p\phi_p = k\ g_p/f_p$ is analytic in Δ and so $k\phi$ is analytic in $\Delta + p$, in a sence of D.Douglas and K.Yan [2]. Therefore

$$\prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)\phi(z,w)$$

is analytic in a neighborhood of $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda$.

If $p \notin \mathcal{Z}(M)$, then there exists a function k in M such that k has no zero in some polydisc Δ_p, centered at p. As in the proof above, $\phi(z, w)$ is analytic in Δ_p and hence ϕ is analytic in $D^2 \backslash \mathcal{Z}(M)$. Thus $\Pi(\beta z - \alpha w)\phi(z, w)$ is analytic in $D^2 \backslash \mathcal{Z}(M) \cap \mathcal{D}_\Lambda^c$. By (2), $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda^c$ is a removable singularity for analytic functions and hence $\psi(z, w) = \Pi(\beta z - \alpha w)\phi(z, w)$ is analytic in D^2. By the proof of [2, Theorem 1], $\psi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\psi \leq -\mathcal{Z}_\partial(M)$ because $d\sigma_\phi = d\sigma_\psi$. By (3), $d\sigma_\psi = 0$ and hence $\psi \in H^\infty(D^2)$. By Lemma, ϕ belongs to $H^\infty(D^2)$ and hence $\mathcal{M}(M) = H^\infty(D^2)$.

Theorem 5. Let Λ be a finite set of $T \times D \cup D \times T$. If M is an invariant subspace of $H^2(D^2)$ which $\mathcal{Z}(M) \supseteq \mathcal{D}_\Lambda$ and satisfies the following (1) $\sim$ (3), then

$$\mathcal{M}(M) = \prod_{(\alpha,\beta)\in\Lambda} (\beta z - \alpha w)^{-1} H^\infty(D^2).$$

(1) For any $p \in \mathcal{Z}(M)$, there exists a function f in M such that f has a zero of order 1 at p.

(2) The 2-dimensional Hausdorff measure of $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda^c$ is zero.

(3) $\mathcal{Z}_\partial(M) = 0$.

Proof. By the proof of Theorem 4, if $\phi \in \mathcal{M}(M)$, then $\Pi(\beta z - \alpha w)\phi(z, w) \in H^\infty(D^2)$ where (α, β) ranges over Λ. Hence $\phi \in \Pi(\beta z - \alpha w)^{-1} H^\infty(D^2)$. Conversely if $\phi \in \Pi(\beta z - \alpha w)^{-1} H^\infty(D^2)$ and $f \in M$, then $f = 0$ on $\mathcal{D}_\Lambda$, hence by the Weierstrass preparation theorem, $\Pi(\beta z - \alpha w)^{-1} f(z, w)$ is analytic in D^2 and ϕ belongs to $\mathcal{M}(M)$.

§4. Two general cases and remarks

Let a and b be two functions in $H^\infty(D)$ with $\|a\|_\infty \leq 1$ and $\|b\|_\infty \leq 1$. For f in $H^p(D^2)$,

$$(\Phi_{ab}^p f)(\lambda) = f(a(\lambda), b(\lambda)) \qquad (\lambda \in D).$$

If $a(\lambda) = \alpha\lambda$ and $b(\lambda) = \beta\lambda$, then Φ_{ab}^p was called a slice map $\Phi_{\alpha\beta}^p$ in the previous sections. For arbitrary pair a and b, we know only very trivial results. It is easy to see that Φ_{ab}^∞ maps $H^\infty(D^2)$ into $H^\infty(D)$. If $\|a\|_\infty < 1$ and $\|b\|_\infty < 1$, then Φ_{ab}^p maps $H^p(D^2)$ into $H^\infty(D)$. In general, $ker\Phi_{ab}^2$ is still an invariant subspace of $H^2(D^2)$, and

$$\mathcal{Z}(ker\Phi_{ab}^2) = \mathcal{D}_{ab} = \{(a(\lambda), b(\lambda)) \in D^2 \; ; \; \lambda \in D\}.$$

The function $b(z) - a(w)$ may not belong to $ker\Phi_{ab}^2$. If $a(\lambda) = \alpha\lambda$ and $b(\lambda) = \beta\lambda$, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$ and hence $b(z) - a(w)$ belongs to $ker\Phi_{ab}^2$. If $a(\lambda) = \lambda$ and $b(\lambda)$ is an inner function, then $(b \circ a)(\lambda) = (a \circ b)(\lambda)$ for $\lambda \in D$ and hence $b(z) - a(w) =$

$b(z) - w$ belongs to $ker\Phi^2_{ab}$. In this case, $\mathcal{Z}_\partial(ker\Phi^2_{ab}) = 0$. For any $p \in \mathcal{D}_{ab}, b(z) - w$ has a zero of order 1 at $p \in \mathcal{D}_{ab}$. If $\phi \in L^\infty(T^2)$ and $(b(z) - w)\phi(z, w) \in H^\infty(D^2)$, then $\phi \in H^\infty(D^2)$. This can be shown as in [4, Proposition 3 and Theorem 7]. This implies (4) of Proposition 3. The proof of the following theorem is almost parallel to that of Theorem 4.

Theorem 6. Let $a(\lambda) = \lambda$ and $b(\lambda)$ be an inner function. If M is an invariant subspace of $H^2(D^2)$ which satisfies the following (1) $\sim$ (3), then $\mathcal{M}(M) = H^\infty(D^2)$.

(1) For any $p \in \mathcal{Z}(M) \cap \mathcal{D}_{ab}$, there exists a function f in M such that f has a zero of order 1 at $p \in \mathcal{Z}(M) \cap \mathcal{D}_{ab}$.

(2) The 2-dimensional Hausdorff measure of $\mathcal{Z}(M) \cap \mathcal{D}^c_{ab}$ is zero.

(3) $\mathcal{Z}_\partial(M) = 0$.

If $a(\lambda) = \lambda$ and $b(\lambda) = cq(\lambda)$ where c is a constant with $| c |< 1$ and q is an inner function, we can show a version of Theorem 5 as Theorem 6 which is that of Theorem 4.

Let D^n be the open unit polydisc in $\mathbf{C}^n$ and T^n be its distinguished boundary. Fix $\alpha = (\alpha_1, \cdots, \alpha_n) \in \overline{D^n}$. For f in $H^p(D^n)$

$$(\Phi^p_\alpha f)(\lambda) = f(\alpha_1\lambda, \cdots, \alpha_n\lambda) \qquad (\lambda \in D).$$

(1), (2) and (3) of Proposition 1 can be proved for arbitrary n. If $\alpha_j \in T$ for some j with $1 \le j \le n$ and $\alpha_i \in D$ for all i with $1 \le i \le n$ and $i \ne j$, we can show that Φ^2_α is an onto map from $H^2(D^n)$ to $H^2(D)$ with $\|\Phi^2_\alpha\| \le \prod_{i \ne j}(1 - |\alpha_j|^2)^{-1}$. This is a generalization of (4) of Proposition 1. Similarly we can generalize Proposition 2. If $\phi \in L^\infty(T^n)$ and $(\alpha_i z_j - \alpha_j z_i)\phi(z_1, \cdots, z_n) \in H^\infty(D^n)$ where $1 \le i \ne j \le n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in T^n$, then $\phi \in H^\infty(D^n)$. This also can be shown as in [4, Proposition 3 and Theorem 7]. $ker\Phi^2_\alpha$ is an invariant subspace and a generalization of (1) and (2) of Proposition 3 is true. Suppose $n > 2$. If M is an invariant subspace of $H^2(D^n), \mathcal{Z}(M) = \mathcal{D}_\alpha = \{(\alpha_1\lambda, \cdots, \alpha_n\lambda) \; ; \; \lambda \in D\}$ for $\alpha \in T^n$ and $\mathcal{Z}_\partial(M) = 0$, then $\mathcal{M}(M) = H^\infty(D^n)$. For it is a result of R.G.Douglas and K.Yan [2, Theorem 1] because the real $2n-2$ dimensional Hausdorff measure of $\mathcal{Z}(M)$ is zero.

Remark. (i) As in Theorem 1 of [2], Theorem 4 can be stated as the following : If M is an invariant subspace of $H^2(D^2)$ which satisfies (1) and (2), then $\phi \in \mathcal{M}(M)$ if and only if $\phi \in N(D^2) \cap L^\infty(T^2)$ and $d\sigma_\phi \le \mathcal{Z}_\partial(M)$. (ii) By Lemma 7 in [2] and Theorem 4, if M and N are quasi-similar invariant subspaces of $H^2(D^2)$ and M satisfies (1) $\sim$ (3) in Theorem 4, then $M \subseteq N$. This is a generalization of Theorem 2 in [2]. Similarly we can generalize Corollaries 9 and 12. (iii) Let M, N be invariat subspaces of $H^2(D^2)$ satisfying (a) the 2-dimensional Hausdorff measures of $\mathcal{Z}(M) \cap \mathcal{D}^c_\Lambda$ and $\mathcal{Z}(N) \cap \mathcal{D}^c_\Lambda$ are zero. (b) $\mathcal{Z}_\partial(M) = \mathcal{Z}_\partial(N)$. (c) M and N satisfy the condition (1) in Theorem 4 about $\mathcal{Z}(M) \cap \mathcal{D}_\Lambda$ and $\mathcal{Z}(N) \cap \mathcal{D}_\Lambda$. If M and N are quasi-similar, then $M = N$.

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