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VERTEX–FACE CORRESPONDENCE IN ELLIPTIC SOLUTIONS OF THE YANG–BAXTER EQUATION

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In this review, we introduce an elliptic R-operator, which is a solution of the Yang–Baxter equation on some function space, and show the vertex–face correspondence for the eliiptic R-operator. As a result, the factorized L-operators for the elliptic R-operator are constructed. Moreover we explain that Belavin's R-matrix, the vertex–face correspondence and the factorized L-operators for it are produced from the elliptic R-operator.

1 Introduction

In 1992, we found a new solution of the Yang–Baxter equation, which is called an elliptic R-operator[12,13,14]. It is an operator on some function space, and is defined by means of the elliptic theta function. Roughly speaking, we obtain the elliptic R-operator by taking the limit $n \to \infty$ of Belavin's R-matrix i.e. the completely $\mathbb{Z}_n$ symmetric R-matrix, but we couldn't get Belavin's R-matrix again from the elliptic R-operator. In 1994, Felder and Pasquier[3] solved this problem. They succeeded to show that Belavin's R-matrix is obtained by restricting the domain of the modified version of the elliptic R-operator to a suitable finite-dimensional subspace. We can regard the elliptic R-operator as a generalization of Belavin's R-matrix.

It is well-known that Belavin's R-matrix satisfies several important properties. In particular, the vertex–face correspondence[1,7] is the one of the most important properties. Recently the author[11] showed that the elliptic R-operator has the vertex–face correspondence which produce that of Belavin's R-matrix.

In this review, we will introduce the results above, and will apply the vertex–face correspondence to get the factorized L-operators of the elliptic R-operator[11] and Belavin's R-matrix[4,6,10].

2 Elliptic R-operator

2.1 Definition

We fix $\tau \in \mathbb{C}$ such that $\mathrm{Im}\ \tau > 0$ and define an open subset $D \subset \mathbb{C}$ by $D = \{z \in \mathbb{C}\ ;\ |\mathrm{Im}\ z| < \frac{\mathrm{Im}\ \tau}{2}\}$. Let $\mathcal{V}^+$ and $\mathcal{V}^-$ be spaces of all functions f holomorphic on D and such that $f(z+1) = \pm f(z)$ for all $z \in D$, respectively.

Similarly let $\mathcal{V}^+\hat{\otimes}\mathcal{V}^+$ and $\mathcal{V}^-\hat{\otimes}\mathcal{V}^-$ be spaces of all functions f holomorphic on $D \times D$ with the property $f(z_1+1, z_2) = f(z_1, z_2+1) = \pm f(z_1, z_2)$ for all $z_1, z_2 \in D$, respectively. We put $\mathcal{V} := \mathcal{V}^{\pm}$.

Now we define an elliptic R-operator $\check{R}(u)$ on $\mathcal{V}\hat{\otimes}\mathcal{V}$. Let $\hbar$ be a complex number such that $\hbar \notin \mathbb{Z} + \mathbb{Z}\tau$ and let $\vartheta_1(z) = \vartheta_1(z, \tau)$ be an elliptic theta function; $\vartheta_1(z) = \sum_{m\in\mathbb{Z}} \exp[\pi\sqrt{-1}(m+\frac{1}{2})^2\tau + 2\pi\sqrt{-1}(m+\frac{1}{2})(z+\frac{1}{2})]$.

Definition 2.1.1 (Elliptic R-operator) *For $f \in \mathcal{V}\hat{\otimes}\mathcal{V}$, we define*

$$(\check{R}(u)f)(z_1, z_2) := \frac{\vartheta_1(u)\vartheta_1(z_{21}+\hbar)}{\vartheta_1(\hbar)\vartheta_1(z_{21})} f(z_2, z_1) + \frac{\vartheta_1(z_{21}-u)}{\vartheta_1(z_{21})} f(z_1, z_2),$$

where $z_{21} := z_2 - z_1$. The complex number u is called a spectral parameter.

As the elliptic theta function $\vartheta_1(z)$ has simple zeros at $z \in \mathbb{Z}+\mathbb{Z}\tau$, the function $\check{R}(u)f$ has the singularities at the points $(z_1, z_2) \in D \times D$ such that $z_1 - z_2 \in \mathbb{Z}$. But we can show that all singularities are removable by the Riemann removable singularity theorem. Hence $\check{R}(u)$ is an operator on $\mathcal{V}\hat{\otimes}\mathcal{V}$.

Let $\mathcal{V}^+\hat{\otimes}\mathcal{V}^+\hat{\otimes}\mathcal{V}^+$ and $\mathcal{V}^-\hat{\otimes}\mathcal{V}^-\hat{\otimes}\mathcal{V}^-$ be spaces of all functions f holomorphic on $D \times D \times D$ and such that

$$f(z_1+1, z_2, z_3) = f(z_1, z_2+1, z_3) = f(z_1, z_2, z_3+1) = \pm f(z_1, z_2, z_3)$$

for all $z_1, z_2, z_3 \in D$, respectively. By the three term equation of $\vartheta_1(z)$ (cf. Whittaker and Watson[16] p.461), we get the following theorem[12,13,14].

Theorem 2.1.2 *$\check{R}(u)$ satisfies the Yang–Baxter equation on $\mathcal{V}\hat{\otimes}\mathcal{V}\hat{\otimes}\mathcal{V}$.*

$$\begin{aligned}&(1 \otimes \check{R}(u_{12}))(\check{R}(u_{13}) \otimes 1)(1 \otimes \check{R}(u_{23}))\\ &\quad = (\check{R}(u_{23}) \otimes 1)(1 \otimes \check{R}(u_{13}))(\check{R}(u_{12}) \otimes 1),\end{aligned}$$

where $u_{ij} = u_i - u_j$.

2.2 Vertex–face correspondence

In what follows $\hbar \in \mathbb{R}\backslash\mathbb{Z}$, and let Λ be a set of sequences $\lambda = (\lambda_i)$ $(i \in \mathbb{Z})$ such that

$$\begin{aligned}&\lambda_i \in D,\\ &\lambda_{ij} := \lambda_i - \lambda_j \notin \mathbb{Z} + \mathbb{Z}\hbar \quad \forall i \neq j \in \mathbb{Z}.\end{aligned}$$

We note that the set Λ is not empty for any $\hbar$. For $\lambda \in \Lambda$ and $i \in \mathbb{Z}$, let $\lambda + \hbar\varepsilon_i$ denote the sequence

$$(\lambda + \hbar\varepsilon_i)_j = \begin{cases} \lambda_j, & j \neq i, \\ \lambda_i + \hbar, & j = i. \end{cases}$$

We note that $\lambda + \hbar\varepsilon_i \in \Lambda$ for all $i \in \mathbb{Z}$ by the definition of Λ.

Definition 2.2.1 (Boltzmann weight of the face model [1,4,5,7,10]) *For λ, $\mu, \mu', \nu \in \Lambda$, Boltzmann weights $\overset{\vee}{W}\begin{bmatrix} & \mu & \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix} \in \mathbb{C}$ of a face model are given as follows. For $\lambda \in \Lambda$, we put*

$$\overset{\vee}{W}\begin{bmatrix} & \lambda + \hbar\varepsilon_i & \\ \lambda & u & \lambda + 2\hbar\varepsilon_i \\ & \lambda + \hbar\varepsilon_i & \end{bmatrix} := \frac{\vartheta_1(u+\hbar)}{\vartheta_1(\hbar)},$$

$$\overset{\vee}{W}\begin{bmatrix} & \lambda + \hbar\varepsilon_i & \\ \lambda & u & \lambda + \hbar(\varepsilon_i + \varepsilon_j) \\ & \lambda + \hbar\varepsilon_i & \end{bmatrix} := \frac{\vartheta_1(-u+\lambda_{ij})}{\vartheta_1(\lambda_{ij})} \quad (i \neq j),$$

$$\overset{\vee}{W}\begin{bmatrix} & \lambda + \hbar\varepsilon_i & \\ \lambda & u & \lambda + \hbar(\varepsilon_i + \varepsilon_j) \\ & \lambda + \hbar\varepsilon_j & \end{bmatrix} := \frac{\vartheta_1(u)\vartheta_1(\hbar+\lambda_{ij})}{\vartheta_1(\lambda_{ij})\vartheta_1(\hbar)} \quad (i \neq j),$$

otherwise we set $\overset{\vee}{W}\begin{bmatrix} & \mu & \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix} := 0.$

Next we define incoming intertwining vectors of the elliptic R-operator.

Definition 2.2.2 (Incoming intertwining vector [11]) *For $\lambda, \mu \in \Lambda$, define an incoming intertwining vector $\bar{\phi}^{\mu}_{\lambda} \in \mathcal{V}^*$ as follows.*

$$\bar{\phi}^{\mu}_{\lambda} f := \begin{cases} f(\lambda_i), & \exists i \in \mathbb{Z} \text{ s.t. } \mu = \lambda + \hbar\varepsilon_i, \\ 0, & \textit{otherwise}. \end{cases}$$

The incoming intertwining vectors are the Dirac delta functions essentially.

By Definition 2.1.1 we can get a vertex–face correspondence for the elliptic R-operator.

Theorem 2.2.3 (Vertex–face correspondence [11]) *For $\lambda, \mu, \nu \in \Lambda$*

$$\bar{\phi}^{\nu}_{\mu} \otimes \bar{\phi}^{\mu}_{\lambda} \overset{\vee}{R}(u) = \sum_{\mu' \in \Lambda} \overset{\vee}{W}\begin{bmatrix} & \mu & \\ \lambda & u & \nu \\ & \mu' & \end{bmatrix} \bar{\phi}^{\nu}_{\mu'} \otimes \bar{\phi}^{\mu'}_{\lambda},$$

where both sides are the operators $\mathcal{V}\hat{\otimes}\mathcal{V} \to \mathbb{C}$.

Since $\check{R}(u)$ satisfies the Yang–Baxter equation (Theorem 2.1.2), we can show that the Boltzmann weights of the face model satisfy the star–triangle relation.

2.3 *Factorized L-operator*

First we define the dual of the incoming intertwining vector of the elliptic R-operator, which is called the outgoing intertwining vector[11]. Let k_1 and k_2 be integers such that $k_1 \leq k_2$, and we set $\mathbf{k} := (k_1, k_2)$ and $k = k_2 - k_1 + 1$. In what follows, the following notation will be in force.

$$\mathcal{V} := \begin{cases} \mathcal{V}^+, & \text{if } k \text{ is even,} \\ \mathcal{V}^-, & \text{if } k \text{ is odd.} \end{cases}$$

Definition 2.3.1 (Outgoing intertwining vector [11]) *For $\lambda, \mu \in \Lambda$ and $u \notin \mathbb{Z} + \mathbb{Z}\tau$, an outgoing intertwining vector $\phi_{\mathbf{k}}(u)^{\mu}_{\lambda} \in \mathcal{V}$ of the elliptic R-operator is defined as follows. For $\lambda \in \Lambda$ and $k_1 \leq i \leq k_2$, we set*

$$\phi_{\mathbf{k}}(u)^{\lambda+\hbar\varepsilon_i}_{\lambda}(z) = \frac{\vartheta_1(u + \lambda_i - z)}{\vartheta_1(u)} \prod_{k_1 \leq j \leq k_2, j \neq i} \frac{\vartheta_1(z - \lambda_j)}{\vartheta_1(\lambda_{ij})},$$

otherwise we put $\phi_{\mathbf{k}}(u)^{\mu}_{\lambda}(z) = 0$.

These outgoing intertwining vectors satisfy the equation below, which is the dual of vertex–face correspondence (Theorem 2.2.3).

Proposition 2.3.2 *For $\lambda, \mu, \nu \in \Lambda$ and $u_1, u_2 \notin \mathbb{Z} + \mathbb{Z}\tau$,*

$$(\check{R}(u_{12})\phi_{\mathbf{k}}(u_1)^{\nu}_{\mu} \otimes \phi_{\mathbf{k}}(u_2)^{\mu}_{\lambda})(z_1, z_2)$$
$$= \sum_{\mu' \in \Lambda} \phi_{\mathbf{k}}(u_2)^{\nu}_{\mu'}(z_1) \otimes \phi_{\mathbf{k}}(u_1)^{\mu'}_{\lambda}(z_2) \check{W} \begin{bmatrix} & \mu' & \\ \lambda & u_{12} & \nu \\ & \mu & \end{bmatrix}.$$

Now we are in the position to construct factorized L-operators for the elliptic R-operator. Let $\mathcal{W}$ be a space of all $\mathbb{C}$-valued functions on Λ, and let $\mathcal{V}\hat{\otimes}\mathcal{W}$ (resp. $\mathcal{W}\hat{\otimes}\mathcal{V}$) be a space of all functions $g : D \times \Lambda \to \mathbb{C}$ (resp. $\Lambda \times D \to \mathbb{C}$) such that $g(\cdot, \lambda) \in \mathcal{V}$ (resp. $g(\lambda, \cdot) \in \mathcal{V}$) for any $\lambda \in \Lambda$. We define a factorized L-operator $\check{L}_{\mathbf{k}}(u) : \mathcal{V}\hat{\otimes}\mathcal{W} \to \mathcal{W}\hat{\otimes}\mathcal{V}$ as follows. For $g \in \mathcal{V}\hat{\otimes}\mathcal{W}$ and $u \notin \mathbb{Z} + \mathbb{Z}\tau$

$$(\check{L}_{\mathbf{k}}(u)g)(\mu, z) := \sum_{\lambda \in \Lambda} \phi_{\mathbf{k}}(u)^{\mu}_{\lambda}(z)(\bar{\phi}^{\mu}_{\lambda} g(\cdot, \lambda)).$$

We define $\mathcal{V}\hat{\otimes}\mathcal{V}\hat{\otimes}\mathcal{W}$ (resp. $\mathcal{W}\hat{\otimes}\mathcal{V}\hat{\otimes}\mathcal{V}$) by a space of all functions $g : D\times D\times\Lambda \to \mathbb{C}$ (resp. $\Lambda\times D\times D \to \mathbb{C}$) such that $g(\cdot,\cdot,\lambda) \in \mathcal{V}\hat{\otimes}\mathcal{V}$ (resp. $g(\lambda,\cdot,\cdot) \in \mathcal{V}\hat{\otimes}\mathcal{V}$) for any $\lambda \in \Lambda$. By means of Theorem 2.2.3 and Proposition 2.3.2, we immediately obtain the following theorem.

Theorem 2.3.3 (Factorized L-operator [11]) *For $u_1, u_2 \notin \mathbb{Z} + \mathbb{Z}\tau$*

$$\begin{aligned}&(1\otimes \check{R}(u_{12}))(\check{L}_{\mathbf{k}}(u_1)\otimes 1)(1\otimes \check{L}_{\mathbf{k}}(u_2))\\ &\quad = (\check{L}_{\mathbf{k}}(u_2)\otimes 1)(1\otimes \check{L}_{\mathbf{k}}(u_1))(\check{R}(u_{12})\otimes 1),\end{aligned}$$

where both sides are the operators $\mathcal{V}\hat{\otimes}\mathcal{V}\hat{\otimes}\mathcal{W} \to \mathcal{W}\hat{\otimes}\mathcal{V}\hat{\otimes}\mathcal{V}$.

3 Belavin's R-matrix

3.1 Method to obtain Belavin's R-matrix from the elliptic R-operator

For $n \in \mathbb{Z}_{>1}$, let $V_n(u)$ be a space of entire functions f of one variable such that

$$\begin{aligned}&f(z+1) = (-1)^n f(z),\\ &f(z+\tau) = (-1)^n \exp(-2\pi\sqrt{-1}(kz-u+\tfrac{n\tau}{2}))f(z).\end{aligned}$$

We note that $V_n(u) \subset \mathcal{V}^+$ if n is even and that $V_n(u) \subset \mathcal{V}^-$ if n is odd.

Theorem 3.1.1 (Felder and Pasquier [3])

$$\check{R}(u_{12})(V_n(u_1+\hbar)\otimes V_n(u_2)) \subset V_n(u_2+\hbar)\otimes V_n(u_1).$$

The space $V_n(u)$ is of n dimensions and a basis is given by

$$\{e_j(u)(z) := \vartheta\begin{bmatrix}\frac{1}{2}-\frac{j}{n}\\ \frac{n}{2}\end{bmatrix}(u-nz, n\tau)\}_{j\in\mathbb{Z}/n\mathbb{Z}}.$$

For $n \in \mathbb{Z}_{>1}$, define a translation operator $T_n(u)$ on the space of all holomorphic functions on $\mathbb{C}$ by

$$(T_n(u)f)(z) := f(z-\frac{u}{n}).$$

$T_n(u)$ maps isomorphically $V_n := V_n(0)$ onto $V_n(u)$. We modify the elliptic R-operator as

$$\check{R}_n(u_{12}) := T_n(u_2+\hbar)^{-1}\otimes T_n(u_1)^{-1}\circ \check{R}(u_{12})\circ T_n(u_1+\hbar)\otimes T_n(u_2)\Big|_{V_n\otimes V_n}.$$

Theorem 3.1.2 (Felder and Pasquier [3]) *$\check{R}_n(u)$ preserves $V_n \otimes V_n$ and obeys the Yang–Baxter equation (Theorem 2.1.2).*

Let $\{e^j\}_{j\in\mathbf{Z}/n\mathbf{Z}} \subset V_n^*$ be the dual basis of $\{e_j := e_j(0)\} \subset V_n$;

$$e^i(e_j) = \delta_{ij}.$$

Now we define an operator $\check{R}_n(u)^*$ on $V_n^* \otimes V_n^*$, the transpose of $\check{R}_n(u)$ on $V_n \otimes V_n$.

$$(\check{R}_n(u)^* e^i \otimes e^j)(e_k \otimes e_\ell) := (e^j \otimes e^i)(\check{R}_n(u) e_\ell \otimes e_k).$$

Theorem 3.1.3 (Felder and Pasquier [3]) *The R-matrix $\check{R}_n(u)^*$ is Belavin's R-matrix [2,4,5].*

3.2 Vertex–face correspondence

We use the notation defined in Section 2.3. For $\lambda, \mu \in \Lambda$, we put $\phi(u)^\mu_\lambda := \bar{\phi}^\mu_\lambda \circ T_k(u + |\lambda|_{\mathbf{k}})\big|_{V_k} \in V_k^*$, where $|\lambda|_{\mathbf{k}} = \lambda_{k_1} + \lambda_{k_1+1} + \cdots + \lambda_{k_2}$. Then

$$\begin{aligned}
\phi(u)^\mu_\lambda &= \sum_{j=0}^{k-1} \bar{\phi}^\mu_\lambda \circ T_k(u + |\lambda|_{\mathbf{k}})(e_j)e^j \\
&= \begin{cases} \sum_{j=0}^{k-1} \vartheta\left[\begin{smallmatrix}\frac{1}{2} - \frac{j}{k} \\ \frac{k}{2}\end{smallmatrix}\right](u + |\lambda|_{\mathbf{k}} - k\lambda_i, k\tau)e^j, & \text{if } \mu = \lambda + \hbar\varepsilon_i \ (k_1 \le \exists i \le k_2), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence the vector $\phi(u)^\mu_\lambda$ is nothing but the outgoing intertwining vector of Belavin's R-matrix [4,5], which was first discovered by Baxter [1], Jimbo, Miwa and Okado [7]. Theorem 2.2.3 leads us to

Theorem 3.2.1 (Vertex–face correspondence [1,7]) *For $\lambda, \mu, \nu \in \Lambda$,*

$$\check{R}_k(u_{12})^* \phi(u_1)^\mu_\lambda \otimes \phi(u_2)^\nu_\mu = \sum_{\mu' \in \Lambda} \phi(u_2)^{\mu'}_\lambda \otimes \phi(u_1)^\nu_{\mu'} \check{W}\begin{bmatrix} & \mu & \\ \lambda & u_{12} & \nu \\ & \mu' & \end{bmatrix}.$$

3.3 Factorized L-operator

First we define the operators $\check{L}_{\mathbf{k}}(u)^\mu_\lambda$ as follows. For $f \in \mathcal{V}$,

$$(\check{L}_{\mathbf{k}}(u)^\mu_\lambda f)(z) := \phi_{\mathbf{k}}(u)^\mu_\lambda(z)\bar{\phi}^\mu_\lambda f.$$

We put $\tilde{L}_{\mathbf{k}}(u)^{\mu}_{\lambda} := T_k(u+|\lambda|_{\mathbf{k}})^{-1}\overset{\vee}{L}_{\mathbf{k}}(u)^{\mu}_{\lambda}T_k(u+|\lambda|_{\mathbf{k}})\Big|_{V_k}$ and denote its transpose as $\tilde{L}^*_{\mathbf{k}}(u)^{\mu}_{\lambda} : V_k^* \to V_k^*$. Theorem 2.2.3 and Proposition 2.3.2 imply

$$\sum_{\mu\in\Lambda} \overset{\vee}{R}_k(u_{12})^*\tilde{L}^*_{\mathbf{k}}(u_1)^{\mu}_{\lambda} \otimes \tilde{L}^*_{\mathbf{k}}(u_2)^{\nu}_{\mu} = \sum_{\mu\in\Lambda} \tilde{L}^*_{\mathbf{k}}(u_2)^{\mu}_{\lambda} \otimes \tilde{L}^*_{\mathbf{k}}(u_1)^{\nu}_{\mu}\overset{\vee}{R}_k(u_{12})^*.$$

We define an operator $\tilde{L}^*_{\mathbf{k}}(u) : V_k^* \otimes \mathcal{W} \to \mathcal{W} \otimes V_k^*$ by

$$\tilde{L}^*_{\mathbf{k}}(u)(e^i \otimes \delta^{\mu}) = \sum_{\lambda\in\Lambda} \delta^{\lambda} \otimes \tilde{L}^*_{\mathbf{k}}(u)^{\mu}_{\lambda} e^i.$$

Here, for $\lambda \in \Lambda$, we set $\delta^{\lambda} \in \mathcal{W}$ as $\delta^{\lambda}(\mu) = \delta_{\lambda\mu}$.

Theorem 3.3.1 (Factorized L-operator [4,6,10]) *For $u_1, u_2 \notin \mathbb{Z} + \mathbb{Z}\tau$*

$$\begin{aligned}&(1 \otimes \overset{\vee}{R}_k(u_{12})^*)(\tilde{L}^*_{\mathbf{k}}(u_1) \otimes 1)(1 \otimes \tilde{L}^*_{\mathbf{k}}(u_2)) \\ &= (\tilde{L}^*_{\mathbf{k}}(u_2) \otimes 1)(1 \otimes \tilde{L}^*_{\mathbf{k}}(u_1))(\overset{\vee}{R}_k(u_{12})^* \otimes 1).\end{aligned}$$

Here both sides are the operators $V_k^* \otimes V_k^* \otimes \mathcal{W} \to \mathcal{W} \otimes V_k^* \otimes V_k^*$.

4 Conclusion

In this review, we introduce the elliptic R-operator which induces Belavin's R-matrix and the several important properties of it. We can regard the elliptic R-operator as an infinite-dimensional generalization [12] of Belavin's R-matrix. So we need to study the structure of Belavin's R-matrix making use of the properties of the elliptic R-operator. What properties of the elliptic R-operator induce the crossing symmetry and the gauge factor of Belavin's R-matrix?

Another important problem is to construct elliptic algebra. This is our motivation to investigate the elliptic R-operator. The quantum groups and the quantum homogeneous spaces are useful for a geometric interpretation of Macdonald's symmetric polynomials [8,9,15]. Macdonald's symmetric polynomials have two parameters, but, in this case, we only dealt with the case of specializing these parameters. Further it is more natural to consider Macdonald's symmetric functions rather than Macdonald's symmetric polynomials. Roughly speaking, Macdonald's symmetric functions are the polynomials of infinite variables. Then we wanted to construct new algebra which controls Macdonald's symmetric functions directly. One candidate is the algebra defined as the L-operator of the elliptic R-operator. In this review, we construct the factorized L-operators of the elliptic R-operator, which can be regarded as representations of the elliptic algebra, but we don't have succeed defining the algebra yet.

Acknowledgments

The author expresses his deep gratitude to the staff of Nankai Institute of Mathematics, especially to Professor Mo-Lin Ge, for their kind hospitality during my stay in Nankai Institute.

References

1. R. Baxter, *Ann. Phys.* **76**, 25 (1973).
2. A. A. Belavin, *Nucl. Phys.* **B180** **[FS2]**, 189 (1981).
3. G. Felder *et al*, *Lett. Math. Phys.* **32**, 167 (1994).
4. K. Hasegawa, *J. Phys. A. Math. Gen.* **26**, 3211 (1993).
5. ______, *J. Math. Phys.* **35**(11), 6158 (1994).
6. B. Hou *et al*, *Phys. Lett. A* **178**, 73 (1993).
7. M. Jimbo *et al*, *Nucl. Phys.* **B300** **[FS22]**, 74 (1988).
8. I. G. Macdonald, *Symmetric functions and Hall polynomials, second edition*, (Clarendon Press, Oxford, 1995).
9. M. Noumi, *Macdonald's symmetric polynomials on quantum homogeneous spaces*, to appear in *Adv. Math.*
10. Y. Quano *et al*, *Mod. Phys. Lett.* **A8**, 1585 (1993).
11. Y. Shibukawa, *Commun. Math. Phys.* **172**, 661–677 (1995)
12. Y. Shibukawa *et al*, *Lett. Math. Phys.* **25**, 239 (1992).
13. ______, in *Int. J. Mod. Phys. A (Proc. Suppl.) 3A*, ed. C. N. Yang *et al* (World Scientific, Singapore, 1993).
14. ______, in *Quantum groups, integrable statistical models and knot theory*, ed. M. L. Ge *et al* (World Scientific, Singapore, 1993).
15. K. Ueno *et al*, in *Quantum groups*, ed. P. P. Kulish (Springer-Verlag, Berlin, 1992).
16. E. T. Whittaker *et al*, *A course of modern analysis, fourth edition*, (Cambridge University Press, Cambridge, 1927).