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CONSISTENCY IN EVOLUTIONS BY CRYSTALLINE CURVATURE

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CONSISTENCY IN EVOLUTIONS BY CRYSTALLINE CURVATURE

MI-HO GIGA AND YOSHIKAZU GIGA

Abstract. Motion of curves by crystalline energy is often considered for "admissible" piecewise linear curves. This is because the evolution of such curves can be described by a simple system of ordinary differential equations. Recently, a generalized notion of solutions based on comparison principle is introduced by the authors. In this note we show that a classical admissible solution is always a generalized solution in our sense.

1. Introduction. Motion by crystalline evergy or crystalline curvature is interpreted as a typical example of geometric evolutions by nonsmooth interfacial energy. Let Γ_t denote an embedded curve in the plane depending on time t. Let $\mathbf{n}$ be the unit normal vector field of Γ_t determining the orientation of Γ_t and let V denote the normal velocity in the direction of $\mathbf{n}$. We consider the equation of Γ_t of the form

$$V = -\frac{1}{\beta(\mathbf{n})}\left(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}((\partial_i\gamma)(\mathbf{n})) + C(t)\right) \quad \text{on} \quad \Gamma_t. \tag{1}$$

Here $\gamma : \mathbf{R}^2 \to \mathbf{R}$ is of the form

$$\gamma(q) = |q|\gamma_0(q/|q|)$$

and γ_0, β are given positive functions defined on the unit circle and $\partial_i\gamma$ denotes the partial derivative $\partial\gamma/\partial q_i$ as a function on $\mathbf{R}^2$. The function $C(t)$ is a given continuous function. An interfacial energy γ_0 is called *crystalline* if its Frank diagram

$$\text{Frank}(\gamma_0) = \{(q_1, q_2) \in \mathbf{R}^2; \gamma(q) = 1\}$$

is a convex polygon. In this case because of jumps of first derivatives of γ, (1) is no longer a usual partial differential equation of coordinate representations of Γ_t.

Taylor [T1] proposed an evolution governed by (1) when γ_0 is crystalline by restricting Γ_t as "admissible" polygon. A system of ordinary differential equation is derived by a variational principle when $\beta = \text{const}\,.\gamma^{-1}, C \equiv 0$. Independently, Angenent and Gurtin [AG] derived the same system for general β and C from the balance of forces and the second law of thermodynamics.

We shall recall their equation when Γ_t is given as a graph of a function. Such a version is given in [GirK 1] for $C \equiv 0$. Let Γ_t be given as a graph of a function $y = u(t, x), x \in \mathbf{R}$. Then (1) becomes

$$u_t = a(u_x)\big[(W'(u_x))_x - C(t)\big] \tag{2}$$

with

$$\begin{cases} a(p) = (1+p^2)^{1/2} M(p), \\ \dfrac{1}{M(p)} = \beta\Big(-\dfrac{p}{(1+p^2)^{1/2}}, \dfrac{1}{(1+p^2)^{1/2}}\Big), \\ W(p) = \gamma(-p, 1) \end{cases} \tag{3}$$

provided that $\mathbf{n}$ is taken upward [GMHG]. If γ_0 is crystalline, then W' is a piecewise constant nondecreasing function whose jump discontinuities consists of finitely many points $p_1 < p_2 < \cdots < p_m$.

We say a function v on $\mathbf{R}$ is *admissible crystal* if (i) v is a piecewise linear continuous function with slopes belong to $P = \{p_i\}_{i=1}^m$; (ii) let p_i be a slope of v in an interval (a_1, a_2) where v_x has jump at a_1 and a_2. Then the slopes of $v(x)$ for $x < a_1$ (near a_1) or for $x > a_2$ (near a_2) are either p_{i+1} or p_{i-1} with $i+1 \le m, i-1 \ge 1$. The graph of such a function is called a Wulff curve in [EGS].

We say $u = u(t,x)$ $(0 < t < T)$ is an *admissible evolving crystal* if $u(t,\cdot)$ is admissible crystal and jumps of u_x move smoothly in time. (This definition is consistent with that in [GG].) For an admissible evolving crystal an evolution equation corresponding to (2) is derived in [T1] and [AG]. It is of the form

$$u_t = a(u_x)\left(\frac{\chi\Delta}{L} - C(t)\right) \quad \text{on} \quad x_j(t) < x < x_{j+1}(t). \tag{4}$$

Here for fixed t, $\{x_j(t)\}$ is a discrete set and it consists of jumps of $u_x(t,\cdot)$ and $u(t,\cdot)$ is linear on $(x_j(t), x_{j+1}(t))$; j runs in either a finite set $\{1, 2, \cdots, d\}$, the set of natural numbers $\mathbf{N} = \{1, 2, \cdots\}$ or the set $\mathbf{Z}$ of integers; in the first two cases we use convention that $x_1 = -\infty$, and in the first case $x_d = +\infty$. The quantity L denotes the length of (x_j, x_{j+1}) i.e.,

$$L = x_{j+1}(t) - x_j(t).$$

If (x_j, x_{j+1}) is an infinite interval, we interpret $1/L$ as zero. The quantity Δ is defined by

$$\Delta = W'(p_k + 0) - W'(p_k - 0)$$

where $u_x = p_k$ on (x_j, x_{j+1}). The quantity χ is called a transition number. It takes the value 1 (resp. -1) if $u(t,\cdot)$ is convex (resp. concave) around (x_j, x_{j+1}); otherwise $\chi = 0$. The quantity $\chi\Delta/L$ is often called a crystalline curvature or weighted curvature [T1,2].

Note that on an admissible evolving crystal the value of a outside P is irrelvant to define the equation (4).

The equation (4) yields a system of ordinary differential equations (ODE) at least for x_j's ([AG], [T1,2], [GirK1,2]). We shall postpone to present this equation. If $\{x_j\}$ is a finite set or $u(t,\cdot)$ is periodic in x, the number of unknown is finite so that a local existence theorem for ODE applies. For example we observe that if initial data is an admissible crystal and periodic, then there is a unique admissible evolving crystal satisfying (4) at least for a short time. However, x_j may agree with x_{j+1} in a finite time. Fortunately, if $x_j(t_0) = x_{j+1}(t_0)$ for some time, then $\chi = 0$ for $(x_j(t), x_{j+1}(t))$, $0 < t < t_0$. In other words *a facet with* $\chi = \pm 1$ *does not disappear.* Moreover, at most three consecutive x_j's may agree at one time. Even at such a time t_0 where some facet disappears, $u(t_0 - 0, \cdot)$ is an admissible crystal. There observations are given in [GirK1] with $C \equiv 0$ but it extends to $C \not\equiv 0$ provided that C is continuous in t.

Note that even if admissible crystalline evolution loses facet at $t = t_0$, one can restart with $u(t_0, \cdot)$ and solve (4) again. By this process a global solution of (4) is obtained. To be precise we say $u = u(t,x)$ $(0 < t < T)$ is *weakly admissible evolving crystal* if u is an admissible evolving crystal for $(t_\ell, t_{\ell+1}), \ell = 0, \cdots, k$ for some $0 < t_0 < t_1 < \cdots < t_k < t_{k+1} = T$ and u is continuous across $t_\ell, \ell = 0, \cdots, k$. (This definition is consistent with that in [GG].) Using this terminology, we obtain for example that there is a unique weakly admissible evolving crystal satisfying (4) globally-in-time if initial data is an admissible.

The main goal of this note is to show that weakly admissible evolving crystal is indeed a generalized solution introduced by the authors [GMHG] (under some conditin for β if $C \not\equiv 0$.) This means our notion of solutions is a natural extension. In [EGS] analogous statement is proved for generalized solutions based on the nonlinear semigroup theory [FG]. However, since the generalized solution in [FG] is only defined for $C \equiv 0$, their statement is restricted for $C \equiv 0$.

Recently, motion by crystalline energy is studied extensively. Instead of mentioning all related articles we only list review articles [T2], [GirK2] and [GMHG].

We conclude this section by deriving a system of ODEs from (4). For simplicity we assume that an admissible evolving crystal $u(t,\cdot)$ $(0 \le t < T)$ is periodic in x (with period ω) so that for some d, $x_{i+d} = x_i + \omega$ for all i in $\mathbf{Z}$. Let $L_j(t)$ denote the length of $(x_{j-1}(t), x_j(t))$, i.e.

(5)
$$L_j(t) = x_j(t) - x_{j-1}(t), \quad i = 1, \cdots, d.$$

Let $R_j(t)$ denote $[x_{j-1}(t), x_j(t)]$ so that the interior $R_j(t)^0 = (x_{j-1}(t), x_j(t))$. Let $(u_t)_j = (u_t)_j(t)$ denote $u_t(t,x)$ for $x \in R_j(t)^0$. Let $(u_x)_j = (u_x)_j(t)$ denote the slope of $u(t,\cdot)$ on $R_j(t)^o$. i.e.,

(6)
$$(u_x)_j = u_x(t,x) \quad \text{for} \quad x_{j-1}(t) < x < x_j(t).$$

Since u is continuous,

(7)
$$dL_j(t)/dt = \rho_j^0 (u_t)_j + \rho_j^{-1}(u_t)_{j-1} + \rho_j^1 (u_t)_{j+1}, \quad j = 1, \cdots, d$$

with

$$\begin{aligned}
\rho_j^0 &= ((u_x)_j - (u_x)_{j-1})^{-1} + ((u_x)_{j+1} - (u_x)_j)^{-1} \\
\rho_j^{-1} &= -((u_x)_j - (u_x)_{j-1})^{-1} \\
\rho_j^1 &= -((u_x)_{j+1} - (u_x)_j)^{-1}.
\end{aligned}$$

This follows from the elementary geometry and does not depend on the special evolution equation (4). We now invoke (4) which is rewritten as

$$(8)\qquad \begin{aligned}(u_t)_j &= a((u_x)_j)(\Lambda_j - C(t)), \quad j = 1,\cdots,d\\ \Lambda_j &= \chi_j\Delta/L_j\end{aligned}$$

with $\Delta = W'((u_x)_j + 0) - W'((u_x)_j - 0)$, where χ_j is the transition number on $R_j^0 = (x_{j-1}, x_j)$. Since $(u_x)_j$ and χ_j are determined initially, the equations (7) and (8) yield a system of ODEs for L_j $(j = 1,\cdots,d)$; note that $(u_0)_t = (u_d)_t$ in (7) so the system (7)-(8) is closed. As in [GirK1] differentiating $u(x_j(t), t)$ in t we get

$$(9)\qquad dx_j/dt = -\big((u_t)_{j+1} - (u_t)_j\big)\big/\big((u_x)_{j+1} - (u_x)_j\big)$$

for $j = 1,\cdots,d$. The function $(u_t)_j$ is computable from (7), (8) so the evolution of x_j is determined by (9).

Derivation of (7), (8), (9) is found in [GirK1], where $C \equiv 0$ and $a(p) = (1+p^2)^{1/2}$ is assumed but it extends to our setting with no essential change. The systems (7), (8) is found in [AG] in a little bit different form ; L_j is replaced by the length of the graph $y = u(t,x)$ on (x_{j-1}, x_j) and $(u_t)_j$ is replaced by the normal velocity. For later convenience by *j-th facet* we mean the graph of $y = u(t,\cdot)$ on $R_j(t)$. The point $(x_j(t), u(t, x_j(t)))$ is called *a corner.*

2. Generalized solutions. We recall from [GMHG] our definition of generalized solution when W' is a piecewise constant function with jumps on P.

Definition(P-faceted). Let Ω be an open interval. A function ϕ in $C(\Omega)$ is called *P-faceted* at x_0 in Ω if ϕ fulfills the following conditions.

There are a closed nontrivial finite interval $I(\subset \Omega)$ containing x_0 and p in P such that ϕ agrees with an affine function

$$\ell_p(x) = p(x - x_0) + \phi(x_0)$$

in I and $\phi(x) \neq \ell_p(x)$ for all $x \in J \setminus I$ with some neighborhood $J(\subset \Omega)$ of I. The interval I is called a *faced region* of ϕ containing x_0. The value p is called the slope of the facet.

Definition(Weighted curvature with driving term). Let c be a constant and x_0 be a point in Ω. For $\phi \in C(\Omega)$ we set the value

$$\Lambda_W(\phi, x_0,; c) = W''(\phi'(x_0))\phi''(x_0) + c (= 0 + c \ \text{ since } \ W'' = 0 \ \text{ outside } \ P)$$

if ϕ is second differentiable at x_0 and $\phi'(x_0) \notin P$ and

$$\Lambda_W(\phi, x_0; c) = \frac{\chi}{L}\Delta_i + c$$

if ϕ in P-faceted at x_0 in Ω with slope p_i, where $\Delta_i = W'(p_i + 0) - W'(p_i - 0)$. Here $L = L(\phi, x_0)$ is the length of the faceted region I containing x_0 and $\chi = \chi(\phi, x_0)$ is the transition number defined by

$$\begin{aligned}\chi &= +1 \quad \text{if} \quad \phi \geq \ell_{p_i} \quad \text{in} \quad J,\\ \chi &= -1 \quad \text{if} \quad \phi \leq \ell_{p_i} \quad \text{in} \quad J,\\ \chi &= 0 \qquad \text{otherwise}\end{aligned}$$

for some neighborhood J of the facet region I.

Definition. A function $\phi \in C^2(\Omega)$ belongs to a class $C_P^2(\Omega)$ if ϕ is P-faceted at x_0 in Ω whenever $\phi'(x_0)$ belongs to P. For $\phi \in C_P^2, \Lambda_W(\phi, x; c)$ is defined for all $x \in \Omega$ so we often write it by $\Lambda_W(\phi; c)(x)$.

Definition(Space of admissible functions). For $Q = (0,T) \times \Omega$ let $A_P(Q)$ be the set of functions on Q of the form

$$\phi(x) + g(t), \quad \phi \in C_P^2(\Omega), \quad g \in C^1(0,T).$$

An element of $A_P(Q)$ is called an admissible function.

We are now in position to define our generalized solution in the viscosity sense.

Definition (Generalized solution). A real valued function u on Q is a (*viscosity*) *subsolution* of

$$u_t - a(u_x)\Lambda_W(u; -C) = 0 \tag{E}$$

if the upper semicontinuous envelope $u^* < \infty$ in $\overline{Q}$ and

$$\psi_t(\hat{t}, \hat{x}) - a(\psi_x(\hat{t}, \hat{x}))\Lambda_W(\psi(\hat{t}, \cdot); -C(\hat{t}))(\hat{x}) \le 0 \tag{$*$}$$

whenever $(\psi, (\hat{t}, \hat{x})) \in A_P(Q) \times Q$ fulfills

$$\max_Q (u^* - \psi) = (u^* - \psi)(\hat{t}, \hat{x}).$$

A (*viscosity*) *supersolution* is defined by replacing $u^*(< \infty)$ by the lower semicontinuous envelope $u_*(> -\infty)$, max by min and the inequality in $(*)$ by the opposite one. If u is sub- and supersolution, u is called a *viscosity solution* or a *generalized solution*. Hereafter we avoid to use the word viscosity because this is nothing to do with viscosity in the sense of physics corresponding to (E). A key result in [GMHG] is

Fundamental Comparison Theorem. *Let u and v be a sub- and supersolution of (E), respectively, where Ω is a bounded open interval. Assume that $C \in C[0,T)$ and $a \in C(\mathbf{R})$ with $a \ge 0$. If $u^* \le v_*$ on the parabolic boundary ($= [0,T) \times \partial\Omega \cup \{0\} \times \bar{\Omega}$) of Q, then $u^* \le v_*$ in Q.*

Existence Theorem. *Suppose that $u_0 \in C(\mathbf{R})$ is periodic with period ω. Then there is a unique global generalized solution $u \in C([0,\infty) \times \mathbf{R})$ of (E) with $u(0,x) = u_0(x)$ and $u(t, x+\omega) = u(t,x)$.*

3. Consistency.

We always assume that a is a nonnegative continuous function and that $C \in C[0,\infty)$. Our goal is to prove :

Theorem. *Assume that u is a weakly admissible evolving crystal satisfying (4) defined in $[0,\infty)\times\mathbf{R}$. If $C\not\equiv 0$, assume that a satisfies*

$$\begin{aligned} a(p) &= \theta a(p_k) + (1-\theta)a(p_{k+1}) \\ &\text{for} \quad p = \theta p_k + (1-\theta)p_{k+1}, 0 \le \theta \le 1, 1 \le k \le m \end{aligned}$$

where $P = \{p_1 < \cdots < p_m\}$. Then u is a generalized solution of (E) (on $[0,\infty)\times\mathbf{R}$).

We shall mainly give a proof for $C \equiv 0$ and point out how to alter the proof for general C.

We shall only prove that u is a subsolution of (E) in $Q = (0,\infty)\times\Omega$ with $\Omega = \mathbf{R}$ since the proof for supersolution is the same. Since $u \in C(\overline{Q})$, the property $u^* < \infty$ in $\overline{Q}$ is trivially satisfied. Let $(\hat{t},\hat{x}) \in Q$ and $\psi \in A_P(Q)$ satisfy

$$\max_Q (u-\psi) = (u-\psi)(\hat{t},\hat{x}).$$

Our goal is to show $(*)$. As usual we may assume $(u-\psi)(\hat{t},\hat{x}) = 0$. By the definition of $A_P(Q)$, ψ is of the form

$$\psi(t,x) = \phi(x) + g(t), \quad \phi \in C_P^2(\Omega), \quad g \in C^1(0,\infty).$$

Note that at $\hat{t}$ some facet may disappear. However, $x_j(t)$ together with its time derivative $\dot{x}_j(t)$ is always continuous on $(\hat{t}-\delta,\hat{t}]$ for sufficiently small $\delta > 0$.

Lemma 1. *Assume that $C = 0$. Assume that $\hat{x}$ is the abscissa of a corner of $u(\hat{t},\cdot)$ i.e. $\hat{x} = x_i(\hat{t}-0)$ for some integer i. Let j be the largest integer such that $\hat{x} = x_j(\hat{t}-0)$. Let $p_{k-1} \in P$ be $(u_x)_{j+1}$, the slope of $u(t,\cdot)$ on $R_{j+1}(t)^o = (x_j(t), x_{j+1}(t))$ for t close to $\hat{t}$ $(t<\hat{t})$; $(u_x)_{j+1}$ is independent of t.*

(i) $p_{k-1} \le \phi'(\hat{x}) \le p_k$.

(ii) If $x_{j-1}(\hat{t}-0) < x_j(\hat{t}-0)$ then $(u_x)_j = p_k$. Moreover, $(u_t)_{j+1}(\hat{t}-0) \le 0$ and $(u_t)_j(\hat{t}-0) \le 0$.

(iii) If $x_{j-1}(\hat{t}-0) = x_j(\hat{t}-0)$ and $(u_x)_j(t) = p_k$ for t close to $\hat{t}$ and $t<\hat{t}$, then $(u_t)_j(\hat{t}-0) = 0$ and $(u_t)_{j+1}(\hat{t}-0) \le 0$.

(iv) It holds that $g'(\hat{t}) \le 0$ provided that either assumption of (ii) or (iii) holds.

Remark. The assumption $C = 0$ is invoked only to prove the statements for u_t and g'.

Proof. (i) Since $u(\hat{t},\cdot)$ is an admissible crystal and $u(\hat{t},\cdot)-\phi$ takes its maximum in Ω at $\hat{x}$, we see $p_{k-1} \le \phi'(\hat{x}) \le p_k$.

(ii) The first assertion is trivial. Since the transition numbers χ_{j+1} and χ_j cannot equal one, from (8) it follows that

$$\begin{aligned} (u_t)_{j+1}(t) &\le -a(p_{k-1})C(t) = 0, \\ (u_t)_j(t) &\le -a(p_k)C(t) = 0 \end{aligned}$$

for t sufficiently close to $\hat{t}$ and $t < \hat{t}$, where we have used $C = 0$.

(iii) In this case j-th facet on $(x_{j-1}(t), x_j(t))$ vanishes at $\hat{t}$. Its transition number χ_j must be zero as pointed out in Section 1. Since $C = 0$, the equation (8) yields $(u_t)_j(\hat{t} - 0) = 0$. Since $(u_x)_j = p_k$, the transition number χ_{j+1} cannot be one so $(u_t)_{j+1}(\hat{t} - 0) \leq 0$ follows.
(iv) We take y such that

$$x_j(t) < y < x_{j+1}(t) \quad \text{for} \quad \hat{t} - \delta < t \leq \hat{t}$$

by taking $\delta > 0$ sufficiently small. Our assumption

$$\max_Q (u - \psi) = (u - \psi)(\hat{t}, \hat{x})$$

implies that

$$u(t, x_j(t)) - u(\hat{t}, \hat{x}) \leq \psi(t, x_j(t)) - \psi(\hat{t}, \hat{x}), \ \hat{t} - \delta < t \leq \hat{t}.$$

The left hand side equals

$$\begin{aligned} &u(t, y) + p_{k-1}(x_j(t) - y) - u(\hat{t}, y) - p_{k-1}(\hat{x} - y) \\ &\quad = u(t, y) - u(\hat{t}, y) + p_{k-1}(x_j(t) - \hat{x}) \end{aligned}$$

for $t \in (\hat{t} - \delta, \hat{t})$. Multiplying the preceeding inequality with $-(t - \hat{t})^{-1}$ and sending t to $\hat{t}$ with $t < \hat{t}$ yields

$$-u_t(\hat{t} - 0, y) - p_{k-1}\dot{x}_j(\hat{t} - 0) \leq -\phi'(\hat{x})\dot{x}_j(\hat{t} - 0) - g'(\hat{t})$$

or

$$g'(t) \leq (u_t)_{j+1}(\hat{t} - 0) + \dot{x}_j(\hat{t} - 0)(p_{k-1} - \phi'(\hat{x})).$$

By (9) we see

$$\dot{x}_j(\hat{t} - 0) = -\{(u_t)_{j+1}(\hat{t} - 0) - (u_t)_j(\hat{t} - 0)\} \Big/ (p_{k-1} - p_k).$$

Using this relation we end up with

$$g'(\hat{t}) \leq (u_t)_{j+1}(\hat{t} - 0)\left\{1 - \frac{\phi'(\hat{x}) - p_{k-1}}{p_k - p_{k-1}}\right\} + (u_t)_j(\hat{t} - 0)\frac{\phi'(\hat{x}) - p_{k-1}}{p_k - p_{k-1}}. \tag{10}$$

Since $p_{k-1} \leq \phi'(\hat{x}) \leq p_k$, in both cases (ii) and (iii) we conclude $g'(\hat{t}) \leq 0$. ∎

We continue to prove our Theorem with $C = 0$.

Step 1. If $\phi'(\hat{x})$ does not belong to P, then $\hat{x}$ must be the abscissa of a corner of $u(\hat{t}, \cdot)$. As pointed out in Section 1, at most three consecutive $x_j(\hat{t} - 0)$'s may agree with $\hat{x}$. If exactly two $x_j(\hat{t} - 0)$'s agree with $\hat{x}$, then $u_x(\hat{t}, \cdot)$ is constant near $\hat{x}$ which leads $\phi'(\hat{x}) \in P$. Thus it only happens either

$$x_{i-1}(\hat{t} - 0) < x_i(\hat{t} - 0) = \hat{x} < x_{i+1}(\hat{t} - 0) \qquad \text{(see Fig.2)}$$

or

$$x_{i-2}(\hat{t}-0) < x_{i-1}(\hat{t}-0) = x_i(\hat{t}-0) = x_{i+1}(\hat{t}-0) < x_{i+2}(\hat{t}-0) \quad \text{(see Fig.4 and 5).}$$

In the second case since $u(\hat{t},\cdot)-\phi$ takes its maximum at $\hat{x}$, $(u_x)_{i+1}(\hat{t}-0) = p_k, (u_x)_i(\hat{t}-0) = p_{k-1}, (u_x)_{i-1}(\hat{t}-0) = p_k$ if $(u_x)_{i+2}(\hat{t}-0) = p_{k-1}$; Fig.5 is excluded. We now apply Lemma 1 (ii), (iii) to get (iv) $g'(\hat{t}) \leq 0$. Since $\phi'(\hat{x})$ does not belong to P, we have $\Lambda_W(\phi, \hat{x}; 0) = 0$ (note that $C = 0$), which now yields (*). (If $C \not\equiv 0$, instead of (iv) it follows from (10) and estimates of $(u_t)_j$ and $(u_t)_{j+1}$ that

$$g'(\hat{t}) \leq -C(\hat{t})a(\phi'(\hat{x}))$$

provided that a satisfies the assumption in our Main Theorem. Note that (10) is still valid for $C \not\equiv 0$.)

Step 2. Assume that ϕ is faceted at $\hat{x}$. Then the situation is divided into four cases.

Case A (No facet disappearing near $\hat{x}$. See Fig.1).
$x_{i-1}(\hat{t}-0) < x_i(\hat{t}-0) < \hat{x} < x_{i+1}(\hat{t}-0) < x_{i+2}(\hat{t}-0)$ (for some integer i).

Case B (No facet disappearing near $\hat{x}$. See Fig.2).
$x_{i-1}(\hat{t}-0) < \hat{x} = x_i(\hat{t}-0) < x_{i+1}(\hat{t}-0)$ (for some integer i).

Case C (Annihilation of one facet near $\hat{x}$. See Fig.3).
$x_{i-2}(\hat{t}-0) < x_{i-1}(\hat{t}-0) = x_i(\hat{t}-0) < x_{i+1}(\hat{t}-0)$
and $x_{i-2}(\hat{t}-0) < \hat{x} < x_{i+1}(\hat{t}-0)$ (for some integer i).

Case D (Annihilation of two facets near $\hat{x}$. See Fig.4 and 5).
$x_{i-1}(\hat{t}-0) = x_i(\hat{t}-0) = x_{i+1}(\hat{t}-0)$,
$x_{i-2}(\hat{t}-0) < x_{i-1}(\hat{t}-0)$, $x_{i+1}(\hat{t}-0) < x_{i+2}(\hat{t}-0)$
and $x_{i-2}(\hat{t}-0) < \hat{x} < x_{i+2}(\hat{t}-0)$ (for some integer i).

We need to compare Λ_W of ϕ and $u(\hat{t},\cdot)$ near $\hat{x}$. We recall a variant of the maximum principle in [GG].

Lemma 2. *Let f be in $C^2_P(\mathbf{R})$. Let h be an admissible crystal, and let c be a real number. Assume that $f \geq h$ on $\mathbf{R}$.*

(i) If $f = h$ around a point $\overline{x}$, then
$\Lambda_W(f, \overline{x}; c) \geq \Lambda_W(h, \overline{x}; c)$.

(ii) If $R(f,\overline{x}) \cap R(h, x') = \{\overline{x}\}$ for some x' and $f(\overline{x}) = h(\overline{x})$ then
$\Lambda_W(f, \overline{x}; c) \geq \Lambda_W(h, x'; c)$
provided that f' on $R(f,\overline{x})$ equals h' on $R(h,x')^\circ$.

The proof is essentially the same as in [GG] so is omitted. The second part of Lemma 2 corresponds to *proper edge-edge touching* in[GG]. We shall abbreviate $\Lambda_W(f,x;0)$ by $\Lambda(f,x)$.

We now prove (*) in each cases.

Case A. Since $u - \psi$ takes its maximum at $(\hat{t},\hat{x})$ and u is smooth near $(\hat{t},\hat{x})$,

$$\phi'(\hat{x}) = (u_x)_i(\hat{t}) \in P, \quad g'(\hat{t}) = (u_t)_i(\hat{t}).$$

Applying Lemma 2 we obtain

$$\begin{aligned} g'(\hat{t}) &- a(\phi'(\hat{x}))\Lambda(\phi,\hat{x}) \\ &\le (u_t)_i(\hat{t}) - a((u_x)_i(\hat{t}))\Lambda(u(\hat{t},\cdot),\hat{x}) = 0, \end{aligned}$$

since u is a weakly admissible evolving crystal satisfying (4).

Case B. From Lemma 1 it follows that $g'(\hat{t}) \le 0$. This implies $(*)$ if $\Lambda(\phi,\hat{x}) \ge 0$. If $\Lambda(\phi,\hat{x}) < 0$ so that $\chi(\phi,\hat{x}) = -1$, then there is $\overline{x} \ne \hat{x}$ in $R(\phi,\hat{x})^o \cap (x_{i-1}(\hat{t}-0), x_{i+1}(\hat{t}-0))$ such that

$$(u-\psi)(\hat{t},\overline{x}) = \max_Q (u-\psi) = 0.$$

For $(\hat{t},\overline{x})$ the situation now becomes Case A, C or D. If $(*)$ holds for $(\hat{t},\overline{x})$, then $(*)$ holds for $(\hat{t},\hat{x})$ since $\Lambda(\phi,\overline{x}) = \Lambda(\phi,\hat{x})$ and $\phi'(\hat{x}) = \phi'(\overline{x})$. If $x_{i-2}(\hat{t}-0) < x_{i-1}(\hat{t}-0)$ and $x_{i+1}(\hat{t}-0) < x_{i+2}(\hat{t}-0)$, it is clear that Case A occurs so the proof is complete without studying Case C and D.

Combining Case A and Case B we have proved :

Corollary. *Assume that u is an admissible evolving crystal satisfying (4) defined in $[0,\infty)\times\mathbf{R}$. If $C \equiv 0$, then u is a generalized solution of (E).*

We continue to study cases C and D.

Case C. As pointed out in Section 1 we know that on disappearing facets the transition number equals zero [GirK1]. It turns out that there are only six possible pictures (cases (I)-$(I\!I\!I)$, (I')-$(I\!I\!I')$ in Fig.3) of the graph of u at the time $t-\varepsilon$ just before $\hat{t}$. We set $p_k = u_x(\hat{t},\hat{x})$ and observe that

$$(u_x)_{i-1}(\hat{t}-0) = (u_x)_{i+1}(\hat{t}-0) = p_k.$$

In Propositions 1 - 3 we consider Case C.

Proposition 1.

$$g'(t) \le \min\{u_t(\hat{t}-0,y); y \in R(\phi,\hat{x})^o \cap R(u(\hat{t},\cdot),\hat{x})^o \quad \textit{with} \quad u_x(\hat{t}-0,y) = p_k\}.$$

Proof. Since $u-\psi$ takes its maximum at $(\hat{t},y)$ with $y \in R(\phi,\hat{x})^o \cap R(u(\hat{t},\cdot),\hat{x})^o$ and $u_x(\hat{t}-0,y) = p_k$,

$$g'(\hat{t}) \le u_t(\hat{t}-0,y).$$

■

We shall always use this property to estimate $g'(t)$.

Proposition 2. *Assume that j-th facet satisfies*

$$\begin{aligned} &\chi_j(\hat{t}-0) = -1, \quad (u_x)_j(\hat{t}-0) = p_k, \\ &R_j(\hat{t}-0) \subset R(u(\hat{t},\cdot),\hat{x}) \quad \textit{and} \quad R_j(\hat{t}-0) \ne R(u(\hat{t},\cdot),\hat{x}). \end{aligned}$$

Then,

(i) $R_j(\hat{t}-0)^o \cap R(\phi, \hat{x})^o = \phi$,
(ii) $\chi(\phi, \hat{x}) \neq -1$.

Proof. (i) Suppose that there is $y \in R_j(\hat{t}-0)^o \cap R(\phi, \hat{x})^o$. Clearly

$$\max_Q (u - \psi) = (u - \psi)(\hat{t}, y).$$

This implies

$$u_t(\hat{t}+0, y) \leq (u_t)_j(\hat{t}-0) = a(p_k)\Lambda_j(\hat{t}-0)$$

where $\Lambda_j(t) = \chi_j(t)\Delta/L_j(t)$ with $\Delta = W'(p_k+0) - W'(p_k-0)$. Since

$$u_t(\hat{t}+0, y) = a(p_k)\Lambda(u(\hat{t}+0, \cdot), y) = a(p_k)\Lambda(u(\hat{t}, \cdot), y),$$

we now obtain

$$\Lambda(u(\hat{t}, \cdot), y) \leq \Lambda_j(\hat{t}-0).$$

Since $\chi_j(\hat{t}-0) = -1$, so that $\Lambda_j(\hat{t}-0) < 0$, this implies $\chi(u(\hat{t}, \cdot), y) = -1$. Dividing both sides of $\Lambda \leq \Lambda_j$ by Δ and χ_j, we obtain

$$L(u(\hat{t}, \cdot), \hat{x}) = L(u(\hat{t}, \cdot), y) \leq L_j(\hat{t}-0).$$

This contradicts the inclusion relation of $R_j(\hat{t}-0)$ and $R(u(\hat{t}, \cdot), \hat{x})$.
(ii) Suppose that $\chi(\phi, \hat{x}) = -1$. Since $u - \psi$ attains its maximum zero at $(\hat{t}, \hat{x})$, $u(\hat{t}, \cdot) \leq \phi$ in $\mathbf{R}$. This implies $\chi(u(\hat{t}, \cdot), \hat{x}) = -1$ and $R(u(\hat{t}, \cdot), \hat{x}) \subset R(\phi, \hat{x})$. Since $R_j(\hat{t}-0) \subset R(u(\hat{t}, \cdot), \hat{x})$ this contradicts (i) so we have proved $\chi(\phi, \hat{x}) \neq -1$. ∎

Let ϕ be faceted at x_0 with slope p, and let $\overline{x}$ be the right (resp. left) boundary point of $R(\phi, x_0)$. The value $\chi^+(\phi, x_0)$ (resp. $\chi^-(\phi, x_0)$) is assigned to one if $\phi \geq \ell_p$ near $\overline{x}$ and to minus one otherwise, where ℓ_p is a linear function of slope p defined in Section 2. For u at $x_j(t)$ we set

$$\chi_j^c(t) = \begin{cases} 1 & \text{if} \quad (u_x)_{j+1} > (u_x)_j, \\ -1 & \text{if} \quad (u_x)_{j+1} < (u_x)_j. \end{cases}$$

Proposition 3. *(i) Assume that $x_i(\hat{t}-0) \leq \hat{x} < x_{i+1}(\hat{t}-0)$ and $\chi_{i+1}^c(\hat{t}-0) = 1$. Then $\chi^+(\phi, \hat{x}) = 1$.*
(ii) Assume that $x_{i-2}(\hat{t}-0) < \hat{x} \leq x_{i-1}(\hat{t}-0)$ and $\chi_{i-2}^c(\hat{t}-0) = 1$. Then $\chi^-(\phi, \hat{x}) = 1$.

Proof. The proof of (ii) parallels that of (i) so we only present the proof of (i). Suppose that $\chi^+(\phi, \hat{x}) = -1$. Then

$$R(\phi, \hat{x}) \supset \{x > \hat{x}; x \in R(u(\hat{t}, \cdot), \hat{x})\}.$$

Since $u(\hat{t}, \cdot) \leq \phi$ in $\mathbf{R}$, the $i+2$-th facet disappears at $t = \hat{t}$, so that $(u_x)_{i+3} = p_k$ $(= u_x(\hat{t}, \hat{x}))$. Since $\chi_{i+1}^c(\hat{t}-0) = 1$ and $\chi_{i+2}(\hat{t}-0) = 0$, this implies $\chi_{i+2}^c(\hat{t}-0) = -1$. If $\chi_{i+3}^c(\hat{t}-0) = -1$, then $\chi_{i+3}(\hat{t}-0) = -1$ which contradicts Proposition 2 (i). So we may assume $\chi_{i+3}^c(\hat{t}-0) = 1$. Since $u(\hat{t}, \cdot) \leq \phi$ on $\mathbf{R}$, the $i+4$-th facet disapears at $t = \hat{t}$ so that $(u_x)_{i+5} = p_k$ and $\chi_{i+4}^c(\hat{t}-0) = -1$. Since both the $i+2$ and $i+4$-th facets disappear,

$\chi_{i+5}(\hat{t}-0) = -1$ holds, which again contradicts Proposition 2 (i). We have thus proved $\chi^+(\phi, \hat{x}) = 1$. ■

We now complete the proof of Theorem in Case C. We shall only present the proof for cases (I)-(III) in Fig.3 since the proof for case (I')-(III') is obtained by changing x by $-x$.

Case (I). By Proposition 2 (i), $R_{i-1}(\hat{t}-0)^o$ does not intersect $R(\phi, \hat{x})^o$. It follows that

$$x_i(\hat{t}-0) \leq \hat{x} < x_{i+1}(\hat{t}-0).$$

By Proposition 2 (ii), we have $\chi(\phi, \hat{x}) \neq -1$, so that $\chi(\phi, \hat{x}) \geq 0$. Since $g'(\hat{t}) \leq (u_t)_{i+1}(\hat{t}-0) = 0$, we observe that $g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) \leq g'(\hat{t}) \leq 0$, which implies $(*)$.

Case (II). As in Case (I) we may assume $\chi(\phi, \hat{x}) \geq 0$ and $R(\phi, \hat{x})^o$ does not intersect $R_{i-1}(\hat{t}-0)^o$.

Suppose that $\chi(\phi, \hat{x}) = 0$. Since $R(\phi, \hat{x})^o$ does not intersect $R_{i-1}(\hat{t}-0)^o$, we have $\chi^-(\phi, \hat{x}) = 1$ and $\chi^+(\phi, \hat{x}) = -1$. However, this contradicts Proposition 3.

We may now assume $\chi(\phi, \hat{x}) = 1$.

(a) If $R_{i+1}(\hat{t}-0)$ includes $R(\phi, \hat{x})$, then $L(\phi, \hat{x}) \leq L_{i+1}(\hat{t}-0)$, so that $\Lambda(\phi, \hat{x}) \geq \Lambda_{i+1}(\hat{t}-0)$(since $\chi_{i+1}(\hat{t}-0) = 1$). We thus observe that

$$g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) \leq (u_t)_{i+1}(\hat{t}-0) - a(p_k)\Lambda_{i+1}(\hat{t}-0) = 0,$$

which implies $(*)$.

(b) If $R_{i+1}(\hat{t}-0)$ does not include $R(\phi, \hat{x})$ (so that $R(\phi, \hat{x}) \setminus R_{i+1}(\hat{t}-0) \neq \phi$) then the $i+2$-th facet must disappear at $t = \hat{t}$, so that $(u_x)_{i+3} = p_k$ and $\chi_{i+3}(\hat{t}-0) \leq 0$. We thus obtain

$$\begin{aligned} g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) &\leq g'(\hat{t}) \\ &\leq (u_t)_{i+3}(\hat{t}-0) = a(p_k)\Lambda_{i+3}(\hat{t}-0) \leq 0, \end{aligned}$$

which implies $(*)$.

Case (III). Suppose that $\chi(\phi, \hat{x}) \leq 0$.Then $\chi^+(\phi, \hat{x}) = -1$ and/or $\chi^-(\phi, \hat{x}) = -1$. This contradicts Proposition 3, so we may assume $\chi(\phi, \hat{x}) = 1$. If $R(\phi, \hat{x})^o$ intersects $R_{i-1}(\hat{t}-0)^o$, then

$$\begin{aligned} g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) &\leq g'(\hat{t}) \\ &\leq (u_t)_{i-1}(\hat{t}-0) = a(p_k)\Lambda_{i-1}(\hat{t}-0) = 0. \end{aligned}$$

Otherwise, $R(\phi, \hat{x})^o$ does not intersect $R_{i-1}(\hat{t}-0)^o$, then we have $(*)$ as in the same way of the proof of Case (II) with $\chi(\phi, \hat{x}) = 1$. This completes the proof for Case C.

Case D. It turns out that there are only two possible pictures (cases (i) and (ii) in Fig.4 and Fig.5, respectively) of the graph of u just before $\hat{t}$. We observe that $\chi_i(\hat{t}-0) = \chi_{i+1}(\hat{t}-0) = 0$ since the transition number of disappearing facets is zero.

Case (i). First, suppose that $x_{i-2}(\hat{t}-0) < \hat{x} < x_{i-1}(\hat{t}-0)$. If $\chi(\phi, \hat{x}) \geq 0$ then

$$g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) \leq g'(\hat{t}) \leq (u_t)_{i-1}(\hat{t}-0) = a(p_k)\Lambda_{i-1}(\hat{t}-0) < 0,$$

where $p_k = (u_x)_{i-1}(\hat{t}-0)$. If $\chi(\phi, \hat{x}) = -1$ then $R(\phi, \hat{x}) \supset R_{i-1}(\hat{t}-0)$, so that $L(\phi, \hat{t}) \geq L_{i-1}(\hat{t}-0)$. We now obtain $\Lambda(\phi, \hat{x}) \geq \Lambda_{i-1}(\hat{t}-0)$ to get

$$g'(\hat{t}) - a(p_k)\Lambda(\phi, \hat{x}) \leq (u_t)_{i-1}(\hat{t}-0) - a(p_k)\Lambda_{i-1}(\hat{t}-0) = 0.$$

Next, we note that the proof when $x_{i+1}(\hat{t}-0) < \hat{x} < x_{i+2}(\hat{t}-0)$ is the same as the preceeding case. It remains to consider the case when $\hat{x} = x_i(\hat{t}-0)$. Since $u(\hat{t},\cdot)-\phi$ takes its maximum at $\hat{x}$, we have $p_{k-1} \le \phi'(\hat{x}) \le p_k$. Since ϕ is faceted at $\hat{x}$ by the assumption of Step 2, either $\phi'(\hat{x}) = p_{k-1}$ or $\phi'(\hat{x}) = p_k$. If $\phi'(\hat{x}) = p_{k-1}$, then by Lemma 2 for $y \in (x_{i+1}(\hat{t}-0), x_{i+2}(\hat{t}-0))$ we have $\Lambda(\phi,\hat{x}) \ge \Lambda(u(\hat{t},\cdot),y) \ge \Lambda_{i+2}(\hat{t}-0)$. We now observe that

$$g'(\hat{t}) - a(p_{k-1})\Lambda(\phi,\hat{x}) \le (u_t)_{i+2}(\hat{t}-0) - a(p_{k-1})\Lambda_{i+2}(\hat{t}-0) = 0,$$

which is the same as (*). If $\phi'(\hat{x}) = p_k$, then the argument similar to the preceeding case leads $g'(\hat{t}) \le (u_t)_{i-1}(\hat{t}-0)$ and $\Lambda(\phi,\hat{x}) \ge \Lambda_{i-1}(\hat{t}-0)$ by Lemma 2. We obtain

$$g'(\hat{t}) - a(p_k)\Lambda(\phi,\hat{x}) \le (u_t)_{i-1}(\hat{t}-0) - a(p_k)\Lambda_{i-1}(\hat{t}-0) = 0,$$

which is the same as (*).

Case (ii). First, suppose that $x_{i-2}(\hat{t}-0) < \hat{x} < x_{i-1}(\hat{t}-0)$. Then we have $\chi(\phi,\hat{x}) \ge 0$ and $R(\phi,\hat{x})^o \cap R_{i+2}(\hat{t}-0)^o = \phi$. As in the proof of Case C (*II*) (a), if $R(\phi,\hat{x}) \subset R_{i-1}(\hat{t}-0)$ then $\chi(\phi,\hat{x}) = 1$, $L(\phi,\hat{x}) \le L_{i-1}(\hat{t}-0)$, so that $\Lambda(\phi,\hat{x}) \ge \Lambda_{i-1}(\hat{t}-0)$. It then follows that

$$g'(\hat{t}) - a(p_k)\Lambda(\phi,\hat{x}) \le (u_t)_{i-1}(\hat{t}-0) - a(p_k)\Lambda_{i-1}(\hat{t}-0) = 0,$$

where $p_k = (u_x)_{i-1}(\hat{t}-0)$. If $R(\phi,\hat{x}) \setminus R_{i-1}(\hat{t}-0) \neq \phi$, then $i-2$-th facet disappears at $t = \hat{t}$, so that $(u_x)_{i-3} = p_k$ and $\chi_{i-3}(\hat{t}-0) \le 0$. We thus obtain

$$g'(\hat{t}) - a(p_k)\Lambda(\phi,\hat{x}) \le g'(\hat{t}) \le (u_t)_{i-3}(\hat{t}-0) = a(p_k)\Lambda_{i-3}(\hat{t}-0) \le 0.$$

Next, if $x_{i+1}(\hat{t}-0) < \hat{x} < x_{i+2}(\hat{t}-0)$, then the proof parallels the preceeding case. Finally, we note that the point $\hat{x}$ does not agree $x_i(\hat{t}-0)$, since $u-\psi$ attains its maximum at $(\hat{t},\hat{x})$. The proof of our main Theorem is now complete. ■

Remark. The proof of our Theorem is simplified if we use our Existence Theorem when u is periodic in x. Indeed, we only have to prove that an admissible evolving crystal solving (4) is a generalized solution. Let u be a weakly admissible evolving crystal solving (4) and let t_0 be the first time that some facets disappear. Let $\overline{u}$ be a generalized solution with $\overline{u}(0,x) = u(0,x)$. By the unique existence theorem $\overline{u}$ has the semigroup property, i.e. $\overline{u}(t,x)$ is a generalized solution with initial data $\overline{u}(t_0,x)$ at $t = t_0$. Since an admissible evolving crystal solving (4) is a generalized solution, $u(t,x) = \overline{u}(t,x)$ for $t < t_0$. By continuity of u and $\overline{u}$, $u(t_0,x) = \overline{u}(t_0,x)$. Since u is a generalized solution for $t > t_0$ where t is close to t_0, by the semigroup property we have $u = \overline{u}$ for such t. Thus u is a generalized solution across the time $t = t_0$. Repeating this argument we conclude that u is a generalized solution in Q.

This argument is of course simpler than our proof for cases C and D. However, the latter has two advantages.

(i) It is a direct proof without appealing any nontrivial results like Existence Theorem.

(ii) The argument is local so it does not require that u is periodic in x. Moreover it is possible to prove a local version of our main Theorem although we do not present its form here.

Remark. Even if C exists, the proof of Step 2 is similar. The proof of Step 1 needs the extra condition for a as we have pointed out at the end of the proof of Step1. This condition guarantees that the corner point stays corner in the weakly sense [G]; if this condition is missing then at the corner point with $\chi_j^c = -1$ (for $C < 0$) u is not a generalized solution. Such a condition is also appeared in [GSS].

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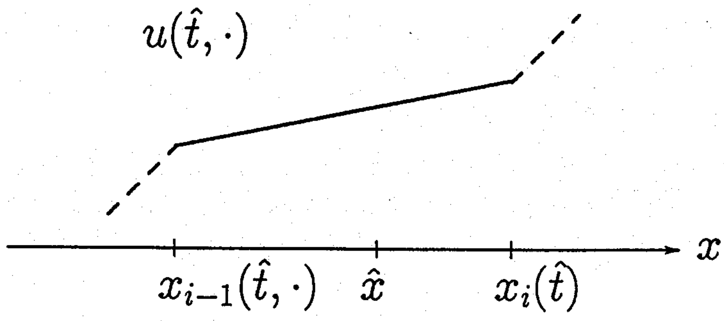

Fig.1. $\hat{x}$ belongs to interior of i-th faceted region $R_i(\hat{t})^\circ$.

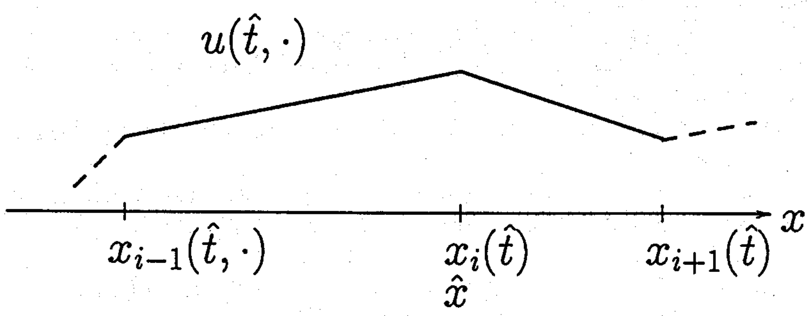

Fig.2. $\hat{x}$ coincides with $x_i(\hat{t})$.

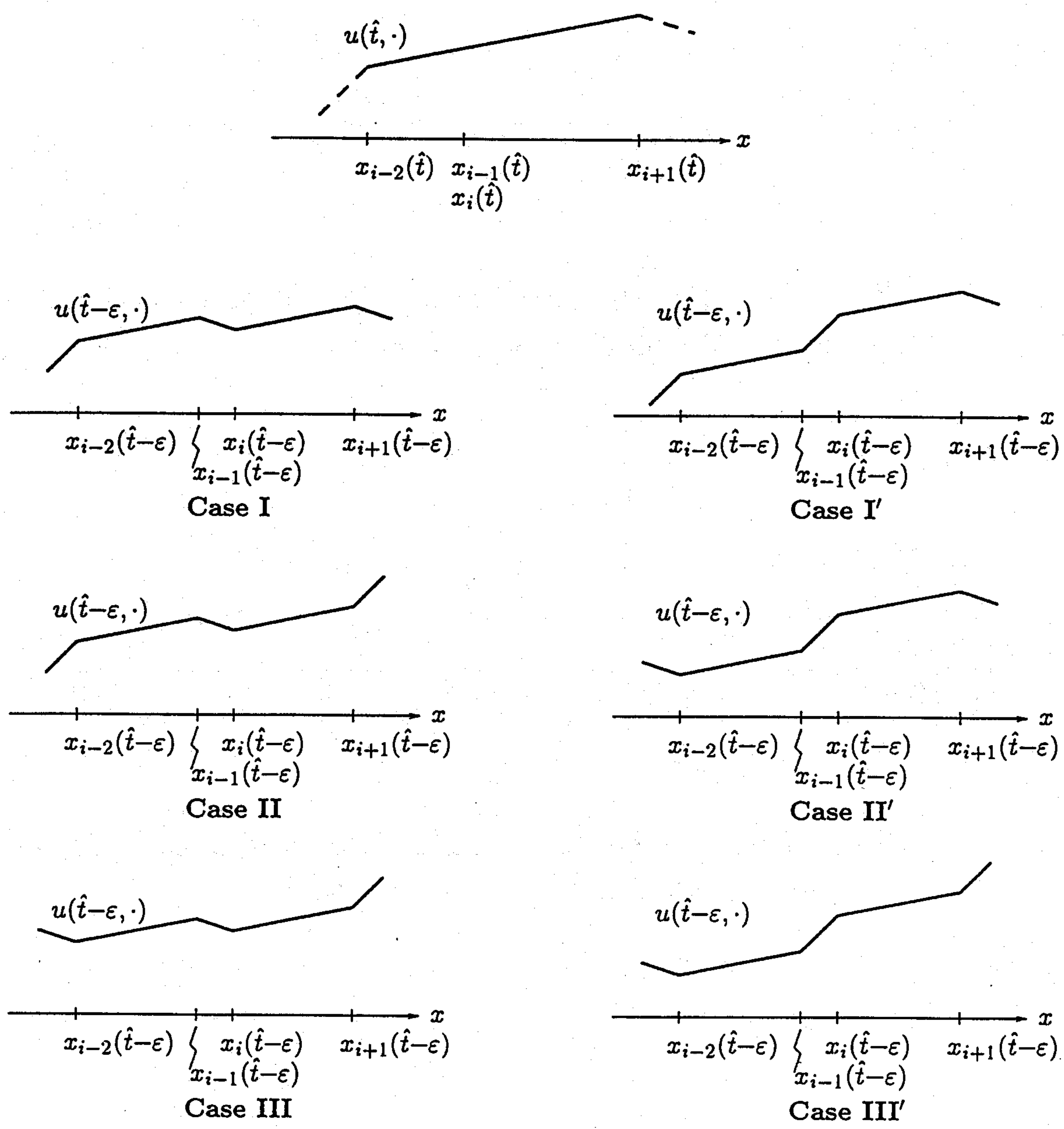

Fig.3. There are 6 cases of i-th facet disappearing without its adjacent neighbors disappearing.

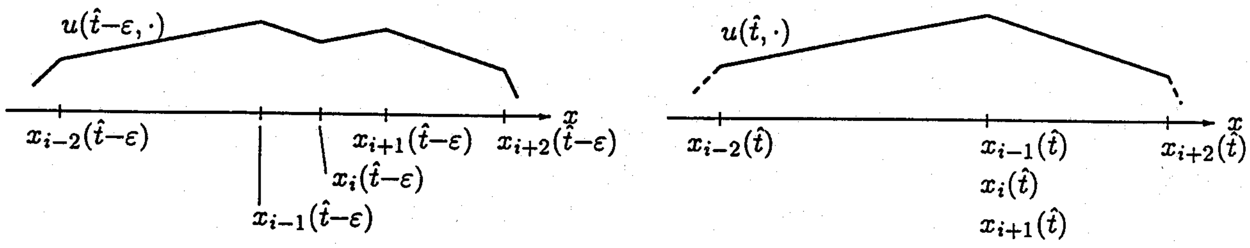

Fig.4. Case(i): i-th and $i+1$-th facets disappear without their adjacent neighbors disappearing.

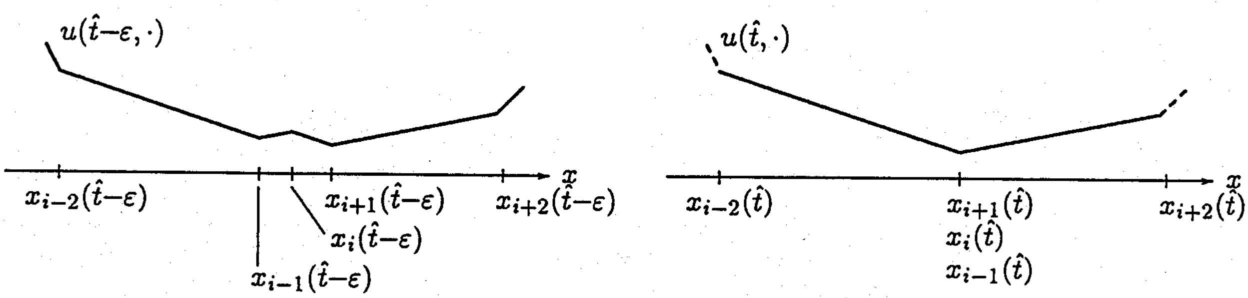

Fig.5. Case(ii): i-th and $i+1$-th facets disappear without their adjacent neighbors disappearing.