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Title	Shadows of moving surfaces
Author(s)	Sun, Wei-Zhi
Citation	Hokkaido University Preprint Series in Mathematics, 318, 1-19
Issue Date	1995-12-1
DOI	10.14943/83465
Doc URL	http://hdl.handle.net/2115/69069
Type	bulletin (article)
File Information	pre318.pdf

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SHADOWS OF MOVING SURFACES

Wei-Zhi Sun

Series #318. December 1995

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SHADOWS OF MOVING SURFACES

WEI-ZHI SUN

Department of Mathematics, North East Normal University, Chang Chun 130024, P.R.CHINA

ABSTRACT. We classify the bifurcation of generic local pictures of shadows for one-parameter families of surfaces in the Euclidean 3-space.

1. INTRODUCTION

In this paper we consider the problem how the bifurcation of shadows for moving surface looks like. A classification of the shadows of generic submanifolds in $\mathbb{R}^n$ was given by Watanabe [12]. In this paper we shall study the normal forms of shadows of one parameter families of surfaces and illustrate how shadows of surfaces change when surfaces move along one parameter in $\mathbb{R}^3$.

One of the motivations for the study of the shadows of surfaces is given in Vision Theory([4],[8]). In [8], Lions et. al. studied the so-called Shape-from-Shading problem. This problem corresponds, roughly speaking, to the reconstruction of a shape (a surface) from the brightness of the two- dimensional image. They studied as an application of the theory of viscosity solutions for various kinds of boundary value problems for a first order Hamilton-Jacobi equation. The boundary in these problems was considered as the edge of the shadows of a surface.

Let $\mathbb{R}^3$ be the Euclidean 3-space with coordinate (x, y_1, y_2). The subset G in $\mathbb{R}^2$ is called the shadow of a surface H in $\mathbb{R}^3$, if G is the image of projection π along a certain direction (for example, x-axis), where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $\pi(x, y_1, y_2) = (y_1, y_2)$.

Let H be a closed surface in $\mathbb{R}^3$. We denote the set of embeddings from H to $\mathbb{R}^3$ by

$$Emb(H, \mathbb{R}^3) = \{i : H \hookrightarrow \mathbb{R}^3 |\ i \text{ is an embedding}\}$$

which is the Borel-space if we adopt the Whitney topology. We consider the following set

$$\mathcal{P} = \{e : H \times I \hookrightarrow \mathbb{R}^3 \times \mathbb{R} | e(p,t) = (i_t(p), t),\ i_t \in Emb(H, \mathbb{R}^3)\},$$

where I is an open interval in $\mathbb{R}$ which includes origin. For any $e \in \mathcal{P}$, e is regarded as a family of elements of $Emb(H, \mathbb{R}^3)$ with a parameter t, and the image $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R}^3 \times \mathbb{R}$.

We suppose that the moving surfaces have the shadow in $\mathbb{R}^2 \times \mathbb{R}$. For any $e \in \mathcal{P}$, the image of $\Pi \circ e$ is called *a shadow* of e, where $\Pi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ is defined

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by $\Pi(x, y_1, y_2, t) = (y_1, y_2, t)$. Hence, our classification of the shadow is the local classification of the bifurcation of the image of $\Pi \circ e$ along the parameter t under the parameterized diffeomorphisms. The precise definition is given as follows.

Definition 1.1. Let D and D' be set germs in $(\mathbb{R}^2 \times \mathbb{R}, 0)$. We say that D and D' are *t-diffeomorphic* if there exist map germs $\hat{\Phi} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\hat{\phi} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\hat{\Phi}(D) = D'$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second components.

Under the above notation, we define $D_t = D \cap (\mathbb{R}^2 \times \{t\})$ and $D'_t = D' \cap (\mathbb{R}^2 \times \{t\})$. If D and D' are $t-$diffeomorphic, $\hat{\Phi}(D_t) = D'_{\hat{\phi}(t)}$, so that the bifurcations of $\{D_t\}_{t \in (\mathbb{R},0)}$ and $\{D'_t\}_{t \in (\mathbb{R},0)}$ along the parameter t are diffeomorphic. Our main result in this paper is the following theorem.

Theorem A. *There exists a residual subset $\mathcal{Q} \subset \mathcal{P}$ with the following property: For any $e \in \mathcal{Q}$ and for any point Y_0 of the shadow $\Pi \circ e(H \times I)$, the number r of singular points of $\Pi \circ e$ in $(\Pi \circ e)^{-1}(Y_0)$ is at most 3 and the set germ of the shadow at Y_0 is $t-$diffeomorphic to one of the following list:*

$r = 1$

${}^pG_k$	normal forms of set germs of the shadows
${}^0G_0$ ${}^0G_2$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^0G_1$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^1G_2^+$ ${}^1G_2^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_3$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid 27y_2^4 - 256y_1^3 - 144y_1y_2^2t$ $+4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

$r = 2$

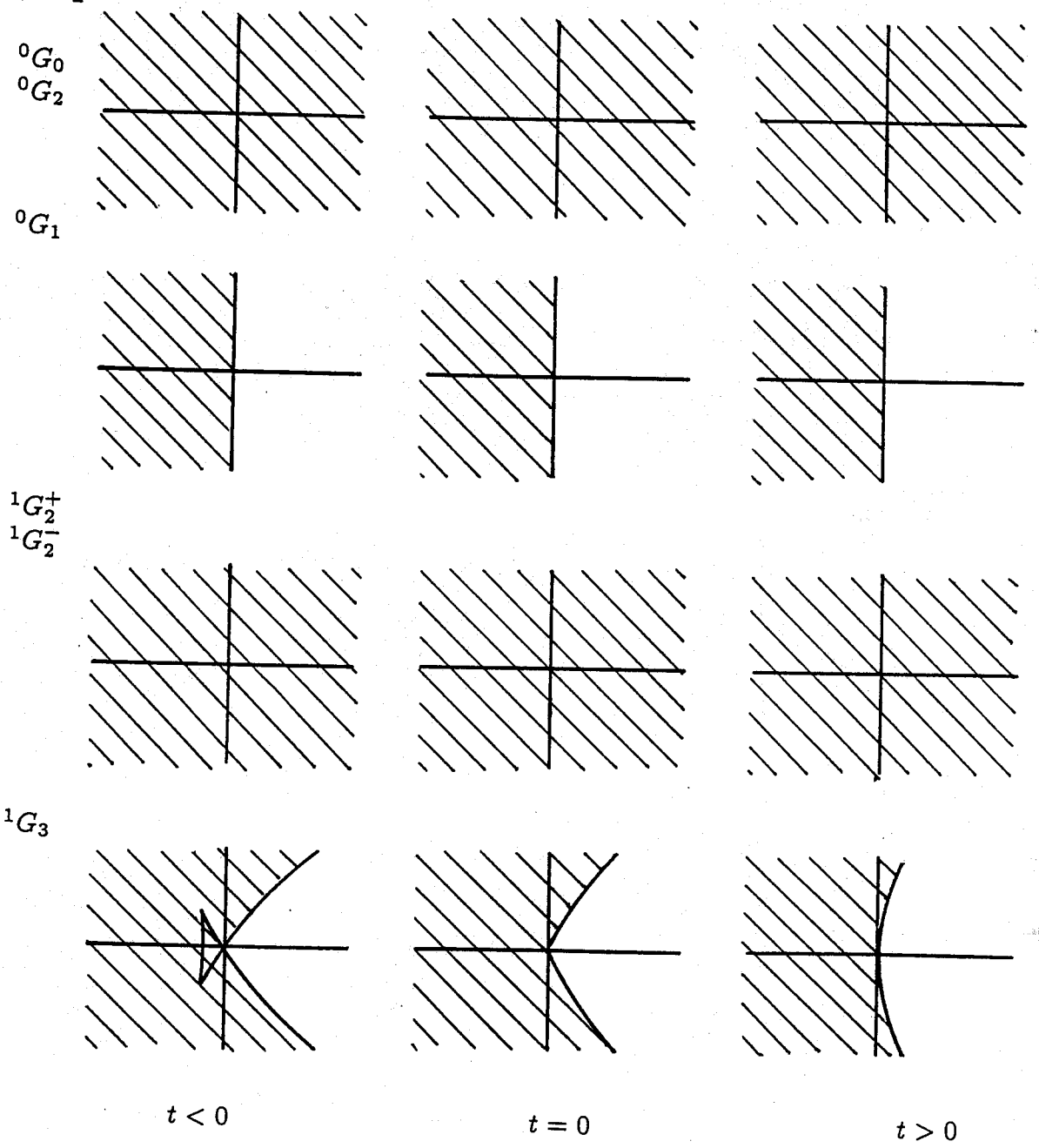
${}^pG_k$	normal forms of set germs of 2-multi- shadows
${}^0G_{1,1}$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup \{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^1G_{1,1}^+$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2^2 + t + y_1 \leqslant 0\}$ $\cup \{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^1G_{1,1}^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2^2 - t + y_1 \leqslant 0\}$ $\cup \{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^1G_{2,1}^+$ ${}^1G_{2,1}^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$ $\cup \{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^1G_{1,2}^+$ ${}^1G_{1,2}^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 + t \leqslant 0\}$ $\cup \{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$

$r = 3$

${}^{p}G_k$	normal forms of set germs of 3-multi- shadows
${}^{1}G^{+}_{1,1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 + y_2 + t \leqslant 0\}$ $\cup\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^{1}G^{-}_{1,1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 + y_2 - t \leqslant 0\}$ $\cup\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$

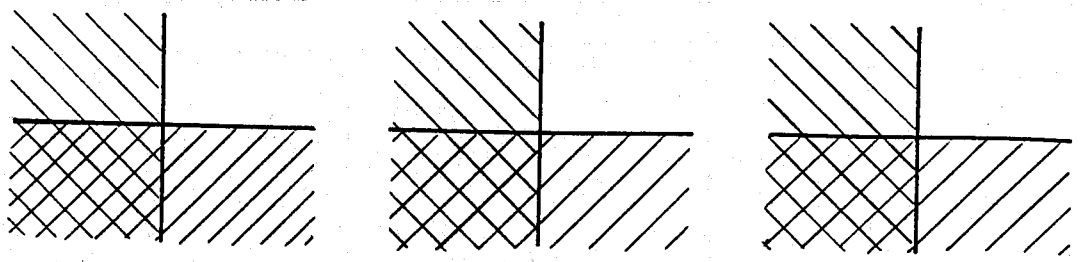
The situation is depicted as follows:

$r = 1$

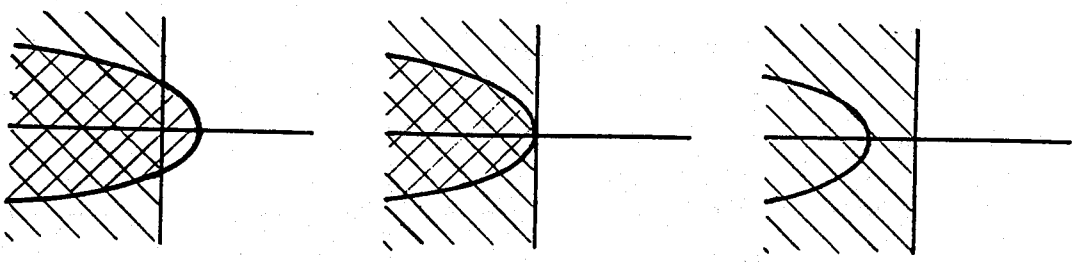


$r = 2$

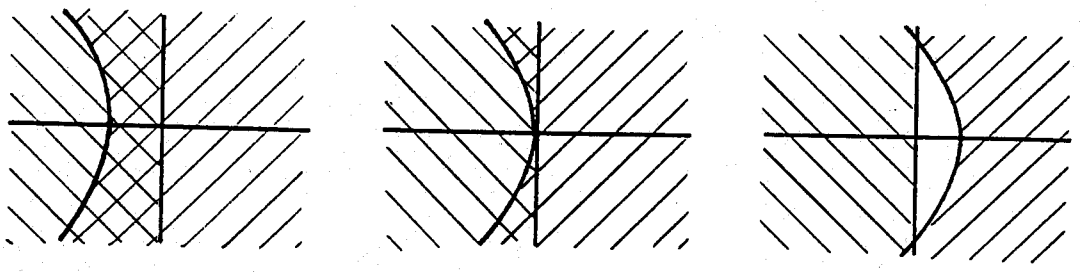
${}^0G_{1,1}$



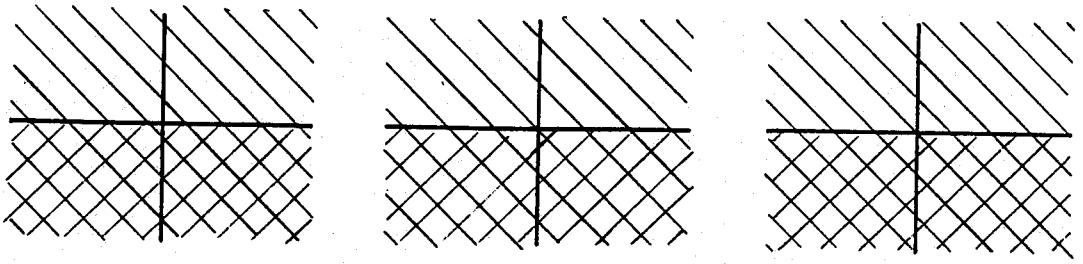
${}^1G^+_{1,1}$



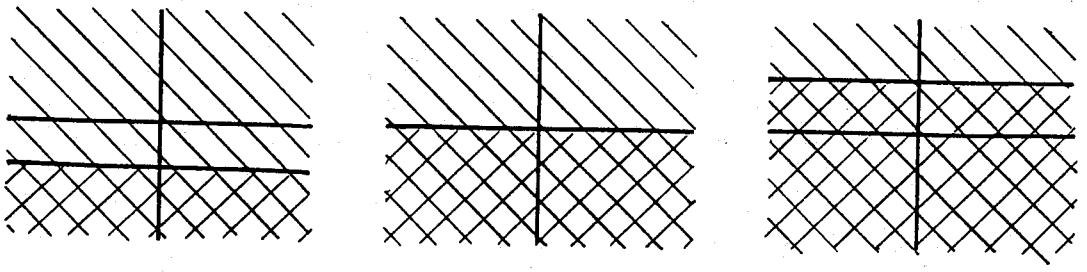
${}^1G^-_{1,1}$



${}^1G^+_{2,1}; {}^1G^-_{2,1}$



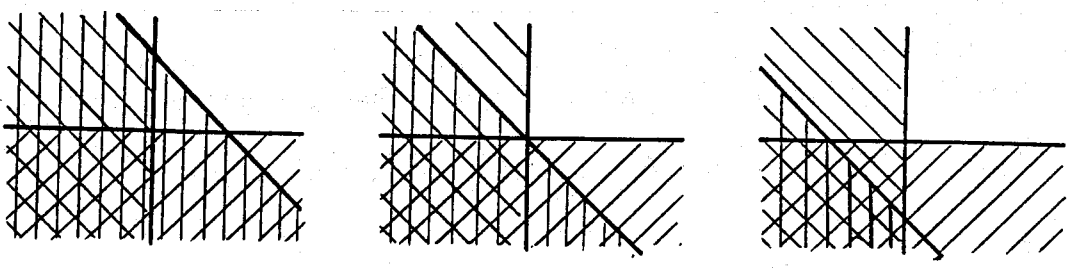
${}^1G^+_{1,2}; {}^1G^-_{1,2}$



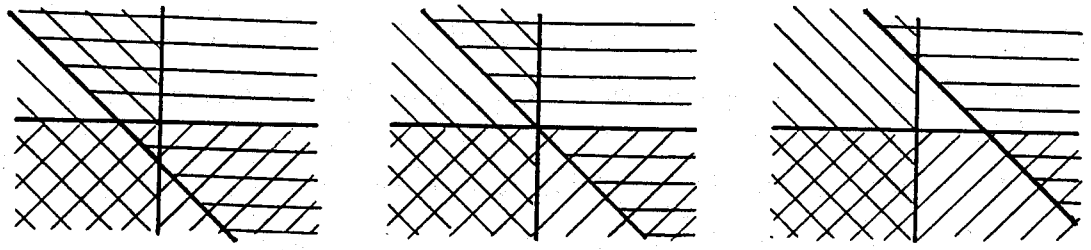
$t < 0$ $\qquad$ $t = 0$ $\qquad$ $t > 0$

$r = 3$

${}^1G^+_{1,1,1}$



${}^1G^-_{1,1,1}$



$t < 0$ $\qquad$ $t = 0$ $\qquad$ $t > 0$

The Theorem A is divided into Theorem 3.1 in §3 and Theorem 4.9 in

The Theorem A is divided into Theorem 3.1 in §3 and Theorem 4.9 in §4. The proof of Theorem 4.9 is almost the same as that of Theorem 3.1, so that we omit the details. The idea of the proof of Theorem 3.1 is summarized as follows: Since the image of e is a hypersurface in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it may be locally considered as a zero point set of a submersion $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. We apply Zakalyukin's classifications([13]) among such function germs up to a certain equivalence relation, which preserves the bifurcation of shadows. We can translate such a classification into the classification of $\Pi_F : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ which corresponds to the local classification of $\Pi \circ e$ around a point. After that we apply the Thom's transversality theorem to detect the generic condition on e. We use the multi-germ version of the above arguments to prove Theorem 4.9.

In §2, we study the local properties of submanifold $e(H \times I)$ around a single point. In §3, we give a proof of Theorem 3.1. In §4, we study the case $r = 2, 3$ and give a proof of Theorem 4.9.

All map germs considered here are differentiable of class C^∞, unless stated otherwise.

2. CLASSIFICATION OF THE LOCAL SHADOWS

In this section we prepare some local theory for the study of shadows.

Let $e \in \mathcal{P}$. For any $(p_0, t_0) \in H \times I$, since $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it follows from the implicit function theorem that there exists a

We call F *a local equation* of e at $e(p_0, t_0)$.

Since we consider the local theory, we may study submersion $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}, 0) \to (\mathbb{R}, 0)$ at the origin.

Definition 2.1. Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and F' are $t-(P-\mathcal{K})-$*equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0)$$

given by

$$\Phi(x, y_1, y_2, t) = (\phi_1(x, y_1, y_2, t), \phi_2(y_1, y_2, t), \phi_3(t))$$

such that

$$\Phi^* < F >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F' >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

where $\mathcal{E}_{(x,y_1,y_2,t)}$ denotes the ring consisting of function germs $(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$.

We remark that the following diagram commutes:

$$\begin{array}{ccc}
(\mathbb{R}, 0) & & (\mathbb{R}, 0) \\
{\scriptstyle F}\uparrow & & \uparrow{\scriptstyle F'} \\
(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow{\Phi} & (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \\
{\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\
(\mathbb{R}^2 \times \mathbb{R}, 0) & \xrightarrow{(\phi_2,\phi_3)} & (\mathbb{R}^2 \times \mathbb{R}, 0) \\
{\scriptstyle \pi_t}\downarrow & & \downarrow{\scriptstyle \pi_t} \\
(\mathbb{R}, t_0) & \xrightarrow{\phi_3} & (\mathbb{R}, t_0')
\end{array}$$

It is clear that $(\phi_2, \phi_3) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\phi_3 : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ are the diffeomorphisms.

Similarly we may define the $t-(P-\mathcal{K})-$*equivalence* for function germs at arbitrary base points. By definition, we have the following proposition.

Proposition 2.2. *Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. If F, F' are $t-(P-\mathcal{K})-$equivalent then $\Pi(F^{-1}(0))$ and $\Pi(F'^{-1}(0))$ are $t-$diffeomorphic.*

Proof. By definition, there exists a diffeomorphism germ $\Phi = (\phi_1, \phi_2, \phi_3)$, such that

$$< F' \circ \Phi >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

so that $F^{-1}(0) = \Phi^{-1}(F'^{-1}(0))$. By the commutative diagram, we obtain

$$(\phi_2, \phi_3)(\Pi(F^{-1}(0))) = \Pi(F'^{-1}(0)).$$

Set $\hat{\Phi} = (\phi_2, \phi_3)$ and $\hat{\phi} = \phi_3$, then we have $\hat{\Phi}(\Pi(F^{-1}(0))) = \Pi(F'^{-1}(0))$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second component. □

For the local case, by Proposition 2.2, it is sufficient to consider the *local shadows* of local equations F, that is, the image of $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. For $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$, we define the subspaces of $\mathcal{E}_{(x,y_1,y_2)}$ by

$$T_e(P-\mathcal{K})(f) = \langle \frac{\partial f}{\partial x}, f\rangle_{\mathcal{E}_{(x,y_1,y_2)}} + \langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}\rangle_{\mathcal{E}_{(y_1,y_2)}}.$$

We also consider their codimensions

$$(P-\mathcal{K})_e - cod(f) = dim_{\mathbb{R}}\mathcal{E}_{(x,y_1,y_2)} \,/\, T_e(P-\mathcal{K})(f).$$

Let $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ, we say that F is a $(P-\mathcal{K}) - versal\ \ deformation$ of $f : (\mathbb{R}\times\mathbb{R}^2\times\{0\}, 0) \to (\mathbb{R}, 0)$ if

$$\langle \frac{\partial F}{\partial t}\ |_{t=0}\rangle_{\mathbb{R}} + T(P-\mathcal{K})_e(f) = \mathcal{E}_{(x,y_1,y_2)}.$$

In [7] Zakalyukin's theorem of the classification is developed to the following theorem which is useful for classification of the local equation.

Theorem 2.3. *Let $F : (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with $(P-\mathcal{K})_e - cod(f) \leqslant 1$, where $f = F|_{\mathbb{R}^2\times\mathbb{R}\times\{0\}}$. If F is $(P-\mathcal{K}) - versal$ deformation of f, then F is generically $t-(P-\mathcal{K}) - equivalent$ to one of germs in the list:*

${}^0A_k$: $\quad x^{k+1} + \sum_{i=1}^{k} y_i x^{i-1} \qquad (0 \leqslant k \leqslant n)$

${}^1A_k$: $\quad x^{k+1} + x^{k-1}(t \pm y_k^2 \pm \cdots \pm y_n^2) + \sum_{i=1}^{k-1} y_i x^{i-1} \qquad (2 \leqslant k \leqslant n+1)$

In the case $n = 2$, by Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let $F : (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with $(P-\mathcal{K})_e - cod(f) \leqslant 1$, where $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$ .If F is a $(P-\mathcal{K}) - versal$ deformation of f, then F is $t-(P-\mathcal{K}) - equivalent$ to one of the following function germs:*

${}^0A_0 : x$
${}^0A_1 : x^2 + y_1$
${}^0A_2 : x^3 + xy_2 + y_1$
${}^1A_2^+ : x^3 + xy_2^2 + tx + y_1$
${}^1A_2^- : x^3 - xy_2^2 + tx + y_1$
${}^1A_3 : x^4 + xy_2 + tx^2 + y_1.$

In order to classifying the shadows, we denote the families of sets of the local shadows by

$${}^pG_k = \{(y_1, y_2, t) \in (\mathbb{R}^2\times\mathbb{R}, 0) | x^{k+1} + \sum_{i=1}^{k-1} y_i x^{i-1} + px^{k-1}(t \pm y_k^2)$$

$$+(1-p)y_k x^{k-1} = 0, \text{for some } x \in (\mathbb{R}, 0)\},$$

where $p = 0, 1, \quad 2p \leqslant k \leqslant p+2, \quad y_k = 0$ as $k > 2$. By Theorem 2.3 we also have the following corollary.

Corollary 2.5. *Let* $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ *be a function germ with* $(P - \mathcal{K})_e - cod(f) \leqslant 1$, *where* $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$. *If* F *is a* $(P - \mathcal{K}) - versal$ *deformation of* f, *then* $\Pi(F^{-1}(0))$ *is* $t-$*diffeomorphism to one of the set germs in the above list* ${}^pG_k$ *(See the following table).*

${}^pG_k$	normal forms of set germs of the shadows
${}^0G_0$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^0G_1$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^0G_2$	$\{(y_1, y_2, 0) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_2^+$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_2^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$
${}^1G_3$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid 27y_2^4 - 256y_1^3 - 144y_1y_2^2t + 4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

Remark. When $p = 1$ and $k = 3$, we observe that $x^4 + tx^2 + xy_2 + y_1$ is $t - (P - \mathcal{K}) - equivalent$ to $x^4 - tx^2 + xy_2 + y_1$.

In order to study the generic properties of $e \in \mathcal{P}$ which respect to the local equation F at $e(p_0, t_0)$, we need some preparations.

Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^∞ germ. In [2] the notion of codimensions of g are defined as follows:

$$(\mathcal{A}) - cod(g) = dim_{\mathbb{R}} \mathfrak{M}_2 \times \mathfrak{M}_2 \,/\, T(\mathcal{A})(g)$$

and

$$(\mathcal{A})_e - cod(g) = dim_{\mathbb{R}} \mathcal{E}_2 \times \mathcal{E}_2 \,/\, T_e(\mathcal{A})(g),$$

where

$$T(\mathcal{A})(g) = \mathfrak{M}_2\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \rangle_{\mathcal{E}_2} + g^*\mathfrak{M}_2 \times g^*\mathfrak{M}_2$$

and

$$T_e(\mathcal{A})(g) = \langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \rangle_{\mathcal{E}_2} + g^*\mathcal{E}_2 \times g^*\mathcal{E}_2.$$

Remark. $T(\mathcal{A})(g)$ and $T(\mathcal{A})_e(g)$ do not depend on the choice of the local coordinates on the source and the target.

In ([1],[6]), we have the following definition. Let $G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a $C^\infty-$ map germ and $g = G|_{\mathbb{R}^2\times\{0\}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. We say that G is an $\mathcal{A} - versal \;\; deformation$ of g if

$$\langle \frac{\partial G}{\partial t} \,|_{t=0} \rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) = \mathcal{E}_2 \times \mathcal{E}_2.$$

We now consider a map germ

$$j_1^\ell G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to J^\ell(\mathbb{R}^2, \mathbb{R}^2) \cong \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2,2)$$

given by

$$j_1^\ell G(x,t) = j^\ell G_t(x).$$

Let $z = j^\ell g(0)$ and $L^\ell(2) \times L^\ell(2)(z)$ be the $\mathcal{A}-$ orbit through z in $J^\ell(2,2)$ (see [2],[5]).

Lemma 2.6. *Suppose that $g = G|_{t=0}$ is $\mathcal{A}$–finitely determined (i.e.,$(\mathcal{A})_e - cod(g) < +\infty$). Under the above notations, for sufficiently large ℓ, the following conditions are equivalent.*

(i) $\tilde{\pi} \circ j_1^\ell G \pitchfork (L^\ell(2) \times L^\ell(2))(z)$.

(ii) *G is an $\mathcal{A}-$versal deformation of g,*

where, $\tilde{\pi} : \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2,2) \to J^\ell(2,2)$ is the canonical projection.

Proof. By the definition of the transversality,

$$\tilde{\pi} \circ j_1^\ell G \pitchfork (L^\ell(2) \times L^\ell(2))(z)$$

if and only if

$$(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R}) + T_z(L^\ell(2) \times L^\ell(2))(z) = T_z J^\ell(2,2).$$

It is also equivalent to the following condition:

(1) $\pi_\ell^{-1}(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R})) + T(\mathcal{A})(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} = \mathfrak{M}_2 \times \mathfrak{M}_2$,

where $\pi_\ell : \mathfrak{M}_2 \times \mathfrak{M}_2 \to J^\ell(2,2)$ is the canonical projection. Therefore we have

$$\pi_\ell^{-1}(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R})) = \langle \pi_\ell^{-1}(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(\frac{\partial}{\partial x_1}),$$

$$\pi_\ell^{-1} d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(\frac{\partial}{\partial x_2}), \pi_\ell^{-1} d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(\frac{\partial}{\partial t})\rangle_{\mathbb{R}} + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1}.$$

Then (1) is equivalent to

(2) $$\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial G}{\partial t}|_{t=0}\rangle_{\mathbb{R}} + T(\mathcal{A})(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} = \mathfrak{M}_2 \times \mathfrak{M}_2.$$

It follows from the definitions of $T(\mathcal{A})(g)$ and $T(\mathcal{A})_e(g)$ that (2) is equivalent to

(3) $$\langle \frac{\partial G}{\partial t} |_{t=0}\rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} = \mathcal{E}_2 \times \mathcal{E}_2.$$

Since g is $\mathcal{A}$-finitely determined, we have

$$\mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} \subset T(\mathcal{A})(g) \quad \text{for some } \ell.$$

Then (3) is equivalent to

$$\langle \frac{\partial G}{\partial t} |_{t=0}\rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) = \mathcal{E}_2 \times \mathcal{E}_2.$$

□

Let $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ such that $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} : (\mathbb{R} \times \mathbb{R}^2 \times \{0\}, 0) \to (\mathbb{R}, 0)$ is a submersion germ. We consider the local projection $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. and $\pi_f = \pi|_{f^{-1}(0)\times\{0\}} : (f^{-1}(0), 0) \to (\mathbb{R}^2 \times \{0\}, 0)$.

By the above remark, $T(\mathcal{A})(\pi_f)$ and $T(\mathcal{A})_e(\pi_f)$ are well-defined.

Under the above notations, we have the following proposition.

Proposition 2.7. *The following conditions are equivalent.*
(i) F *is a* $(P-\mathcal{K})-$*versal deformation of* f.
(ii) $\pi_2 \circ \Pi_F$ *is an* $\mathcal{A}-$*versal deformation of* π_f.
Here $\pi_2 : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *is the canonical projection.*

Proof. Since f is a submersion, we may suppose that $\frac{\partial F}{\partial y_1} \neq 0$ (for the case $\frac{\partial F}{\partial x} \neq 0$ or $\frac{\partial F}{\partial y_2} \neq 0$ are similar), then we write that $F(x, y_1, y_2, t) = y_1 - h(x, y_2, t)$, where $h : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $f(x, y_1, y_2) = F(x, y_1, y_2, 0) = y_1 - h_0(x, y_2)$, where $h_0(x, y_2) = h(x, y_2, 0)$. Set $G_F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ given by $G_F(x, y_2, t) = (h(x, y_2, t), y_2)$ and $g_f(x, y_2) = (h_0(x, y_2), y_2)$. Then $G_F = \pi_2 \circ \Pi_F$ and $g_f = \pi_f$. We consider map germ $I_{h_0} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$I_{h_0}(x, y_2) = (x, h_0(x, y_2), y_2)$$

and we also consider the *pull-back* homomorphism

$$I_{h_0}^* : \mathcal{E}_{(x,y_1,y_2)} \to \mathcal{E}_{(x,y_2)}.$$

Then $ker I_{h_0}^* = \langle y_1 - h_0(x, y_2) \rangle \mathcal{E}_{(x,y_1,y_2)}$ and

$$I_{h_0}^*(T(P-\mathcal{K})_e(f)) = \langle \frac{\partial h_0}{\partial x} \rangle \mathcal{E}_{(x,y_2)} + \langle 1, \frac{\partial h_0}{\partial y_2} \rangle I_{h_0}^* \mathcal{E}_{(y_1,y_2)} \tag{4}$$

We now verify the following equality

$$\mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) = \langle (\frac{\partial h_0}{\partial x}, 0) \rangle \mathcal{E}_{(x,y_2)} + \langle (1,0), (\frac{\partial h_0}{\partial y_2}, 0) \rangle I_{h_0^*} \mathcal{E}_{(y_1,y_2)} \tag{5}$$

Since for any $(\zeta, 0) \in \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f)$, by the definition of $T(\mathcal{A})_e(g_f)$ and (4), we may assume that

$$(\zeta, 0) = (\xi \frac{\partial h_0}{\partial x}, 0) + (\lambda \frac{\partial h_0}{\partial y_2}, \lambda) + (I_{h_0}^* \eta_1, I_{h_0}^* \eta_2)$$

here $\eta_1, \eta_2 \in \mathcal{E}_{(y_1,y_2)}$ and $\xi, \lambda \in \mathcal{E}_{(x,y_2)}$. hence $(\zeta, 0) = (\xi \frac{\partial h_0}{\partial x} - (I_{h_0}^* \eta_2) \frac{\partial h_0}{\partial y_2} + (I_{h_0}^* \eta_1) \cdot 1, 0) \in \langle (\frac{\partial h_0}{\partial x}, 0) \rangle \mathcal{E}_{(x,y_2)} + \langle (\frac{\partial h_0}{\partial y_2}, 0), (1,0) \rangle I_{h_0}^* \mathcal{E}_{(y_1,y_2)}$, that is $(\zeta, 0) \in$ the right hand side of (5). The converse is similar, so we omit it.

By (4) and (5), we have

$$\begin{aligned} \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) &= \langle (\frac{\partial h_0}{\partial x}, 0) \rangle \mathcal{E}_{(x,y_2)} + \langle (1,0), (\frac{\partial h_0}{\partial y_2}, 0) \rangle I_{h_0}^* \mathcal{E}_{(y_1,y_2)} \\ &= I_{h_0}^*(T(P-\mathcal{K})_e(f) \times \{0\} \end{aligned}$$

Then

$$I_{h_0}^* T(P-\mathcal{K})_e(f) \cong I_{h_0}^* T(P-\mathcal{K})_e(f) \times \{0\} = \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f),$$

and $I^*_{h_0}$ induces an $\mathbb{R}$-isomorphism:

$$\mathcal{E}_{(x,y_1,y_2)} \diagup T(P-\mathcal{K})_e(f) \cong \mathcal{E}_{(x,y_2)} \times \{0\} \diagup \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f).$$

On the other hand, since $g_f(x,y_2) = (h_0(x,y_2), y_2)$, it is clear that

$$\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f).$$

Then

$$\begin{aligned}
&\mathcal{E}_{(x,y_1,y_2)} \diagup T(P-\mathcal{K})_e(f) \\
&\cong \mathcal{E}_{(x,y_2)} \times \{0\} \diagup T(\mathcal{A})_e(g_f) \cap \mathcal{E}_{(x,y_2)} \times \{0\} \\
&\cong \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f) \diagup T(\mathcal{A})_e(g_f) \\
&= \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} \diagup T(\mathcal{A})_e(g_f).
\end{aligned}$$

If $\mathcal{E}_{(x,y_1,y_2)} = T(P-\mathcal{K})_e(f)$, by the above equality we have $\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = T(\mathcal{A})_e(g_f)$. Hence (i) holds if and only if (ii) holds. On the other hand, since

$$\frac{\partial G_F}{\partial t} \mid_{t=0} = (\frac{\partial h}{\partial t} \mid_{t=0}, 0) \in \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)}$$

and

$$\frac{\partial F}{\partial t} \mid_{t=0} = -\frac{\partial h}{\partial t} \mid_{t=0} \in \mathcal{E}_{(x,y_2)},$$

the condition

$$dim_{\mathbb{R}} \mathcal{E}_{(x,y_1,y_2)} \diagup T(P-\mathcal{K})_e(f) = 1$$

is equivalent to

$$\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} \diagup T(\mathcal{A})_e(g_f) = 1.$$

In this case, F is a $(P-\mathcal{K})-$versal deformation of f if and only if $\frac{\partial F}{\partial t} \mid_{t=0} \notin T(P-\mathcal{K})_e(f)$. Moreover

$$I^*_{h_0}(\frac{\partial F}{\partial t} \mid_{t=0}) = I^*_{h_0}(\frac{-\partial h}{\partial t} \mid_{t=0}) = (\frac{-\partial h}{\partial t} \mid_{t=0}, 0) = \frac{-\partial G_F}{\partial t} \mid_{t=0}$$

so that

$$\frac{\partial F}{\partial t} \mid_{t=0} \notin T(P-\mathcal{K})_e(f) \text{ if and only if } \frac{-\partial G_F}{\partial t} \mid_{t=0} \notin T(\mathcal{A})_e(g_f)$$

The last condition is equivalent to G_F is an $\mathcal{A}-$versal deformation of g_f.

For other case($\frac{\partial F}{\partial y_2} \neq 0$ or $\frac{\partial F}{\partial x} \neq 0$), the proof is similar, so that we omit these. $\square$

3. Generic property of shadows of the moving surface

In this section we use Thom's k-transversal theorem to show generic property of shadows of the moving surface.

Theorem 3.1. *There exists a dense subset $\mathcal{Q} \subset \mathcal{P}$ with the property: for any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, such that the set germ of the shadow of $e(H \times I)$ at $\Pi \circ e(p_0, t_0)$ is t-diffeomorphic to one of the list ${}^pG_k$*

$$
\begin{aligned}
{}^pG_k = \{(y_1, y_2, t) \in (\mathbb{R}^2 \times \mathbb{R}, 0) | x^{k+1} + \sum_{i=1}^{k-1} y_i x^{i-1} + p x^{k-1}(t \pm y_k^2) \\
+ (1-p) y_k x^{k-1} = 0, \textit{for some } x \in (\mathbb{R}, 0)\}
\end{aligned}
$$

where $p = 0, 1$, and $2p \leqslant k \leqslant p + 2$.

Proof. Take ℓ to be sufficiently large. Let $\hat{S}_j$ be the subsets of $J^\ell(2,2)$ which is the $\mathcal{A}$-orbits through $z = j^\ell(h)_{(0,0)}$ with $(\mathcal{A}) - cod(h) = j$, where $j = 0, 1, 2, 3$. Let Σ be the component of $\cup_{j=0}^3 \hat{S}_j$ in $J^\ell(2,2)$(That is, Σ is the *disjoint union* of the $\mathcal{A}$-orbits through$j^\ell(h)$ with $(\mathcal{A}) - cod(h) \geqslant 4$). Then we have

$$J^\ell(2,2) = \hat{S}_0 \cup \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3 \cup \Sigma.$$

Now we consider the subsets $S_j = H^2 \times \mathbb{R}^2 \times \hat{S}_j$ in $J^\ell(H, \mathbb{R}^2)$. For any $e \in \mathcal{P}$, we definite the $\ell - jet - extension$ map $j_1^\ell e : H \times I \to J^\ell(H, \mathbb{R}^3)$ given by

$$j_1^\ell e(p,t) := j^\ell(i_t(p)),$$

where $i_t = e|_{H \times \{t\}}$.

We also consider the projection ${}^\ell\pi : J^\ell(H, \mathbb{R}^3) \to J^\ell(H, \mathbb{R}^2)$ by

$${}^\ell\pi(j^\ell h(x)) = j^\ell(\Pi \circ h(x))$$

for $h : (H, p_0) \to (\mathbb{R}^3, h(p_0))$ and $\Pi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$.

Since ${}^\ell\pi$ is a submersion and $S_j (j = 0, 1, 2, 3)$ are submanifolds of $J^\ell(H, \mathbb{R}^2)$, ${}^\ell\pi^{-1}(S_j)$ are submanifolds in $J^\ell(H, \mathbb{R}^3)$ and

$$\text{codim of } S_j = \text{codim of } {}^\ell\pi^{-1}(S_j) \qquad (j = 0, 1, 2, 3)$$

Moreover, we can show that

$$j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j) \text{ if and only if } j_1^\ell(\Pi \circ e) \pitchfork S_j.$$

Set

$$\hat{\mathcal{Q}}_j := \{e \in \mathcal{P} |\ j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j)\}, \quad (j = 0, 1, 2, 3).$$

and

$$\mathcal{Q}_\Sigma := \{e \in \mathcal{P} | j_1^\ell(e) \cap {}^\ell\pi^{-1}(\Sigma) = \phi\},$$

By [11], $\mathcal{Q}_\Sigma$ is an algebraic subset of $J^\ell(2,2)$ of codimension $\geqslant 4$.

It follows from Thom's $k-$ transversal Theorem(see [10]), that $\hat{\mathcal{Q}}_j$ are residual subsets of $\mathcal{P}$.

Finally we set

$$\mathcal{Q} = (\cap_{j=0}^{3} \hat{\mathcal{Q}}_j) \cap \mathcal{Q}_\Sigma \subset \mathcal{P}$$

Then $\mathcal{Q}$ is a residual subset in $\mathcal{P}$.

For any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, there exists a neighbourhood U of $e(p_0, t_0)$ and a local equation $F : (U, e(p_0, t_0)) \to (\mathbb{R}, 0)$ of e at $e(p_0, t_0)$, so that $F^{-1}(0) = U \cap e(H \times I)$. Without the loss of generality, $e(p_0, t_0)$ is assumed to be the origin, so that we consider a submersion germ $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. Under the above notation, we may have the following identification:

$$j_1^\ell \Pi \circ e = j_1^\ell \pi_2 \circ \Pi_F.$$

Since $e \in \mathcal{Q}$, $j_1^\ell \pi_2 \circ \Pi_F$ is transversal to S_j. It follows from lemma 2.6, $\pi_2 \circ \Pi_F$ is an $\mathcal{A}-$ *versal deformation* of f. Moreover, by the Proposition 2.7 F is $P - \mathcal{K}-$versal deformation of $f = F|_{\mathbb{R} \times \mathbb{R}^2 \times \{t_0\}}$. Hence we may apply Corollary 2.5 to get the result. □

4. CLASSIFICATION OF MULTI-SHADOWS OF THE MOVING SURFACE

In this section, we consider the local shadows of a generic submanifold $e(H \times I)$ in $\mathbb{R}^3 \times \mathbb{R}$ around r-points $e(p_1, t_0), ..., e(p_r, t_0)$. The all results in this section are the multi-germ version of results in §2, 3, so that we omit the detail of the proofs.

Let $e \in \mathcal{P}$ and $p_1, ..., p_r \in H$, where $p_i \neq p_j$ as $i \neq j$, for $i, j = 1, ..., r$, $e_i : (H \times I, (p_i, t_0)) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, (x_i, 0, 0))$ is a germ of e at $(p_i, t_0)(i = 1, ..., r)$. The r-multiple $e_1 \times \cdots \times e_r$ is called an *r-multi-germ* of $e \in \mathcal{P}$ at $(p_1, t_0), ..., (p_r, t_0)$.

As in §2, for each e_i, there exists a neighborhood U_i of $e(p_i, t_0)$ in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ and a local equation $F_i : (U_i, e(p_i, t_0)) \to (\mathbb{R}, 0)$ such that $F_i^{-1}(0) = U_i \cap e(H \times I)$. The r-multiple $F_1 \times \cdots \times F_r$ is called an *r-multi-local equation* of the r -multi-germ $e_1 \times \cdots \times e_r$ of $e \in \mathcal{P}$ at $(p_1, t_0), ..., (p_r, t_0)$. The r-multi-local equation $F_1 \times \cdots \times F_r$ express the properties of the submanifold $e(H \times I)$ around r points $e(p_1, t_0), ..., e(p_r, t_0)$.

We first consider the local case.

Definition 4.1. Let F_i and $F_i' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs$(i = 1, ..., r)$. We say that $F_1 \times ... \times F_r$ and $F_1' \times ... \times F_r'$ are *$t - (P - \mathcal{K})_r -$ equivalent* if there exists a diffeomorphism germ

$$\Phi_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \quad i = 1, ..., r$$

of the form

$$\Phi_i(x, y_1, y_2, t) = (\phi_1^i(x, y_1, y_2, t), \phi_2(y_1, y_2, t), \phi_3(t)),$$

such that

$$\Phi_i^* \langle F_i' \rangle_{\mathcal{E}(x, y_1, y_2, t)} = \langle F_i \rangle_{\mathcal{E}(x, y_1, y_2, t)}, \quad i = 1, ..., r.$$

Similarly we can define $(P - \mathcal{K})_r$-equivalence for $f_1 \times ... \times f_r = F_1 \times ... \times F_r|_{t=0}$(See [6],[7]).

Firstly, we give tools for classifications of multi-germs under the $t-(P-\mathcal{K})_r$-equivalence. We consider the subset of $\mathcal{E}^r_{(x,y_1,y_2)}=\mathcal{E}_{(x,y_1,y_2)}\times\cdots\times\mathcal{E}_{(x,y_1,y_2)}$ given by

$$T(P-\mathcal{K})_r(f_1\times...\times f_r)=\langle\frac{\partial f_1}{\partial x}\rangle_{\mathfrak{M}_{(x,y_1,y_2)}}\times\cdots\times\langle\frac{\partial f_r}{\partial x}\rangle_{\mathfrak{M}_{(x,y_1,y_2)}}$$

$$+\langle f_1\rangle_{\mathcal{E}_{(x,y_1,y_2)}}\times\cdots\times\langle f_r\rangle_{\mathcal{E}_{(x,y_1,y_2)}}+\langle\frac{\partial(f_1\times...\times f_r)}{\partial y_1},\frac{\partial(f_1\times...\times f_r)}{\partial y_2}\rangle_{\mathfrak{M}_{(y_1,y_2)}}$$

and define

$$(P-\mathcal{K})_r-cod(f_1\times...\times f_r)=dim_{\mathbb{R}}\mathfrak{M}^r_{(x,y_1,y_2)}\,/\,T(P-\mathcal{K})_r(f_1\times...\times f_r),$$

where

$$\mathfrak{M}^r_{(x,y_1,y_2)}=\mathfrak{M}_{(x,y_1,y_2)}\times\cdots\times\mathfrak{M}_{(x,y_1,y_2)}.$$

We also consider the subset of $\mathcal{E}^r_{(x,y_1,y_2)}=\mathcal{E}_{(x,y_1,y_2)}\times\cdots\times\mathcal{E}_{(x,y_1,y_2)}$ given by

$$T((P-\mathcal{K})_e)_r(f_1\times...\times f_r)=\langle\frac{\partial f_1}{\partial x},f_1\rangle_{\mathcal{E}_{(x,y_1,y_2)}}\times\cdots\times\langle\frac{\partial f_r}{\partial x},f_r\rangle_{\mathcal{E}_{(x,y_1,y_2)}}$$

$$+\langle\frac{\partial(f_1\times...\times f_r)}{\partial y_1},\frac{\partial(f_1\times...\times f_r)}{\partial y_2}\rangle_{\mathcal{E}_{(y_1,y_2)}}$$

and define

$$((P-\mathcal{K})_e)_r-cod(f_1\times...\times f_r)=dim_{\mathbb{R}}\mathcal{E}^r_{(x,y_1,y_2)}\,/\,T((P-\mathcal{K})_e)_r(f_1\times...\times f_r).$$

We call that $F_1\times...\times F_r$ is the $(P-\mathcal{K})_r$-versal deformation of $f_1\times...\times f_r$ if

$$\mathcal{E}^r_{(x,y_1,y_2)}=\langle\frac{\partial(F_1\times...\times F_r)}{\partial t}|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}\rangle_{\mathbb{R}}+T((P-\mathcal{K})_e)_r(f_1\times...\times f_r).$$

Similarly we may define $T(\mathcal{K})_r(f_{0,1}\times...\times f_{0,r})$, $(\mathcal{K})_r-cod(f_{0,1}\times...\times f_{0,r})$ and $(\mathcal{K})_r-versal$ deformation of $f_{0,1}\times...\times f_{0,r}$, where $f_{0,i}=f_i|_{\mathbb{R}\times\{0\}\times\{0\}}$ and $i=1,...,r$ (See [2],[3], [9]).

Lemma 4.2. $\sum_{i=1}^r\mathcal{K}-cod(f_{0,i})\leqslant(P-\mathcal{K})_r-cod(f_1\times...\times f_r)\leqslant((P-\mathcal{K})_e)_r-cod(f_1\times...\times f_r)+2$.

where

$$\mathcal{K}-cod(f_{0,i})=dim_{\mathbb{R}}\mathfrak{M}_x\,/\,\langle\frac{\partial f_{0,i}}{\partial x}\rangle_{\mathfrak{M}_x}+\langle f_{0,i}\rangle_{\mathcal{E}_x}.$$

Like as the case when $r=1$, Zakalyukin's theorem([13]) is the key of our classification.

In order to state Zakalyukin's theorem we define the *discriminant* set of r-multi germs $G_1\times...\times G_r$, where $G_i:(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0)\to(\mathbb{R},0)$ is a function-germ$(i=1,...,r)$ as follows:

$$D_G=\cup_{i=1}^r D_{G_i}$$

where

$$D_{G_i}=\{u\in\mathbb{R}^2\times\mathbb{R}|G_i(x,u_1,u_2,u_3)=\frac{\partial G_i}{\partial u_1}(x,u)=\frac{\partial G_i}{\partial u_2}(x,u)=\frac{\partial G_i}{\partial u_3}(x,u)=0,$$

$$\text{for some } x\in(\mathbb{R},0)\}$$

We will utilize the following result for $n=2$.

Proposition 4.3. *(Zakalyukin's Theorem [13]) Let* $t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ *be a submersion germ which is defined on the* $(u_{11}, ..., u_{1m_1}, ..., u_{r1}, ..., u_{rm_r}, u_1, ..., u_\mu)$ *- space, where* $\mu = 3 - \sum_{i=1}^{r} m_i$. *Suppose that*

$$\frac{\partial t}{\partial u_{1m_1}} \neq 0, ..., \frac{\partial t}{\partial u_{rm_r}} \neq 0$$

and

$$t|_{u_{11}=...=u_{rm_r}=0}$$

is a Morse function germ. Then there exists a diffeomorphism germ $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ *preserving the discriminant set* D_F *such that* $t \circ \phi = u_1$ *or* $t \circ \phi = \pm u_{1m_1} \pm ... \pm u_{rm_r} \pm u_1^2 \pm ... \pm u_\mu^2$.

Remark. The submersion t which satisfies the assumption of the Proposition 4.3 is generic.

If $\mathcal{K} - cod(f_{0,i}) = 0$, the germ $f_{0,i}$ is non-singular, so that the image of the neighbourhood of such a point by the projection Π is contained in the inside of shadows. Hence, we may only consider germs with $\mathcal{K} - cod(f_{0,i}) \geqslant 1$. Moreover, we will classify multi-germs with $((P - \mathcal{K})_e)_r - cod(f_1 \times ... \times f_r) \leqslant 1$(see [7]), then we assume that $r \leqslant 3$ by Lemma 4.2.

We can prove the following multi-germ version of Theorem 2.3 by exactly the same way as the single germ case.

Theorem 4.4. *Let* $F_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ *be function germs with* $((P - \mathcal{K})_e)_r - cod(f_1 \times ... \times f_r) \leqslant 1$, *where* $f_i = F_i|_{\mathbb{R} \times \mathbb{R}^2 \times \{0\}}$ *for* $i = 1, ..., r$ *and* $1 < r \leqslant 3$. *If* $F_1 \times ... \times F_r$ *is a* $((P - \mathcal{K})_e)_r -$*versal deformation of* $f_1 \times ... \times f_r$, *then* $F_1 \times ... \times F_r$ *is* $t - (P - \mathcal{K})_r -$ *equivalent to one of the following* r*-multi-germs:*

$${}_2^0A_{1,1} : (x^2 + y_1, \quad x^2 + y_2)$$

$${}_2^1A_{1,1}^+ : (x^2 + t + y_1 + y_2^2, \quad x^2 + y_1)$$

$${}_2^1A_{1,1}^- : (x^2 + t - y_1 - y_2^2, \quad x^2 + y_1)$$

$${}_2^1A_{2,1}^+ : (x^3 + tx + xy_2 + y_1, \quad x^2 + y_2)$$

$${}_2^1A_{2,1}^- : (x^3 + tx - xy_2 + y_1, \quad x^2 + y_2)$$

$${}_2^1A_{1,2}^+ : (x^2 + t + y_2, \quad x^3 + xy_2 + y_1)$$

$${}_2^1A_{1,2}^- : (x^2 + t - y_2, \quad x^3 + xy_2 + y_1)$$

$${}_3^1A_{1,1,1}^+ : (x^2 + t + y_1 + y_2, \quad x^2 + y_1, x^2 + y_2)$$

$${}_3^1A_{1,1,1}^- : (x^2 + t - y_1 - y_2, \quad x^2 + y_1, x^2 + y_2).$$

We can denote the above list to the following form

$${}_r^pA_{k_1k_2k_3} = ({}^pA_{k_1}, {}^pA_{k_2}, {}^pA_{k_3}),$$

where

$$^pA_{k_1} = x^{k_1+1} + \sum_{i=1}^{k_1-1} y_i x^{i-1} + px^{k_1-1}(t \pm y_{k_1+k_2-1} \pm y_{k_1+k_2+k_3-1} \pm y^2_{k_1+k_2+k_3})$$

$$+(1-p)y_{k_1}x^{k_1-1},$$

$$^pA_{k_2} = x^{k_2+1} + p\sum_{i=1}^{k_1+k_2-1} y_{k_1-1+i}x^{i-1} + (1-p)\sum_{i=1}^{k_1+k_2-1} y_{k_1+i}x^{i-1},$$

$$^pA_{k_3} = x^{k_3+1} + p\sum_{i=1}^{k_1+k_2+k_3-1} y_{k_1+k_2-1+i}x^{i-1} + (1-p)\sum_{i=1}^{k_1+k_2+k_3-1} y_{k_1+k_2+i}x^{i-1},$$

for $p = 0,1.$ $r = 2,3.$ $r \leqslant \sum_{i=1}^r k_i \leqslant 3$ and $y_{k_1+k_2-1} = 0$ if $k_1k_2 = 0$, $y_{k_1+k_2+k_3-1} = 0$ if $k_1k_2k_3 = 0$, $^pA_{k_{i+1}} = 0$ if $i = r$.

Proposition 4.5. *If $F_1 \times ... \times F_r$, $F'_1 \times ... \times F'_r$ are $t-(P-\mathcal{K})-$equivalent, then $\Pi(\cup_{i=1}^r F_i^{-1}(0))$, $\Pi(\cup_{i=1}^r {F'}_i^{-1}(0))$ are t-diffeomorphic, where $\Pi : \mathbb{R}\times\mathbb{R}^2\times\mathbb{R} \to \mathbb{R}^2\times\mathbb{R}$ is the canonical projection.*

As a corollary of Theorem 4.4 and Proposition 4.5 we have the following result.

Corollary 4.6. *Under the same assumptions of Theorem 4.4, $\Pi(\cup_{i=1}^r F_i^{-1}(0))$ is t-diffeomorphism to one of following the list*

$$^0_2G_{1,1} \quad ^1_2G^+_{1,1} \quad ^1_2G^-_{1,1} \quad ^1_2G^+_{2,1} \quad ^1_2G^-_{2,1} \quad ^1_2G^+_{1,2} \quad ^1_2G^-_{1,2} \quad ^1_3G^+_{1,1,1} \quad ^1_3G^-_{1,1,1},$$

where

$$^p_2G_{k_1,k_2} = \Pi(^pA_{k_1} = 0) \cup \Pi(^pA_{k_2} = 0),$$

and

$$^p_3G_{k_1,k_2,k_3} = \Pi(^pA_{k_1} = 0) \cup \Pi(^pA_{k_2} = 0) \cup \Pi(^pA_{k_3} = 0)$$

for $p = 0,1.$ $2 \leqslant \sum_{i=1}^r k_i \leqslant 3$

In order to show that the list in corollary 4.5 is a generic local classification of shadows of e, we need some preparations.

Let $g_i, g'_i : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be map-germs $(i = 1,...,r)$. We say that $g_1\times\cdots\times g_r$ and $g'_1 \times \cdots \times g'_r$ are $\mathcal{A}_{r,1} - equivalent$ if there exist diffeomorphism germs

$$\Phi, \Psi_i : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0) \quad (i = 1,\ldots,r)$$

such that

$$\Phi \circ g_i = g'_i \circ \Psi_i. \quad \text{for } i = 1,...,r.$$

We define the tangent space of $\mathcal{A}_{r,1} - orbit$ of $g_1 \times \cdots \times g_r$ is follows.

$$T(\mathcal{A})_{r,1}(g_1 \times \cdots \times g_r) = \langle\frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2}\rangle_{\mathfrak{M}_2} \times ... \times \langle\frac{\partial g_r}{\partial x_1}, \frac{\partial g_r}{\partial x_2}\rangle_{\mathfrak{M}_2} + g^{(r*)}\Delta_r(\mathfrak{M}_2 \times \mathfrak{M}_2),$$

where

$$g^{(r*)} = (g_1^* \times g_1^*) \times ... \times (g_r^* \times g_r^*) : (\mathcal{E}_2 \times \mathcal{E}_2)^r \to (\mathcal{E}_2 \times \mathcal{E}_2)^r,$$

and

$$\Delta_r(\mathfrak{M}_2 \times \mathfrak{M}_2) = \{((h_1, h_2), \ldots, (h_1, h_2)) \in (\mathfrak{M}_2 \times \mathfrak{M}_2)^r | h_1, h_2 \in \mathfrak{M}_2\}.$$

We also consider the codimensions of $T(\mathcal{A})_{r,1}(g_1 \times \cdots \times g_r)$ as follows:

$$\mathcal{A}_{r,1} - cod(g_1 \times \cdots \times g_r) = (\mathfrak{M}_2 \times \mathfrak{M}_2)^r \,/\, T(\mathcal{A})_r(g_1 \times \cdots \times g_r).$$

Similarly, we define the tangent space of $(\mathcal{A}_e)_{r,1} - orbit$ of $(g_1 \times \cdots \times g_r)$ is follows:

$$T(\mathcal{A}_e)_{r,1}(g_1 \times \cdots \times g_r) = \langle \frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2} \rangle_{\mathcal{E}_2} \times ... \times \langle \frac{\partial g_r}{\partial x_1}, \frac{\partial g_r}{\partial x_2} \rangle_{\mathcal{E}_2} + g^{(r*)}\Delta_r(\mathcal{E}_2 \times \mathcal{E}_2),$$

where

$$\Delta_r(\mathcal{E}_2 \times \mathcal{E}_2) = \{((h_1, h_2), \ldots, (h_1, h_2)) \in (\mathcal{E} \times \mathcal{E}_2)^r | h_1, h_2 \in \mathcal{E}_2\}.$$

The codimension is defined by

$$(\mathcal{A}_e)_{r,1} - cod(g_1 \times \cdots \times g_r) = (\mathcal{E}_2 \times \mathcal{E}_2)^r \,/\, T(\mathcal{A}_e)_{r,1}(g_1 \times \cdots \times g_r).$$

Remark. *$T(\mathcal{A})_{r,1}(g_1 \times \cdots \times g_r)$ and $T((\mathcal{A})_e)_{r,1}(g_1 \times \cdots \times g_r)$ depend only the equivalent class under the $\mathcal{A}_{r,1}-$ equivalence.*

We say that $G_1 \times \cdots \times G_r$ is a $\mathcal{A}_{r,1} - versal$ deformation of $g_1 \times \cdots \times g_r$ if

$$\langle \frac{\partial(G_1 \times \cdots \times G_r)}{\partial t}|_{t=0} \rangle_{\mathbb{R}} + T(\mathcal{A}_e)_{r,1}(g_1 \times \cdots \times g_r) = \mathcal{E}_2 \times \mathcal{E}_2$$

Let $F_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, , 0) \to (\mathbb{R}, 0)$ be a submersion-germ, for each $i = 1, ..., r$. Let $f_1 \times \cdots \times f_r = F_1 \times \cdots \times F_r|_{\mathbb{R} \times \mathbb{R}^2 \times \{0\}}$. We consider local projections

$$\Pi_{F_i} = \Pi|_{F_i^{-1}(0)} : (F_i^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$$

defined by

$$\Pi_{F_i}(x, y_1, y_2, t) = (y_1, y_2, t), \qquad \text{for } i = 1, ..., r$$

and

$$\pi_{f_i} = \pi|_{f_i^{-1}(0)} : (f_i^{-1}(0), 0) \to (\mathbb{R}^2, 0)$$

defined by

$$\pi_{f_i}(x, y_1, y_2) = (y_1, y_2), \qquad \text{for } i = 1, ..., r$$

By the above remark, $T(\mathcal{A})_{r,1}(\pi_{f_1} \times \cdots \times \pi_{f_r})$ and $T((\mathcal{A})_e)_{r,1}(\pi_{f_1} \times \cdots \times \pi_{f_r})$ are well–defined.

Proposition 4.7. *The following conditions are equivalent.*
(i) $F_1 \times \cdots \times F_r$ *is* $(P-\mathcal{K})_r -$ *versal deformation of* $f_1 \times \cdots \times f_r$.
(ii) $\Pi_{F_1} \times \cdots \times \Pi_{F_r}$ *is* $\mathcal{A}_{r,1} -$ *versal deformation of* $\pi_{f_1} \times \cdots \times \pi_{f_r}$.

Let $G_i : (\mathbb{R}^2 \times \mathbb{R}, , 0) \to (\mathbb{R}^2, 0)$ be map-germs for $i = 1, \ldots, r$ and $g_i = G_i|_{\mathbb{R}^2 \times \{0\}}$. Now we consider a map germ

$$j_1^\ell G : (\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}, 0) \to J^\ell(\mathbb{R}^2, \mathbb{R}^2) \times \cdots \times J^l(\mathbb{R}^2, \mathbb{R}^2) \qquad (r - \text{times})$$

given by

$$j_1^\ell G(x_1, ..., x_r, t) := (j^\ell G_{1,t}(x_1), ..., j^\ell G_{r,t}(x_r))$$

and the canonical projection

$$\pi : J^l(\mathbb{R}^2, \mathbb{R}^2) \times \cdots \times J^l(\mathbb{R}^2, \mathbb{R}^2) \to J^l(2,2) \times \cdots \times J^l(2,2) \qquad (r - \text{times}).$$

Set $z_i = j^\ell g_i(0)$ for $i = 1, ..., r$.

Proposition 4.8. *Suppose that* $(\mathcal{A}_e)_{r,1} - cod(g_1 \times \cdots \times g_r) < +\infty$. *For sufficiently large* ℓ, *the following conditions are equivalent.*
(i) $\pi \circ j_1^\ell G \pitchfork$ *for sufficiently* ℓ. $\mathcal{A}_{r,1}-$ *orbit through* $(z_1, \ldots, z_r)$
(ii) $G_1 \times \cdots \times G_r$ *is an* $\mathcal{A}_{r,1}-$*versal deformation of* $g_1 \times \cdots \times g_r$,
where $G(x_1, \ldots, x_r, t) = (G_1(x_1, t), \ldots, G_r(x_r, t))$.

Theorem 4.9. *There exists a residual subset* $\mathcal{Q} \subset \mathcal{P}$ *with the following property: For any* $e \in \mathcal{Q}$ *and for any point* Y_0 $(Y_0 \in \Pi \circ e(H \times I))$ *of the shadow of* e, *the germ of* $\Pi \circ e(H \times I)$ *at* Y_0 *is* $t-$ *diffeomorphic to one of the germs in the list in Corollary 4.6.*

proof. We apply Thom's multi-jet transversality theorem (c.f., [3]) like as the proof of the Theorem 3.1. Let $\hat{S}_j$ be the $\mathcal{A}-$ orbit through $z = j^\ell h(0)$ in $J^\ell(2,2)$ with $\mathcal{A} - cod(h) = j$, where $j = 0, 1, 2, 3$. we consider a submanifold

$$S(j_1, ..., j_r) = (\mathbb{R}^2)^{(r)} \times \Delta_r(\mathbb{R}^2) \times \hat{S}_{j_1} \times \cdots \times \hat{S}_{j_r}$$

in the multi-jet space ${}_rJ^\ell(\mathbb{R}^2, \mathbb{R}^2)$, where $\Delta_r(\mathbb{R}^2) = \{(X, \ldots, X) \in (\mathbb{R}^2)^r | X \in \mathbb{R}^2\}$. By the remark after Proposition 4.3, we may assume that $r = 1, 2, 3$. We have already proved for the case $r = 1$ by Theorem 3.1, so that we consider the case $r = 2, 3$. We remark that

$$codim S(j_1, \ldots, j_r) = 2r - 2 + j_1 + \cdots + j_r.$$

Since $dim H^{(r)} \times I = 2r+1$, by the multi-jet transversality theorem, we may consider the following case

(1) $r = 3$, $\qquad j_1 = j_2 = j_3 = 1.$

(2) $r = 2$,

$$\begin{cases} j_1 = 1, j_2 = 2 \\ j_1 = 2, j_2 = 1 \\ j_1 = 1, j_2 = 1 \end{cases}$$

For each case, we also consider the$\mathcal{A}_{r,1}-$orbit through each germs in the case (1) or (2). We also apply multi-transversality theorem for these orbits, we have the result by exactly the same way as the proof of Theorem 3.1, so we omit the details. □

Acknowledgments.

This work was done during the author's stay at Hokkaido University. The author would like to thank Professor S. Izumiya very much for his help and guidance. The author would like to thank Professor G. Ishikawa for the helpful conversation. The author also want to thank Professors T. Suwa, I. Nakai, Dr.Y. Kurokawa, T. Tsukada and Department of Mathematics, Faculty of Science, Hokkaido University for their kind hospitality. Finally, the author would like to especially thank Professor T. Fukuda for his encouragement and support.

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