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An Outer Function And Several Important Functions In Two Variables

by

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Abstract. In one variable, an outer function has several important properties and a function with one of the properties is an outer function. In two variables, the situation is very different and we study these functions.

§1. Introduction

Let m be the normalized Lebesgue measure of the torus T^2, the distinguished boundary of the open unit bidisc D^2 in the space $\mathbf{C}^2$ of two-complex variables (z,w). Let Z be the set of all integers, Z_+ the set of all nonnegative integers, $Z^2 = Z \times Z$ and $Z_+^2 = Z_+ \times Z_+$. For $1 \le p \le \infty$, $L^p = L^p(T^2, m)$ denotes the Lebesgue space and $H^p = H^p(T^2, m) = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0 \text{ if } (\ell, n) \notin Z_+^2\}$, that is, H^p denotes the usual Hardy space on T^2. Let $H_0^p = \{f \in H^p \ ; \ \int f dm = 0\}$ and $K_0^p = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0$ if $-(\ell, n) \in Z_+^2\}$. Then $K_0^p = \{f \in L^q \ ; \ \int fg dm = 0 \text{ if } g \in H^q\}$ where $1/p + 1/q = 1$. Let $\mathcal{H}$ be the set of all trigonometric polynomials in H^p, $\mathcal{H}_0 = H_0^p \cap \mathcal{H}$ and $\mathcal{K}_0$ the set of all trigonometric polynomials in K_0^p

The function h in H^p is called an outer function if

$$\int \log | h | dm = \log | \int h dm | > -\infty.$$

In this paper we study the properties of an outer function in function theory, operator theory, functional analysis and prediction theory. Outer functions have been studied by W.Rudin [13], K.Izuchi [7], K.Izuchi-Y.Matsugu [8] and R.Cheng [2]. In two variables, an outer function is too strong in some sence. If h is a nonzero function in H^p, then the following inequalities are valid. m_z and m_w denote the normalized Lebesgue measures on the unit circle $T = T_z$ and $T = T_w$. $T^2 = T \times T$ and $m = m_z \times m_w$.

$$\int_{T^2} \log | h | dm \ge \int_T (\log | \int_T h dm_z |) dm_w, \int_T (\log | \int_T h dm_w |) dm_z \ge \log | \int_{T^2} h dm |.$$

For any measurable set $E = E_z$ and $E = E_w$ in T_z and T_w, respectively,

$$\int_{T \times E} \log | h | dm \ge \int_E (\log | \int_T h dm_z |) dm_w$$

and

$$\int_{E \times T} \log | h | dm \ge \int_E (\log | \int_T h dm_w |) dm_z.$$

The h is called z-outer for $E = E_w$ if

$$\int_{T \times E} \log | h | dm = \int_E (\log | \int_T h dm_z |) dm_w$$

and the h is called w-outer for $E = E_z$ if

$$\int_{E \times T} \log | h | dm = \int_E (\log | \int_T h dm_w |) dm_z.$$

If h is z-outer for T_w, then h is simply z-outer and a w-outer function is defined similarly. If h is z-outer and w-outer, then h is called weakly outer. See [2],[4] and [8] for these definitions. Suppose $h(z,w) = f(z) - g(w)$ where f and g are nonzero functions in $H^p(T)$ and $f \not\equiv g$. Put $E_w = \{\zeta \in T_w \ ; \ \|f\|_\infty \le | g(\zeta) |\}$ and $E_z = \{\zeta \in T_z \ ; \ | f(\zeta) | \ge \|g\|_\infty\}$.

Then h is z-outer for E_w and h is w-outer for E_z. If $\|f\|_\infty \leq |g|$ *a.e.* on T_w, then h is z-outer and if $|f| \geq \|g\|_\infty$ *a.e.* on T_z, then h is w-outer. Hence if $|f| = 1$ *a.e.* on T_z and $|g| = 1$ *a.e.* on T_w, then h is weakly outer. Moreover if $f(z) - \int g dm_w$ or $\int f dm_z - g(w)$ is outer, then h is outer.

All results in this paper are related to z-outer functions for E_w and w-outer functions for E_z that were just defined above. In Section 3, we study functions whose absolute values have harmonic properties. In Section 4, we study invariant subspaces which are generated by a function. Previously such invariant subspaces were studied in [4], [7] and [8]. We can progress more than they did. In Section 5, we study extreme points of a unit ball of H^1. Such a research is in [5] and [17]. We give a few new results. Sections 4 and 5 are strongly dependent on Section 3. In Section 6, we study the relation between a prediction error and weight. We get a necessary and sufficient condition for an outer function using that. In Section 2, we give four lemmas which will be used in the latter section.

In this paper, for a subset S in L^p, $[S]_p$ denotes the norm closure of S in L^p if $p \neq \infty$ and the weak star closure if $p = \infty$.

§2. Lemmas

We give lemmas which will be used in this paper. Put

$$H_z^p = \{f \in L^p \ ; \ \hat{f}(\ell, n) = 0 \ \text{ if } \ \ell < 0\} \text{ and } H_z^p \cap \bar{H}_z^p = \mathcal{L}_w^p,$$

then $H_z^\infty \cap \bar{H}_z^\infty = \mathcal{L}_w^\infty$ is a commutative von Neumann algebra. If $\mathcal{E}^w$ denotes the conditional expectation from L^∞ to $\mathcal{L}_w^\infty$, then $\mathcal{E}^w$ is multiplicative on H_z^∞ and $H_z^\infty + \bar{H}_z^\infty$ are weak star dense in L^∞. This implies that H_z^∞ is an extended weak-$*$Dirichlet algebra with respect to $\mathcal{E}^w$. Hence we can use the general theory of an extended weak-$*$Dirichlet algebra in [9].

Let χ_E be a characteristic function. Then $\chi_E \in H_z^\infty$ if and only if $\chi_E = \chi_{T\times G}$ for some measurable set $G \subset T$. $\mathcal{L}_w^p$ is isometrically isomorphic to $L^p(T, m_w)$. In fact, $F \in \mathcal{L}_w^p$ and $F(z,w) = f(w)$ is mapped to $f(w) \in L^p(T, m_w)$. For example, $F(z,w) = \chi_{T\times G}$ is mapped to χ_G. Let $1/p + 1/q = 1$. If $F \in \mathcal{L}_w^p$ and $G \in \mathcal{L}_w^q$, then

$$\int_{T^2} FG dm = \int_T fg dm_w$$

where $F(z,w) = f(w)$ and $G(z,w) = g(w)$.

Lemma 1. If F is a function in L^p, then

$$(\mathcal{E}^w F)(z,w) = \int_T F(z,w) dm_z \qquad a.e. w \text{ on } T$$

Proof. We may assume $p = 1$. Put

$$(\Phi F)(w) = \int_T F(z,w)dm_z,$$

then Φ is a norm continuous linear operator from L^1 onto $L^1(T, m_w)$. It is easy to see that $(\mathcal{E}^w F)(z,w) = (\Phi F)(w)$ if F is a polynomial of $z, \bar{z}, w$ and $\bar{w}$. This implies the lemma.

Lemma 2. Suppose h is a nonzero function in H^p. h is z-outer for $E = E_w$ if and only if

$$[\chi_{T\times E} h H_z^\infty]_p = \chi_{T\times E} H_z^p.$$

Proof. If h is z-outer for E, put $\tilde{h} = \chi_{T\times E} h + \chi_{T\times E^c}$. Then

$$\begin{aligned}
&\int \log \mid \tilde{h} \mid dm \\
= &\int_{T\times E} \log \mid h \mid dm + \int_{T\times E^c} \log \mid 1 \mid dm \\
= &\int_E (\log \mid \int_T h dm_z \mid) dm_w + \int_{E^c} (\log \mid \int_T 1 dm_z \mid) dm_w \\
= &\int_T (\log \mid \int_T \tilde{h} dm_z \mid) dm_w.
\end{aligned}$$

Hence, by Lemma 1

$$\int \log \mid \tilde{h} \mid dm = \int \log \mid \mathcal{E}^w(\tilde{h}) \mid dm > 0.$$

By [9, Corollary 3], $\mathcal{E}^w(\log \mid \tilde{h} \mid) \geq \log \mid \mathcal{E}^w(\tilde{h}) \mid$ $\quad a.e.m$ on T^2 and so by the equality above $\mathcal{E}^w(\log \mid \tilde{h} \mid) = \log \mid \mathcal{E}^w(\tilde{h}) \mid$ $\quad a.e.m$ on T^2. Thus

$$\int exp\mathcal{E}^w(\log \mid \tilde{h} \mid) dm = \int \mid \mathcal{E}^w(\tilde{h}) \mid dm$$

and hence by [9, Corollary 5] $[\tilde{h} H_z^\infty]_p = H_z^p$ for $p = 1$. For $p \neq 1$, we can show it similarly if we use the idea in [15]. Since $\chi_{T\times E} \in H_z^p$, this implies the lemma.

Lemma 3. If F is a function in H_z^p, then

$$\int_T F(z,w)dm_w \text{ belongs to } H^p(T_z).$$

Proof. By definition, $\mathcal{E}^z F \in \mathcal{L}_z^p$ and for $\ell \geq 1$

$$\int (\mathcal{E}^z F) z^\ell dm = \int F z^\ell dm = 0$$

because $F \in H_z^p$. Lemma 1 completes the proof.

It is clear that $H_z^p \cap H_w^p = H^p$. The following lemma tells us more.

Lemma 4. If F is in H_z^p and $\chi_{E\times T} F$ is in H_w^p for $m_z(E) > 0$, then F belongs to H^p.

Proof. If $G \subset E$, then $\chi_{G\times T} F \in H_w^p$ because $\chi_{G\times T} \in H_w^p$. Hence for $\ell \geq 1$,

$$\int_G\left(\int_T F(z,w)w^\ell dm_w\right)dm_z = \int(\chi_{G\times T}F)w^\ell dm = 0$$

and so

$$\int_T F(z,w)w^\ell dm_w = 0 \qquad a.e. z \text{ on } E.$$

For $\ell \geq 1$, $w^\ell F \in H^p_z$ because $F \in H^p_z$. By Lemma 3, $\int_T F(z,w)w^\ell dm_w \in H^p(T_z)$ and it is zero on a positive measure set E. Thus $\int_T F(z,w)w^\ell dm_w = 0$ $a.e.z$ on T for all $\ell \geq 1$. Hence for all $\ell \geq 1$ and all $n \in Z$

$$\int Fw^\ell z^n dm = \int_T z^n\left(\int_T F(z,w)w^\ell dm_w\right)dm_z = 0.$$

Thus F belongs to H^p_w and so H^p.

All results about H^p_z, $\mathcal{L}^p_w$ and $\mathcal{E}^w$ in this section are also true for H^p_w, $\mathcal{L}^p_z$ and $\mathcal{E}^z$.

§3. Functions which have harmonic properties

In one variable, it is known that each one of the following three properties (A), (B) and (C) is equivalent to that h is outer. Suppose h is a nonzero function in H^p.

(A) If $|h| \geq |g|$ a.e. on T^2 and $g \in H^p$, then $|h| \geq |g|$ on D^2.

(B) If $|h| = |g|$ a.e. on T^2 and $g \in H^p$, then $|h| \geq |g|$ on D^2.

(C) If $|h| \geq |g|$ a.e. on $T^2, g \in H^p$ and $h^{-1}g$ is a real function on T^2, then $|h| \geq |g|$ on D^2.

The property (A) implies (B) and (C). We don't know whether (B) implies (A). The property (C) does not imply (A).

Theorem 1. Suppose h is a function in H^p. If h is z-outer and w-outer for some measurable set E with $m_z(E) > 0$, it has the property (A).

Proof. If $|h| \geq |g|$ a.e. on T^2 and $g \in H^p$, then we can write $g = \phi h$ for some ϕ in L^∞. Since h is z-outer and w-outer for E, by Lemma 2 $[hH^\infty_z]_p = H^p_z$ and $[\chi_{E\times T}hH^\infty_w]_p = \chi_{E\times T}H^p_w$. Hence there exist $\{h_n\}_{n=1}^\infty$ in H^∞_z and $\{k_n\}_{n=1}^\infty$ in H^∞_w such that

$$\int |hh_n - 1|^p\, dm \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{E\times T} |hk_n - 1|^p\, dm \to 0 \quad \text{as } n \to \infty.$$

This implies that $\phi \in H^p_z$ and $\chi_{E\times T}\phi \in H^p_w$ because $\phi h \in H^p_z \cap H^p_w$. By Lemma 4, ϕ belongs to $H^p \cap L^\infty = H^\infty$.

By Theorem 1, if h is outer or more generally weakly outer, then h has the property (A) and hence the property (C). Theorem 1 implies that the property (A) or (C) does not imply that h is weakly outer. Under the condition that h is w-outer, we can give a necessary and sufficient condition for the property (C). Note that $h_j(z) = \int_T h(z,w)\bar{w}^j dm_w$ belongs to $H^p(T_z)$ by Lemma 3.

Theorem 2. Suppose h is a z-outer function in H^p. h has the property (C) if and only if the common inner divisor of $\{h_j(w)\}_{j=0}^\infty$ is constant, where $h_j(w) = \int_T h(z,w)\bar{z}^j dm_z$ $\in H^p(T_w)$ for $j = 0, 1, 2, \cdots$.

Proof. If $g = \phi h,\ g \in H^p$ and $|\ \phi\ | \leq 1$ $a.e.$ on T^2, then ϕ belongs to H^p_z by Lemma 2 as in the proof of Theorem 1. If ϕ is a real function, then $\phi = \phi(w) \in \mathcal{L}^\infty_w$. If the common inner divisor of $\{h_j(w)\}_{j=0}^\infty$ is constant, then for each j

$$\phi(w)h_j(w) = \int_T \phi(w)h(z,w)\bar{z}^j dm_z = \int_T g(z,w)\bar{z}^j dm_z$$

belongs to $H^p(T_w)$ and hence $\phi \in H^\infty(T_w)$. This implies that h has the property (C). Conversely suppose h has the property (C). If $\{h_j(z)\}_{j=0}^\infty$ has a non-constant common inner divisor $q(z)$, put $\phi(z,w) = (\overline{q(w)} + q(w))/2$ on T^2, then $g = \phi h$ belongs to H^p and $|\ h\ | \geq |\ g\ |$ $a.e.$ on T^2. Since $h^{-1}g = \phi$ is a real function on T^2, by the property (C) of h, $|\ \tilde{h}\ | \geq |\ \tilde{g}\ |$ on D^2 and hence $\tilde{\phi}(z,w) = \tilde{g}(z,w)/\tilde{h}(z,w)$ is bounded on D^2. It is easy to see that $\tilde{\phi}$ is holomorphic in each variable separately. By a theorem of Hartogs ($cf.$[13, p2]), $\tilde{\phi}$ is bounded analytic on D^2. ϕ is the radial limit of $\tilde{\phi}$ and hence ϕ belongs to H^∞. This contradicts that q is non-constant.

§4. Functions generating invariant subspaces

If h is a function in H^2 and $[hH^\infty]_2 = H^2$, then h is called a generator. This means that h is a cyclic vector of two multiplication operators S_z and S_w. If h is a generator, then h is outer [13, Theorem 4.4.6]. Not like the situation in one variable, the convetse is not true [13, Theorem 4.4.8]. If $[hK_0^\infty]_2 = K_0^2$, then h is calld a $\perp$ generator because $K_0^2 = (H^2)^\perp = L^2 \ominus H^2$. This means that h is a cyclic vector of multiplication operators $\{S_z^\ell S_w^m\ ;\ (\ell, m) \notin Z_+^2 \setminus \{0\}\}$. If h is a generator, then h is a $\perp$ generator. However the converse is not true. In one variable, a $\perp$ generator is a generator.

For a nonzero function h in H^2, put $M_h = [hH^\infty]_2$ then M_h is an invariant subspace in H^2. Set $M_h^\times = [\cup_{n=0}^\infty \bar{z}^n M_h]_2 \cap [\cup_{n=0}^\infty \bar{w}^n M_h]_2$, then $M_h^\times$ is also an invariant subspace and $M_h \subset M_h^\times \subset H^2$. It may happen that $M_h^\times = M_h$, $M_h^\times = H^2$ or $M_h \subsetneq M_h^\times \subsetneq H^2$. Set

$$\mathcal{M}(M_h) = \{\phi \in L^\infty \ ; \ \phi M \subset H^2\}.$$

In [11], $M_h^\times$ and $\mathcal{M}(M_h)$ are studied in some detail. For example, $\mathcal{M}(M_h) = \mathcal{M}(M_h^\times)$ and $\mathcal{M}(M_h)$ can be described by using $M_h^\times$. Moreover $M_h^\times$ has a simpler structure than M_h.

Proposition 3. Suppose h is a nonzero function in H^2.

(1) $[M_h + \cup_{n=0}^\infty \bar{z}^n w M_h]_2 \cap [M_h + \cup_{n=0}^\infty \bar{w}^n z M_h]_2 = H^2$ if and only if h is outer.

(2) $M_h^\times = H^2$ if and only if h is weakly outer.

(3) If h is weakly outer, then h is a $\perp$ generator.

Proof. (2) it is easy to see that $H_z^p = [\cup_{n=0}^\infty \bar{w}^n H^p]_p$ and $H_w^p = [\cup_{n=0}^\infty \bar{z}^n H^p]_p$. If h is weakly outer, by Lemma 2 $[hH_z^\infty]_2 = H_z^2$ and $[hH_w^\infty]_2 = H_w^2$. Hence $M_h^\times = H^2$. Conversely if $M_h^\times = H^2$, then $[hH_z^\infty]_2 \supseteq H^2$ and $[hH_w^\infty]_2 \supseteq H^2$ because $[hH_z^\infty]_2 \cap [hH_w^\infty]_2 = H^2$. Thus $[hH_z^\infty]_2 = H^2$ and $[hH_w^\infty]_2 = H^2$. By Lemma 2, h is weakly outer. (1) Put $B_z^p = [H^p + \cup_{n=0}^\infty \bar{z}^n w H^p]_p$ and $B_w^p = [H^p + \cup_{n=0}^\infty \bar{w}^n z H^p]_p$ then B_z^∞ and B_w^∞ are weak-$*$Dirichlet algebras in L^∞. If h is outer, by [15] $[hB_z^\infty]_2 = B_z^2$ and $[hB_w^\infty]_2 = B_w^2$. Hence $[hB_z^\infty]_2 \cap [hB_w^\infty]_2 = H^2$. By the definitions of B_z^∞ and B_w^∞, this implies the 'if' part. The converse can be proved as in the 'only if' part of (2). (3) Since $K_0^2 = wH_z^2 + zH_w^2$, $[hK_0^\infty]_2 = K_0^2$ because $[hH_z^\infty]_2 = H_z^2$ and $[hH_w^\infty]_2 = H_w^2$.

Theorem 4. Suppose h is a nonzero function in H^2.

(1) $\mathcal{M}(M_h) = H^\infty$ if and only if h has the property (A).

(2) $\mathcal{M}(M_h) = H^\infty$ if and only if $[hK_0^\infty]_1 = K_0^1$. In particular, if h is a $\perp$ generator, then $\mathcal{M}(M_h) = H^\infty$.

(3) If h is z-outer and w-outer for some measurable set E with $m_z(E) > 0$, then $\mathcal{M}(M_h) = H^\infty$.

Proof. (1) Suppose $\mathcal{M}(M_h) = H^\infty$. If $| h | \geq | g |$ *a.e.* on T^2 and $g \in H^2$, then $\phi = h^{-1} g \in L^\infty$. Since $g = \phi h \in H^2$, by hypothesis ϕ belongs to H^∞ and hence $| h | \geq | g |$ on D^2. Conversely suppose h has the property (A). If $\phi \in \mathcal{M}(M_h)$ is nonzero and $\|\phi\|_\infty \leq 1$, then $g = \phi h$ belongs to H^2 and so $| h | \geq | g |$ *a.e.* on T^2. By the hypothesis on h, $| h | \geq | g |$ on D^2. As the proof of the 'only if' part of Theorem 2, we can show that ϕ belongs to H^∞.

(2) By [12, Theorem 7], $\mathcal{M}(M_h) = \bar{Q}_z H_z^\infty \cap \bar{Q}_w H_w^\infty$ and $h = Q_z g_z = Q_w g_w$ where Q_z and Q_w are unimodular in H_z^∞ and H_w^∞, respectively, and $[g_z H_z^\infty]_2 = H_z^2$ and $[g_w H_w^\infty]_2 = H_w^2$. If $\mathcal{M}(M_h) = H^\infty$, then $Q_z(zH_z^1) + Q_w(wH_w^1)$ is dense in K_0^1 because $(H_z^\infty)^\perp \cap L^1 = zH_z^1, (H_w^\infty)^\perp \cap L^1 = wH_w^1$ and $(H^\infty)^\perp \cap L^1 = K_0^1$. Hence $h(zH_z^\infty) + h(wH_w^\infty)$ is dense in K_0^1 because $[g_z H_z^\infty]_1 = H_z^1$ and $[g_w H_w^\infty]_1 = H_w^1$. This implies that $[hK_0^\infty]_1 = K_0^1$ because $zH_z^\infty + wH_w^\infty$ is dense in K_0^∞. The proof is reversible and hence if $[hK_0^\infty]_1 = K_0^1$ then $\mathcal{M}(M_h) = H^\infty$. (3) is a result of Theorem 1 and (1).

Suppose $h(z,w) = w - g(z)$ and $| g | \leq 1$ *a.e.* on T. By (3) of Theorem 4, if $E = \{z \in T \ ; \ | g(z) | = 1\}$ and $m_z(E) > 0$, then $\mathcal{M}(M_h) = H^\infty$. This is a result of K.Takahashi which was proved in the different method. In fact, he showed a nice, necessary and sufficient condition for $\mathcal{M}(M_h) = H^\infty$, that is, $\int_T \log(1- \mid g \mid) dm_z = -\infty$.

§5. Extreme points of a unit ball of H^1

In [17], it is noted that if h is an outer function in H^1, then $h/\|h\|_1$ is an extreme point of a unit ball S of H^1. In one variable, the converse is also true [3]. However K.Yabuta [16] showed that $\frac{\pi}{4}(z+w)$ is an extreme point of S. Of course, $z+w$ is not outer but $z+w$ is weakly outer. In general, M.Hasumi [6] showed that if h is weakly outer in H^1, then $h/\|h\|_1$ is an extreme point of S. However the converse is not true by (3) of Theorem 5. (1) of Theorem 5 is due to M.Hasumi [5, Proposition 2]

Theorem 5. Suppose h is a function in S.

(1) h is extreme point of S if and only if $hK_0^\infty + h\bar{K}_0^\infty + \mathbf{C}$ is dense in L^1.

(2) h is an extreme point of S if and only if h has the property (C).

(3) If h is z-outer and w-outer for some measurable set E with $m_z(E) > 0$, then h is an extreme point of S.

(4) Suppose h is a z-outer function in H^1. h is an extreme point of S if and only if the common inner divisor of $\{h_j(w)\}_{j=0}^\infty$ is constant where $h_j(w) = \int_T h(z,w)\bar{z}^j dm_z$ for $j = 0, 1, 2, \cdots$.

Proof. (2) Suppose h is an extreme point of S. Suppose $\mid h \mid \geq \mid g \mid$ *a.e.* on T^2, $g \in H^1$ and $h^{-1}g$ is a real function. Put $\phi = h^{-1}g$, then ϕ is a real function in L^∞. If ϕ is not constant, then $\|\phi - a\|_\infty \neq 0$ for $a = \int \phi dm$. Put $\psi = (\phi - a)/\|\phi - a\|_\infty$, then ψ is a real function, $\psi h \in S$ and $\int \psi dm = 0$. This contradicts [5, Proposition 1] and hence h has the property (C). Conversely suppose h has the property (C). If h is not an extreme point of S, then by [5, Proposition 1] there exists a non-constant real function ψ in L^∞ such that $\psi h \in S$ and $\int \psi dm = 0$. Put $\phi = \psi/\|\psi\|_\infty$ and $g = \phi h$, then $g \in H^1$ and $\mid h \mid \geq \mid g \mid$ *a.e.* on T^2. This contradicts the property (C) of h. Thus h is an extreme point of S. (3) By Theorem 1, h has the property (A) and so (C). Hence (2) implies that h is an extreme point of S. (4) is a result of Theorem 2 and (2).

§6. Weights of Szegő's theorem

For each nonnegative function $W \in L^1$, put

$$S(W) = \inf_{g \in \mathcal{H}_0} \int_{T^2} |1 - g|^2 W dm.$$

In one variable, G.Szegő [15] showed that

$$S(W) = \exp \int \log W dm.$$

In two variables, the author [10] studied and showed that the quantity is a mixed one of an arithmetic mean and a geometric one of W in some special cases. If $W^{-1} \in L^1$, then

$$S(W^{-1})^{-1} \leq \exp \int \log W dm \leq S(W).$$

For arbitrary $W \in L^1$,

$$\lim_{\varepsilon \to 0} S((W + \varepsilon)^{-1})^{-1} \leq \exp \int \log W dm \leq S(W).$$

If $W^{-1} \in L^1$, it is easy to see that $\lim_{\varepsilon \to 0} S((W+\varepsilon)^{-1})^{-1} = S(W^{-1})^{-1}$ (This was pointed out by Professor K.Takahashi). In general, $\lim_{\varepsilon \to 0} S((W + \varepsilon)^{-1})^{-1} \neq S(W^{-1})^{-1}$ may happen. In this section, we show that the difference is that of outer functions and generators. In one variable,

$$\lim_{\varepsilon \to 0} S((W + \varepsilon)^{-1})^{-1} = \exp \int \log W dm = S(W).$$

Theorem 6. Suppose h is a nonzero function in H^2.
(1) $h = qg$ for some inner function q and some generator g if and only if

$$S(|h|^2) = \exp \int \log |h|^2 dm.$$

(2) $h = qg$ for some inner function q and some outer function g if and only if

$$\lim_{\varepsilon \to 0} S((|h|^2 + \varepsilon)^{-1})^{-1} = \exp \int \log |h|^2 dm.$$

Proof. (1) was shown in [10]. (2) For $W \in L^1$, put

$$S^{\perp}(W) = \inf_{g \in \mathcal{K}_0} \int |1 - g|^2 W dm.$$

The author [9] showed that $S(W^{-1})^{-1} = S^{\perp}(W)$ when $W \in L^\infty$ and $W^{-1} \in L^1$. R.Cheng [2] showed this for arbitrary $W \in L^1$ with $W^{-1} \in L^1$. Hence for arbitrary $W \in L^1$ $S((W+\varepsilon)^{-1})^{-1} = S^{\perp}(W + \varepsilon)$ if $\varepsilon > 0$. If $h = qg$ for some inner q and some outer g, then $S^{\perp}(|h|^2) = S^{\perp}(|g|^2) = |\int gdm|^2$ because g is a $\perp$ generator by (3) of Proposition 3. By definition of g, $|\int gdm|^2 = \exp \int \log |h|^2 dm$ and hence $S^{\perp}(|h|^2) = \exp \int \log |h|^2 dm$. By what was just proved above,

$$\lim_{\varepsilon\to 0} S((|\ h\ |^2+\varepsilon)^{-1})^{-1} = \lim_{\varepsilon\to 0} S^{\perp}(|\ h\ |^2+\varepsilon) = \exp\int \log |\ h\ |^2\, dm$$

because

$$\exp\int \log |\ h\ |^2\, dm = S^{\perp}(|\ h\ |^2) \le S^{\perp}(|\ h\ |^2+\varepsilon) \le \exp\int \log(|\ h\ |^2+\varepsilon) dm.$$

Conversely if $\lim_{\varepsilon\to 0} S((|\ h\ |^2+\varepsilon)^{-1})^{-1} = \exp\int \log |\ h\ |^2\, dm$, then $\lim_{\varepsilon\to 0} S^{\perp}(|\ h\ |^2+\varepsilon) = \exp\int \log |\ h\ |^2\, dm$. For any $g \in \mathcal{K}_0, S^{\perp}(|\ h\ |^2+\varepsilon) \le \int |\ 1-g\ |^2\ (|\ h\ |^2+\varepsilon) dm$ and so $\lim_{\varepsilon\to 0} S^{\perp}(|\ h\ |^2+\varepsilon) \le \int |\ 1-g\ |^2 |\ h\ |^2\, dm$. Hence $\lim_{\varepsilon\to 0} S^{\perp}(|\ h\ |^2+\varepsilon) = S^{\perp}(|\ h\ |^2)$ and so $S^{\perp}(|\ h\ |^2) = \exp\int \log |\ h\ |^2\, dm$. Now we will show that $|\ h\ | = |\ g\ |$ for some outer function g in H^2. For $\alpha = (\alpha_1, \alpha_2) \in Z^2$ put $|\ \alpha\ |_r = \alpha_1 - r\alpha_2$ where r is a real number. For $1 < p < \infty$, $\mathcal{L}_r^p$ denotes the space of all $f \in L^p$ whose Fourier coefficients $\hat{f}(\alpha) = 0$ for $\alpha \in Z^2$ with $|\ \alpha\ |_r \ne 0$. If r is irrational then $\mathcal{L}_r^p$ is trivial but if r is rational then $\mathcal{L}_r^p$ is non-trivial. $\mathcal{L}_r^\infty$ is a commutative von Neuman algebra. Let $\mathcal{E}^r$ denote the conditional expectation from L^∞ to $\mathcal{L}_r^\infty$. Let $H_r^\infty = \{f \in L^\infty\ :\ \hat{f}(\alpha) = 0 \text{ if } \alpha \in Z^2 \text{ and } |\ \alpha\ |_r < 0\}$, then $H_r^\infty\ \cap\ \bar{H}_r^\infty = \mathcal{L}_r^\infty$, $\mathcal{E}^r$ is multiplicative on H_r^∞ and $H_r^\infty + \bar{H}_r^\infty$ is weak-*dense in L^∞. That is, H_r^∞ is an extended weak-*Dirichlet algebra with respect to $\mathcal{E}^r$ [8]. By [10, Lemma 1 and (3) of Proposition 1], for any r with $-\infty < r < 0$

$$\exp\int \log |\ h\ |^2\, dm \ge \{\int \exp \mathcal{E}^r(-\log |\ h\ |^2) dm\}^{-1} \ge S^{\perp}(|\ h\ |^2).$$

This implies that $\exp -\int \log |\ h\ |^2\, dm = \int \exp \mathcal{E}^r(-\log |\ h\ |^2) dm$ and hence $\mathcal{E}^r(\log |\ h\ |^2) = \int \log |\ h\ |^2\, dm$ for any r with $-\infty < r < 0$. Thus $(\log |\ h\ |^2)^\wedge(\alpha) = 0$ if $|\ \alpha\ |_r = 0$ and $-\infty < r < 0$. By [13, $p73$ and $p77$] there exists an outer function $g \in H^2$ such that $|\ h\ |^2 = |\ g\ |^2$ (see [10, Lemma 4]).

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