FACTORIZATION OF SELF-ADJOINT OPERATORS BY ABSTRACT DIRAC OPERATORS AND ITS APPLICATION TO SECOND QUANTIZATIONS ON BOSON FERMION FOCK SPACES

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FACTORIZATION OF SELF-ADJOINT OPERATORS BY ABSTRACT DIRAC OPERATORS AND ITS APPLICATION TO SECOND QUANTIZATIONS ON BOSON – FERMION FOCK SPACES

Asao Arai*

We develop an abstract theory on factorization of nonnegative self-adjoint operators in terms of abstract Dirac operators. As an application, we obtain a necessary and sufficient condition for a second quantization on the abstract Boson-Fermion Fock space to be the square of a Dirac type operator and a result on uniqueness of such Dirac type operators.

I. INTRODUCTION

In a previous paper [1], we introduced Dirac type operators acting in the abstract Boson-Fermion Fock space, the tensor product of the Boson Fock space $\mathcal{F}_{\rm b}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ and the Fermion Fock space $\mathcal{F}_{\rm f}(\mathcal{K})$ over a Hilbert space $\mathcal{K}$, and developed index theory for them (cf. also [2] where special classes of Dirac type operators are considered). These Dirac type operators are not only interesting in their own right, but also in applications to physics, since they have concrete realizations in models of supersymmetric quantum field theory [1, 3, 4].

A fundamental class of Dirac type operators in [1] is the class of *free* Dirac operators. This class is parametrized by densely defined closed linear operators S from $\mathcal{H}$ to $\mathcal{K}$. We denote by Q_S the free Dirac operator corresponding to S. In [1] it is shown that $Q_S^2 = d\Gamma_{\rm b}(S^*S) \otimes I + I \otimes d\Gamma_{\rm f}(SS^*)$, where $d\Gamma_{\rm b}(\cdot)$ and $d\Gamma_{\rm f}(\cdot)$ denote the second quantizations on $\mathcal{F}_{\rm b}(\mathcal{H})$ and $\mathcal{F}_{\rm f}(\mathcal{K})$, respectively [13, §VIII.10].

Since the Boson-Fermion Fock space has a natural orthogonal decomposition into two closed subspaces (see §3.1), one can define a Dirac type operator (in an abstract sense) acting there as a self-adjoint operator anticommuting with the grading operator with respect to (w.r.t.) the decomposition. The operator Q_S is an example of such Dirac type operators. As is well known, a second quantization of the form $\Delta(A, B) := d\Gamma_{\rm b}(A) \otimes I + I \otimes d\Gamma_{\rm f}(B)$ with A and B nonnegative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, may be regarded as an infinite dimensional Laplacian if $\dim \mathcal{H} = \infty$. From this point of view, it is interesting to ask if $\Delta(A, B)$ is factorized as the square of a Dirac type operator on the Boson-Fermion Fock space. As the result on the free Dirac operator Q_S described above shows, a sufficient condition for $\Delta(A, B)$ to be the square of a Dirac type operator is given by that there exists a densely defined closed linear operator S from

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$\mathcal{H}$ to $\mathcal{K}$ such that $A = S^*S$ and $B = SS^*$. In this case, $\Delta(A, B) = Q_S^2$. In this paper, we consider a necessary condition for $\Delta(A, B)$ to be the square of a Dirac type operator and uniqueness of such Dirac type operators. A result on this problem is given in Theorem 3.3.

In considering the problem of factorization of second quantizations on the Boson-Fermion Fock space by Dirac type operators, we first develop an abstract theory of factorization of nonnegative self-adjoint operators on a graded Hilbert space $\mathcal{X}$, in which a notion of abstract Dirac operator can be defined. This is done in Section II. The abstract approach makes it possible to clearly understand general aspects associated with the factorization problem of Laplace type operators by Dirac type ones in both finite and infinite dimensional spaces. We give a characterization for abstract Dirac operators whose square are equal each other and for their strong anticommutativity (Theorem 2.1). Moreover, we formulate a necessary and sufficient condition for a nonnegative self-adjoint operator on $\mathcal{X}$ to be the square of an abstract Dirac operator (Theorem 2.3 and Proposition 2.4). Also a result on uniqueness for such abstract Dirac operators is obtained (Theorem 2.3(iii)).

In Section III, we apply the abstract theory in Section II to the factorization problem of the second quantization $\Delta(A, B)$. We first give a characterization for Dirac type operators Q on the Boson-Fermion Fock space such that $Q^2 = Q_S^2$ (Theorem 3.1). Then we obtain a necessary and sufficient condition for operators S and T such that $Q_T^2 = Q_S^2$ (Proposition 3.2). Finally we present a solution to the problem of factorizing $\Delta(A, B)$ in terms of Dirac type operators (Theorem 3.3).

II. FACTORIZATION OF SELF-ADJOINT OPERATORS BY ABSTRACT DIRAC OPERATORS

Let $\mathcal{X}$ be a complex Hilbert space and γ be a grading operator on $\mathcal{X}$, i.e., γ is a bounded self-adjoint operator satisfying $\gamma^2 = I, \gamma \neq \pm I$, so that the spectrum of γ is $\{\pm 1\}$, where I denotes identity. We say that a linear operator Q on $\mathcal{X}$ is an *abstract Dirac operator* w.r.t. γ if it is self-adjoint and

$$\gamma Q \subset -Q\gamma. \tag{2.1}$$

We remark that condition (2.1) implies that $\gamma D(Q) = D(Q)$, where $D(\cdot)$ denotes operator domain. Hence (2.1) is in fact equivalent to the condition that $\gamma Q = -Q\gamma$ (operator equality)[1].

In this section, we consider a factorization problem of self-adjoint operators on $\mathcal{X}$ in the following sense: Given a nonnegative self-adjoint operator H on $\mathcal{X}$, find a necessary and sufficient condition for H to be the square of an abstract Dirac operator w.r.t. γ.

[1] The notion of abstract Dirac operator given here is different from that in [16, p.139]. One can obtain a weaker notion of abstract Dirac operator by replacing "self-adjoint" by "symmetric" in the definition above. This weaker notion may be useful in considering Dirac type operators with singularities (e.g., [7,8]). Spectral theory for self-adjoint extensions for such Dirac operators, which may have connections with results presented in this section, is given in [10].

To solve the problem, we first note that $\mathcal{X}$ admits the orthogonal decomposition

$$\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_- = \left\{ \begin{pmatrix} \psi \\ \phi \end{pmatrix} \middle| \psi \in \mathcal{X}_+, \phi \in \mathcal{X}_- \right\} \tag{2.2}$$

with $\mathcal{X}_\pm$ the eigenspaces of γ with eigenvalues ± 1. Relatively to the decomposition (2.2), every linear operator on $\mathcal{X}$ is represented as a 2×2 matrix with entries being linear operators. For example, we have

$$\gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.3}$$

Let Q be an abstract Dirac operator w.r.t. γ. Then (2.1) and the self-adjointness of Q imply that there exists a unique densely defined closed linear operator $A_Q : \mathcal{X}_+ \to \mathcal{X}_-$ such that

$$Q = \begin{pmatrix} 0 & A_Q^* \\ A_Q & 0 \end{pmatrix} \tag{2.4}$$

with $D(A_Q) = D(Q) \cap \mathcal{X}_+$ and $D(A_Q^*) = D(Q) \cap \mathcal{X}_-$.

We say that two self-adjoint operators S and T on a Hilbert space *strongly anticommute* if $e^{itS}T \subset Te^{-itS}$ for all $t \in \mathbf{R}$ [12][2].

The following theorem gives a characterization for abstract Dirac operators whose square are equal each other and for their strong anticommutativity.

THEOREM 2.1. *Let Q_1 be an abstract Dirac operator w.r.t. γ.*
(i) Suppose that Q_2 is an abstract Dirac operator w.r.t. γ such that

$$Q_1^2 = Q_2^2. \tag{2.5}$$

Then there exists a partial isometry Θ on $\mathcal{X}$ with initial space including $(\ker Q_1)^\perp$ such that

$$\Theta\gamma = \gamma\Theta \tag{2.6}$$

and

$$Q_2 = \Theta Q_1. \tag{2.7}$$

The operator Θ is uniquely determined by the condition that $\ker \Theta = \ker Q_1$.

(ii) Conversely, suppose that there exists a partial isometry Θ on $\mathcal{X}$ with initial space including $(\ker Q_1)^\perp$ such that (2.6) and

$$\Theta Q_1 = Q_1 \Theta^* \tag{2.8}$$

[2] In the terminology of [17, 15, 12, 5, 6], the self-adjoint operators S and T having this property are simply said to anticommute.

hold. Then the operator Q_2 defined by (2.7) is an abstract Dirac operator w.r.t. γ such that (2.5) holds.

(iii) In each case of part (i) and (ii), Q_1 and Q_2 strongly anticommute if and only if

$$\Theta Q_1^2 \subset -Q_1^2\Theta^*. \tag{2.9}$$

Remarks. (a) The self-adjointness of Q_2 and (2.7) imply (2.8).

(b) Part (iii) gives a characterization for abstract Dirac operators Q_1 and Q_2 w.r.t. γ which satisfy $Q_1^2 = Q_2^2$ and strongly anticommute. In the context of supersymmetric quantum mechanics (e.g., [9]), this result completely characterizes the structure of Hilbert space representations of a supersymmetry algebra.

(c) An example of Θ satisfying (2.6), (2.8) and (2.9) is given by $\Theta = i\gamma$. Hence $Q_2 := i\gamma Q_1$ and Q_1 strongly anticommute and satisfy (2.5). In supersymmetric quantum mechanics, the operator Q_2 in this example is called a supercharge associated with Q_1 [16, p.140][3].

Proof. (i) By the assumption, we have $L := |Q_1| = |Q_2|$. By the polar decomposition, we can write $Q_j = U_jL, j = 1,2$, with U_j a partial isometry whose initial space include $(\ker L)^\perp = (\ker Q_1)^\perp = (\ker Q_2)^\perp$. Since γ and Q_j anticommutes, it follows that γ anticommutes with U_j. We have $L = U_1^*Q_1$. Hence $Q_2 = \Theta Q_1$ with $\Theta = U_2U_1^*$. It is easy to see that Θ is a partial isometry with initial space including $(\ker Q_1)^\perp$ and (2.6) holds. The uniqueness of Θ follows from the fact that $R(Q_1)$ (the range of Q_1) is dense in $(\ker Q_1)^\perp$.

(ii) The self-adjointness of Q_2 follows from (2.8). Since $D(Q_2) = D(Q_1)$, it follows that γ leaves $D(Q_2)$ invariant. For all $\psi \in D(Q_1) = D(Q_2)$, we have $\gamma Q_1\psi = -Q_1\gamma\psi$. Hence $\Theta\gamma Q_1 = -Q_2\gamma\psi$. By (2.6), the left hand side is equal to $\gamma Q_2\psi$. Hence $\gamma Q_2\psi = -Q_2\gamma\psi$. Thus Q_2 is an abstract Dirac operator w.r.t. γ. It is easy to see that (2.5) holds.

(iii) Suppose that Q_1 and Q_2 strongly anticommute. Then, for all $\psi, \phi \in D(Q_1) = D(Q_2)$ and $t \in \mathbf{R}$, we have $(Q_1\psi, e^{itQ_2}\phi) = (e^{itQ_2}\psi, Q_1\phi)$. Differenciating this equation in t at $t = 0$, we obtain

$$(Q_1\psi, Q_2\phi) + (Q_2\psi, Q_1\phi) = 0. \tag{2.10}$$

Putting (2.7) into this equation and using (2.8), we obtain for all $\psi, \phi \in D(Q_1^2)$

$$(\Theta Q_1^2\psi, \phi) = -(\Theta^*\psi, Q_1^2\phi),$$

which implies that $\Theta^*\psi \in D(Q_1^2)$ and $-Q_1^2\Theta^*\psi = \Theta Q_1^2\psi$. Thus (2.9) follows.

Conversely, suppose that (2.9) holds. Then, using (2.8), we see that (2.10) holds for all $\psi \in D(Q_1^2)$ and $\phi \in D(Q_1) = D(Q_2)$. Since $D(Q_1^2)$ is a core of Q_1, for every $\psi \in D(Q_1)$, there exists a sequence $\{\psi_n\}_{n=1}^\infty \subset D(Q_1^2)$ such that $\psi_n \to \psi, Q_1\psi_n \to$

[3]In [16], however, the notion of strong anticommutativity is not explicitly used.

$Q_1\psi$ $(n \to \infty)$. Then, by (2.7), we have $Q_2\psi_n \to Q_2\psi$ $(n \to \infty)$. Hence we obtain (2.10) for all $\psi, \phi \in D(Q_1) = D(Q_2)$. Then, applying [6, Theorem 6.3], we conclude that Q_1 and Q_2 strongly anticommute. ∎

For the reader's convenience, we cite a well known fact as a lemma (e.g., [11, p.334], [16, p.143, Theorem 5.5]).

LEMMA 2.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $A : \mathcal{H} \to \mathcal{K}$ be a densely defined closed linear operator. Then A^*A and AA^* are nonnegattive self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, and there exists a unique partial isometry $U : \mathcal{H} \to \mathcal{K}$ with initial space $(\ker A)^\perp$ and final space $(\ker A^*)^\perp$ such that*

$$A = U(A^*A)^{1/2} = (AA^*)^{1/2}U.$$

We have

$$A^* = U^*(AA^*)^{1/2} = (A^*A)^{1/2}U^*.$$

In particular,

$$UA^*AU^* = AA^*. \tag{2.11}$$

Remark. Relation (2.11) implies that A^*A restricted to $(\ker A)^\perp$ is unitarily equivalent to AA^* restricted to $(\ker A^*)^\perp$, a well known fact (e.g., [16, p.144]).

Let H be a nonnegative self-adjoint operator on $\mathcal{X}$. The following theorem gives a necessary condition for H to be the square of an abstract Dirac operator as well as a complete characterization of abstract Dirac operators whose square are equal to H.

THEOREM 2.3. *Suppose that there exists an abstract Dirac operator Q w.r.t. γ such that $H = Q^2$. Then:*

(i) H is reduced by $\mathcal{X}_\pm$. We denote the reduced parts of H to $\mathcal{X}_\pm$ by $H_\pm$.

(ii) There exists a unique partial isometry U from $\mathcal{X}_+$ to $\mathcal{X}_-$ with initial space $(\ker H_+)^\perp$ and final space $(\ker H_-)^\perp$ such that

$$Q = \begin{pmatrix} 0 & H_+^{1/2}U^* \\ UH_+^{1/2} & 0 \end{pmatrix} \tag{2.12}$$

and

$$UH_+^{1/2} = H_-^{1/2}U. \tag{2.13}$$

In particular,

$$UH_+U^* = H_-. \tag{2.14}$$

(iii) For every abstract Dirac operator Q' w.r.t. γ such that $H = Q'^2$, there exists a partial isometry Θ on $\mathcal{X}$ with initial space including $(\ker H)^\perp$ such that (2.6) and

$$Q' = \Theta Q \tag{2.15}$$

hold. The operator Θ *is uniquely determined by the condition that* $\ker \Theta = \ker H$.

Proof. The contents of part (i) and (ii) of this theorem are essentially same as the content of Lemma 2.2 in [9]. But, for the sake of completeness, we prove them (in a slightly different way). Since Q is an abstract Dirac operator, Q is uniquely written as (2.4) with A_Q a densely defined closed linear operator from $\mathcal{X}_+$ to $\mathcal{X}_-$. Hence we have

$$H = \begin{pmatrix} A_Q^* A_Q & 0 \\ 0 & A_Q A_Q^* \end{pmatrix},$$

which implies that H is reduced by $\mathcal{X}_\pm$ with $H_+ = A_Q^* A_Q$ and $H_- = A_Q A_Q^*$. This proves part (i).

By Lemma 2.2, there exists a unique partial isometry $U : \mathcal{X}_+ \to \mathcal{X}_-$ with initial space $(\ker A_Q)^\perp = (\ker H_+)^\perp$ and final space $(\ker A_Q^*)^\perp = (\ker H_-)^\perp$ such that $A_Q = U H_+^{1/2} = H_-^{1/2} U$. Hence (2.12), (2.13) and (2.14) hold. Thus part (ii) follows.

By Theorem 2.1, there exists a partial isometry Θ on $\mathcal{X}$ with initial space including $(\ker Q)^\perp$ such that (2.6) and (2.15) hold. Since $\ker Q = \ker H$, the desired result follows. ∎

Remark. In Theorem 2.3, we have also $H_+^{1/2} U^* = U^* H_-^{1/2}$, so that (2.12) can also be written

$$Q = \begin{pmatrix} 0 & U^* H_-^{1/2} \\ U H_+^{1/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & U^* H_-^{1/2} \\ H_-^{1/2} U & 0 \end{pmatrix}.$$

The converse to Theorem 2.3 is given as follows.

PROPOSITION 2.4. *Suppose that* H *is reduced by* $\mathcal{X}_\pm$ *and there exists a partial isometry* $U : \mathcal{X}_+ \to \mathcal{X}_-$ *with initial space including* $(\ker H_+)^\perp$ *such that (2.14) holds. Then the operator* Q *defined by (2.12) is an abstract Dirac operator w.r.t.* γ *satisfying* $H = Q^2$.

Proof. It is easy to see that $A := U H_+^{1/2}$ is a densely defined closed linear operator from $\mathcal{X}_+$ to $\mathcal{X}_-$ with $D(A) = D(H_+^{1/2})$. Since $A^* = H_+^{1/2} U^*$, it follows that the operator Q defined by (2.12) is self-adjoint. It is obvious that (2.1) is satisfied. Hence Q is an abstract Dirac operator w.r.t. γ. Note that $(\ker H_+)^\perp = (\ker H_+^{1/2})^\perp$. Hence $U^* U = I$ on $R(H_+^{1/2})$, so that $A^* A = H_+^{1/2} U^* U H_+^{1/2} = H_+$. By (2.14), we have $AA^* = U H_+ U^* = H_-$. Thus $H = Q^2$. ∎

III. APPLICATION TO SECOND QUANTIZATIONS ON BOSON-FERMION FOCK SPACES

3.1 Dirac type operators and second quantizations on the abstract Boson-Fermion Fock space

Let $\mathcal{H}$ be a Hilbert space. Then the Boson (symmetric) Fock space $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ over $\mathcal{H}$ is defined as

$$\mathcal{F}_{\mathrm{b}}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{H}),$$

where $\mathcal{F}_n(\mathcal{H})$ is the n-fold symmetric tensor product of $\mathcal{H}$ ($\mathcal{F}_0(\mathcal{H}) := \mathbf{C}$). Let $\mathcal{K}$ be a Hilbert space and $\bigwedge^p(\mathcal{K})$ be the p-fold antisymmetric tensor product of $\mathcal{K}$ ($\bigwedge^0(\mathcal{K}) := \mathbf{C}$). Then the Fermion (antisymmetric) Fock space $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ over $\mathcal{K}$ is defined by

$$\mathcal{F}_{\mathrm{f}}(\mathcal{K}) = \bigoplus_{p=0}^{\infty} \bigwedge^p(\mathcal{K}).$$

The tensor product of these two Fock spaces

$$\mathcal{F}(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{f}}(\mathcal{K}) = \bigoplus_{n,p=0}^{\infty} \mathcal{F}_n(\mathcal{H}) \otimes \bigwedge^p(\mathcal{K})$$

is called the *Boson-Fermion Fock space* over $< \mathcal{H},\mathcal{K} >$. This Hilbert space admits a natural orthogonal decomposition:

$$\mathcal{F}(\mathcal{H},\mathcal{K}) = \mathcal{F}_+(\mathcal{H},\mathcal{K}) \oplus \mathcal{F}_-(\mathcal{H},\mathcal{K}) \tag{3.1}$$

with

$$\mathcal{F}_+(\mathcal{H},\mathcal{K}) = \bigoplus_{p=0}^{\infty} \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \bigwedge^{2p}(\mathcal{K}), \quad \mathcal{F}_-(\mathcal{H},\mathcal{K}) = \bigoplus_{p=0}^{\infty} \mathcal{F}_{\mathrm{b}}(\mathcal{H}) \otimes \bigwedge^{2p+1}(\mathcal{K}).$$

The grading operator w.r.t. the decomposition (3.1) is given by

$$\Gamma = P_+ - P_-$$

with $P_\pm$ the orthogonal projections onto $\mathcal{F}_\pm(\mathcal{H},\mathcal{K})$.

We say that a linear operator Q on $\mathcal{F}(\mathcal{H},\mathcal{K})$ is *Dirac type* if it is self-adjoint and

$$\Gamma Q \subset -Q\Gamma. \tag{3.2}$$

A class of Dirac type operators on $\mathcal{F}(\mathcal{H},\mathcal{K})$ is constructed as follows [1][4]. Let $a(f), f \in \mathcal{H}$, and $b(u), u \in \mathcal{K}$, be the annihilation operators on $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ [14, §X.7] and

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[4] In [1], the Boson Fock space is realized as an L^2-space with a Gaussian measure, which is called the Q-space representation. Here we use the original form of the Boson Fock space. Translating mathematical expressions in the original Boson Fock space into those in the Q-space representation is easy.

on $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$ [1, §2.2], respectively [$a(f)$ (resp. $b(u)$) is *antilinear in* f (resp. u)]. These operators are naturally extended to operators on $\mathcal{F}(\mathcal{H},\mathcal{K})$ as

$$A(f) = a(f) \otimes I, \quad B(u) = I \otimes b(u).$$

Let $\Omega_{\mathrm{b}} = \{1,0,0,\cdots\} \in \mathcal{F}_{\mathrm{b}}(\mathcal{H})$ and $\Omega_{\mathrm{f}} = \{1,0,0,\cdots\} \in \mathcal{F}_{\mathrm{f}}(\mathcal{K})$ be the Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ and in $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$, respectively. The *vacuum* in $\mathcal{F}(\mathcal{H},\mathcal{K})$ is defined by

$$\Omega = \Omega_{\mathrm{b}} \otimes \Omega_{\mathrm{f}}.$$

For a linear operator T on a Hilbert space, we denote by $C^\infty(T)$ the set of C^∞-vectors of T: $C^\infty(T) = \cap_{n=1}^\infty D(T^n)$.

We denote by $\mathcal{C}(\mathcal{H},\mathcal{K})$ the set of densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$. Let $S \in \mathcal{C}(\mathcal{H},\mathcal{K})$ and

$$\mathcal{D}_S = \mathcal{L}\left\{ A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_m)^*\Omega \,\middle|\, n,m \geq 0, f_j \in C^\infty(S^*S), j = 1,\cdots,n, \right.$$
$$\left. u_k \in C^\infty(SS^*), k = 1,\cdots,m \right\},$$

where $\mathcal{L}\{\cdots\}$ means the algebraic linear span of elements in the set $\{\cdots\}$. The subspace $\mathcal{D}_S$ is dense in $\mathcal{F}(\mathcal{H},\mathcal{K})$, since $C^\infty(S^*S)$ and $C^\infty(SS^*)$ are dense in $\mathcal{H}$ and $\mathcal{K}$, respectively. In [1] we showed that there exists a unique closed linear operator d_S on $\mathcal{F}(\mathcal{H},\mathcal{K})$ such that $\mathcal{D}_S$ is a core for d_S and

$$d_S\Psi = \begin{cases} 0 & ; \; n = 0 \\ \sum_{j=1}^n A(f_1)^* \cdots \widehat{A(f_j)}^* \cdots A(f_n)^* B(Sf_j)^* B(u_1)^* \cdots B(u_m)^*\Omega & ; \; n \geq 1 \end{cases}$$

for all $\Psi \in \mathcal{D}_S$ of the form $\Psi = A(f_1)^* \cdots A(f_n)^* B(u_1)^* \cdots B(u_m)^*\Omega$, where $\widehat{A(f_j)}^*$ indicates the omission of $A(f_j)^*$. We have

$$d_S^2 = 0. \tag{3.3}$$

Let

$$Q_S = d_S + d_S^*. \tag{3.4}$$

Then it is shown that Q_S is a Dirac type operator on $\mathcal{F}(\mathcal{H},\mathcal{K})$ [1]. A particular property of Q_S is that, for each $n,p = 0,1,2,\cdots$, it maps $[\mathcal{F}_n(\mathcal{H}) \otimes \bigwedge^p(\mathcal{K})] \cap D(Q_S)$ to $\mathcal{F}_{n-1}(\mathcal{H}) \otimes \bigwedge^{p+1}(\mathcal{K}) \bigoplus \mathcal{F}_{n+1}(\mathcal{H}) \otimes \bigwedge^{p-1}(\mathcal{K})$, where we set $\mathcal{F}_{-1}(\mathcal{H}) = \{0\}, \bigwedge^{-1}(\mathcal{K}) = \{0\}$.

A direct application of Theorem 2.1 to Q_S gives the following.

THEOREM 3.1. *(i) Suppose that there exists a Dirac type operator Q on $\mathcal{F}(\mathcal{H},\mathcal{K})$ such that*

$$Q^2 = Q_S^2. \tag{3.5}$$

Then there exists a partial isometry Θ on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with initial space including $(\ker Q_S)^\perp$ *such that*

$$\Theta\Gamma = \Gamma\Theta \tag{3.6}$$

and

$$Q = \Theta Q_S. \tag{3.7}$$

The operator Θ is uniquely determined by the condition that $\ker \Theta = \ker Q_S$.

(ii) Conversely, suppose that there exists a partial isometry Θ on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ with initial space including $(\ker Q_S)^\perp$ *such that (3.6) and*

$$\Theta Q_S = Q_S \Theta^* \tag{3.8}$$

hold. Then the operator Q defined by (3.7) is a Dirac type operator on $\mathcal{F}(\mathcal{K}, \mathcal{H})$ and (3.5) holds.

(iii) In each case of part (i) and (ii), Q and Q_S strongly anticommute if and only if

$$\Theta Q_S^2 \subset -Q_S^2 \Theta^*. \tag{3.9}$$

Remark. We can show that

$$\ker Q_S = \bigoplus_{n,p=0}^{\infty} [\otimes_{\mathrm{s}}^n \ker S] \otimes [\otimes_{\mathrm{as}}^p \ker S^*],$$

where $\otimes_{\mathrm{s}}^n$ (resp. $\otimes_{\mathrm{as}}^p$) denotes n-fold symmetric (resp. p-fold antisymmetric) tensor product, see [1].

EXAMPLE 3.1. Let $\Theta = i\Gamma$. Then Θ satisfies (3.6), (3.8) and (3.9). Hence $Q'_S := i\Gamma Q_S$ and Q_S strongly anticommute, satisfying ${Q'_S}^2 = Q_S^2$.

We denote by $d\Gamma_{\mathrm{b}}(\cdot)$ (resp.$d\Gamma_{\mathrm{f}}(\cdot)$) the second quantization on $\mathcal{F}_{\mathrm{b}}(\mathcal{H})$ (resp. $\mathcal{F}_{\mathrm{f}}(\mathcal{K})$)[13, §VIII.10]. For nonnegative self-adjoint operators A and B acting in $\mathcal{H}$ and $\mathcal{K}$, respectively, we define a nonnegative self-adjoint operator on $\mathcal{F}(\mathcal{H}, \mathcal{K})$ by

$$\Delta(A, B) = d\Gamma_{\mathrm{b}}(A) \otimes I + I \otimes d\Gamma_{\mathrm{f}}(B). \tag{3.10}$$

We call it the second quantization associated with the pair $< A, B >$.

We set

$$\Delta_S = \Delta(S^*S, SS^*), \tag{3.11}$$

the second quantization $\Delta(A, B)$ with $A = S^*S,\ B = SS^*$. It was shown in [1] that

$$\Delta_S = Q_S^2 = d_S^* d_S + d_S d_S^*. \tag{3.12}$$

Thus the operator Δ_S, which is a special type of second quantizations on $\mathcal{F}(\mathcal{H},\mathcal{K})$, is the square of a Dirac type operator. Relation (3.12) and the mapping property of Q_S mentioned above show that, in the case $\dim \mathcal{H} = \infty$, Q_S is a natural infinite dimensional analogue of finite dimensional Dirac operators.

The following proposition gives a characterization for any pair $< S, T >$ with $S, T \in \mathcal{C}(\mathcal{H},\mathcal{K})$ such that $Q_T^2 = Q_S^2$.

PROPOSITION 3.2. *Let $S, T \in \mathcal{C}(\mathcal{H},\mathcal{K})$. Then, $Q_T^2 = Q_S^2$ if and only if there exist partial isometries $\theta_{\mathcal{K}}$ on $\mathcal{K}$ with initial space including $(\ker S^*)^\perp$ and $\theta_{\mathcal{H}}$ on $\mathcal{H}$ with initial space including $(\ker S)^\perp$ such that*

$$T = \theta_{\mathcal{K}} S, \quad T^* = \theta_{\mathcal{H}} S^*. \tag{3.13}$$

Proof. Suppose that $Q_T^2 = Q_S^2$. Then, by (3.12), we have $\Delta_T = \Delta_S$, which implies that $T^*T = S^*S$, $TT^* = SS^*$. Hence, putting $L = (T^*T)^{1/2} = (S^*S)^{1/2}$, we have from Lemma 2.2 that there exist partial isometries $U_S, U_T : \mathcal{H} \to \mathcal{K}$ with initial spaces $(\ker S)^\perp = (\ker T)^\perp$ and final spaces $(\ker S^*)^\perp = (\ker T^*)^\perp$ such that $T = U_T L$, $S = U_S L$. Since $U_S^* S = L$, it follows that $T = \theta_{\mathcal{K}} S$ with $\theta_{\mathcal{K}} = U_T U_S^*$. It is easy to see that $\theta_{\mathcal{K}}$ is a partial isometry on $\mathcal{K}$ with initial space $(\ker S^*)^\perp$. Similarly we can show that that there exists a partial isometry $\theta_{\mathcal{H}}$ on $\mathcal{K}$ with initial space $(\ker S)^\perp$ such that the second equation of (3.13) holds.

Conversely, suppose that there exists a partial isometry $\theta_{\mathcal{K}}$ on $\mathcal{K}$ (resp. $\theta_{\mathcal{H}}$ on $\mathcal{H}$) with initial space including $(\ker S^*)^\perp$ (resp. $(\ker S)^\perp$) such that (3.13) holds. Then $T^*T = S^*\theta_{\mathcal{K}}^*\theta_{\mathcal{K}} S = S^*S$. Since $T^{**} = T$, (3.13) implies that $T = S\theta_{\mathcal{H}}^*$. Using this relation, we can show that $TT^* = SS^*$. Thus $\Delta_T = \Delta_S$, i.e., $Q_T^2 = Q_S^2$. ∎

EXAMPLE 3.2. Let $S \in \mathcal{C}(\mathcal{H},\mathcal{K})$ $(S \neq 0)$. Then, for all $\alpha \in [0, 2\pi)$, the operator $T := e^{i\alpha} S$ is in $\mathcal{C}(\mathcal{H},\mathcal{K})$ and trivially satisfies (3.13) with $\theta_{\mathcal{K}} = e^{i\alpha} I$ and $\theta_{\mathcal{H}} = e^{-i\alpha} I$. Hence, putting

$$Q_S(\alpha) = Q_{e^{i\alpha} S},$$

we have $Q_S(\alpha)^2 = Q_S^2$. We have for all $z \in \mathbf{C}$

$$d_{zS} = z d_S, \quad d_{zS}^* = z^* d_S^*.$$

Hence

$$Q_S(\alpha) = e^{i\alpha} d_S + e^{-i\alpha} d_S^*. \tag{3.14}$$

In [1] we showed that $Q_S(\pi/2) = i(d_S - d_S^*)$ anticommutes with Q_S in the sense of quadratic form (hence, by [6, Theorem 6.3], $Q_S(\pi/2)$ strongly anticommutes with Q_S). The same applies to $Q_S(3\pi/2) = -i(d_S - d_S^*)$. In what follows, by applying Theorem 3.1(iii), we want to show that $Q_S(\alpha)$ strongly anticommutes with Q_S only if $\alpha = \pi/2$ or $\alpha = 3\pi/2$.

By Theorem 3.1, there exists a partial isometry Θ on $\mathcal{F}(\mathcal{H},\mathcal{K})$ with initial space including $(\ker Q_S)^\perp$ such that (3.6) and

$$Q_S(\alpha) = \Theta Q_S \tag{3.15}$$

hold. It follows from this fact that

$$\Theta = Q_S(\alpha) Q_S L_S^{-1} \quad \text{on } R(\Delta_S),$$

where $L_S = \Delta_S \restriction D(\Delta_S) \cap (\ker \Delta_S)^\perp$. By (3.14), (3.4) and (3.3), we obtain

$$\Theta = (e^{i\alpha} d_S d_S^* + e^{-i\alpha} d_S^* d_S) L_S^{-1} \quad \text{on } R(\Delta_S). \tag{3.16}$$

Eq.(3.15) implies that $Q_S(\alpha) = Q_S \Theta^*$. Multiplying the both sides by Q_S from the left and computing $Q_S Q_S(\alpha)$, we obtain

$$e^{-i\alpha} d_S d_S^* + e^{i\alpha} d_S^* d_S = \Delta_S \Theta^*.$$

This formula, (3.16) and Theorem 3.1(iii) imply that $Q_S(\alpha)$ and Q_S strongly anticommute if and only if

$$e^{i\alpha} d_S d_S^* + e^{-i\alpha} d_S^* d_S \subset -e^{-i\alpha} d_S d_S^* - e^{i\alpha} d_S^* d_S,$$

which is equaivalent to that $\cos\alpha \cdot \Delta_S = 0$. By the assumption that $S \neq 0$, we have $\Delta_S \neq 0$. Hence $Q_S(\alpha)$ and Q_S strongly anticommute if and only if $\alpha = \pi/2$ or $\alpha = 3\pi/2$.

We remark that the result just shown can be proven more easily by using the fact that, if two self-adjoint operators K and L on a Hilbert space strongly anticommute, then $(K\psi, L\phi) + (L\psi, K\phi) = 0$ for all $\psi, \phi \in D(K) \cap D(L)$ (see the proof of Theorem 2.1(iii)).

3.2 Factorization of second quantizations by Dirac type operators

The following theorem gives a characterization of a second quantization on the Boson-Fermion Fock space $\mathcal{F}(\mathcal{H},\mathcal{K})$ which is written as the square of a Dirac type operator with a certain mapping property.

THEOREM 3.3. *Suppose that there exists a Dirac type operator $Q(A,B)$ on $\mathcal{F}(\mathcal{H},\mathcal{K})$ such that*

$$\Delta(A,B) = Q(A,B)^2 \tag{3.17}$$

and $Q(A,B)$ maps $[\mathcal{F}_1(\mathcal{H}) \otimes \bigwedge^0(\mathcal{K})] \cap D(Q(A,B))$ into $\mathcal{F}_0(\mathcal{H}) \otimes \bigwedge^1(\mathcal{K})$. Then there exists an operator $S \in \mathcal{C}(\mathcal{H},\mathcal{K})$ such that $A = S^*S$, $B = SS^*$, so that

$$\Delta(A,B) = \Delta_S = Q_S^2.$$

Moreover, there exists a partial isometry Θ on $\mathcal{F}(\mathcal{H},\mathcal{K})$ with initial space including $(\ker Q_S)^\perp$ such that (3.6) and

$$Q(A,B) = \Theta Q_S \tag{3.18}$$

hold. The operator Θ *is uniquely determined by the condition that* $\ker\Theta = \ker Q_S$.

Proof. It is easy to see that $\Delta(A,B)$ is reduced by $\mathcal{F}_\pm(\mathcal{H},\mathcal{K})$ and its reduced parts to $\mathcal{F}_\pm(\mathcal{H},\mathcal{K})$, which we denote $\Delta_\pm(A,B)$, respectively, are given by

$$\Delta_+(A,B) = d\Gamma_{\mathrm{b}}(A)\otimes I + \sum_{p=0}^{\infty} I\otimes d\Gamma_{\mathrm{f}}^{(2p)}(B),$$

$$\Delta_-(A,B) = d\Gamma_{\mathrm{b}}(A)\otimes I + \sum_{p=0}^{\infty} I\otimes d\Gamma_{\mathrm{f}}^{(2p+1)}(B),$$

where $d\Gamma_{\mathrm{f}}^{(p)}(B)$ is the self-adjoint operator on $\bigwedge^p(\mathcal{K})$ defined as follows:

$$d\Gamma_{\mathrm{f}}^{(0)}(B) = 0, \quad d\Gamma_{\mathrm{f}}^{(p)}(B) = \sum_{j=1}^{p} I\otimes\cdots\otimes I\otimes \overset{j}{\breve{B}}\otimes I\cdots\otimes I, \quad p\geq 1.$$

Hence, by (3.17) and Theorem 2.3, there exists a unique partial isometry U from $\mathcal{F}_+(\mathcal{H},\mathcal{K})$ to $\mathcal{F}_-(\mathcal{H},\mathcal{K})$ with initial space $(\ker\Delta_+(A,B))^\perp$ and final space $(\ker\Delta_-(A,B))^\perp$ such that

$$Q(A,B) = \begin{pmatrix} 0 & \Delta_+(A,B)^{1/2}U^* \\ U\Delta_+(A,B)^{1/2} & \end{pmatrix} \tag{3.19}$$

and

$$U\Delta_+(A,B)^{1/2} = \Delta_-(A,B)^{1/2}U. \tag{3.20}$$

Let $Q_{1,0}(A,B)$ be the operator obtained by restricting $Q(A,B)$ to the subspace $[\mathcal{F}_1(\mathcal{H})\otimes \bigwedge^0(\mathcal{K})]\cap D(Q(A,B))$. Note that $\Delta_+(A,B)$ is reduced by $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$ and its reduced part is equal to $A\otimes I$. Hence $\Delta_+(A,B)^{1/2}$ also is reduced by $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$ and its reduced part is given by $A^{1/2}\otimes I$. By this fact and (3.19), we have $Q_{1,0}(A,B) = U_{1,0}A^{1/2}\otimes I$, where $U_{1,0}$ is the restriction of U to $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$. The present assumption implies that $R(Q_{1,0}(A,B))\subset \mathcal{F}_0(\mathcal{H})\otimes\bigwedge^1(\mathcal{K})$. Hence it follows that $U_{1,0}$ is a partial isometry from $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$ to $\mathcal{F}_0(\mathcal{H})\otimes\bigwedge^1(\mathcal{K})$ with initial space $(\ker A^{1/2}\otimes I)^\perp = (\ker A\otimes I)^\perp$. Every vector in $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$ (resp. $\mathcal{F}_0(\mathcal{H})\otimes\bigwedge^1(\mathcal{K})$) is of the form $\psi\otimes\Omega_{\mathrm{f}}, \psi\in\mathcal{H}$ (resp. $\Omega_{\mathrm{b}}\otimes u, u\in\mathcal{K}$). Hence we can define uniquely an operator $V:\mathcal{H}\to\mathcal{K}$ by

$$U_{1,0}(\psi\otimes\Omega_{\mathrm{f}}) = \Omega_{\mathrm{b}}\otimes V(\psi). \tag{3.21}$$

Then it follows that V is a partial isometry with initial space $(\ker A)^\perp$. On the other hand, restricting Eq.(3.20) to $\mathcal{F}_1(\mathcal{H})\otimes\bigwedge^0(\mathcal{K})$, we have

$$U_{1,0}A^{1/2}\otimes I = I\otimes B^{1/2}U_{1,0}.$$

This fact and (3.21) imply that

$$VA^{1/2} = B^{1/2}V.$$

We now define an operator $S : \mathcal{H} \to \mathcal{K}$ by $S = VA^{1/2}$. It is easy to see that S is in $\mathcal{C}(\mathcal{H},\mathcal{K})$ with $S^* = A^{1/2}V^*$. Hence $S^*S = A$, $SS^* = B$. Thus we obtain the first half of the theorem. The second half is just a simple application of Theorem 2.3(iii) (note that $\ker \Delta(A,B) = \ker Q(A,B) = \ker Q_S$). ∎

In the case of Dirac type operators $Q(A,B)$ satisfying (3.17), but not having the mapping property described in the assumption of Theorem 3.3, the conclusion of Theorem 3.3 does not hold in general. A simple counter example is given as follows.

EXAMPLE 3.3. Let A_1 and A_2 be strongly commuting, nonnegative self-adjoint operators[5] on $\mathcal{H}$ such that $A_1 + A_2 \neq 0$. Then it is obvious that there exist no operators $S \in \mathcal{C}(\mathcal{H},\mathcal{K})$ such that

$$Q_S^2 = \Delta(A_1 + A_2, 0) = d\Gamma_{\rm b}(A_1 + A_2) \otimes I.$$

But, in the following, we show that $\Delta(A_1 + A_2, 0)$ is written as the square of a Dirac type operator which does not have the mapping property of $Q(A,B)$ in Thereom 3.3.

It is not so difficult to show that $d\Gamma_{\rm b}(A_1)$ and $d\Gamma_{\rm b}(A_2)$ strongly commute. Let

$$\mathcal{F}_{{\rm f},+}(\mathcal{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^{2p}(\mathcal{K}), \quad \mathcal{F}_{{\rm f},-}(\mathcal{K}) := \bigoplus_{p=0}^{\infty} \bigwedge^{2p+1}(\mathcal{K})$$

and $E_\pm$ be the orthogonal projections from $\mathcal{F}_{\rm f}(\mathcal{K})$ onto $\mathcal{F}_{{\rm f},\pm}(\mathcal{K})$. Let U be a unitary operator from $\mathcal{F}_{{\rm f},+}(\mathcal{K})$ to $\mathcal{F}_{{\rm f},-}(\mathcal{K})$ and define

$$\gamma_1 = E_-UE_+ + E_+U^*E_-, \quad \gamma_2 = i(E_-UE_+ - E_+U^*E_-).$$

Then $\gamma_j, j = 1,2$, are self-adjoint, unitary on $\mathcal{F}_{\rm f}(\mathcal{K})$, satisfying

$$\{\gamma_j, \gamma_k\} = 2\delta_{jk}, \; j,k = 1,2,$$

where $\{X,Y\} = XY + YX$. Hence $\{\gamma_j\}_{j=1}^2$ is a self-adjoint representation of the two-dimensional Clifford algebra on $\mathcal{F}_{\rm f}(\mathcal{K})$. Applying [5, Theorem 3.4], we see that the self-adjoint operators $d\Gamma_{\rm b}(A_1)^{1/2} \otimes \gamma_1$ and $d\Gamma_{\rm b}(A_2)^{1/2} \otimes \gamma_2$ on $\mathcal{F}(\mathcal{H},\mathcal{K})$ strongly anticommute. Hence, by [17, Theorem 2.1], the operator

$$Q := d\Gamma_{\rm b}(A_1)^{1/2} \otimes \gamma_1 + d\Gamma_{\rm b}(A_2)^{1/2} \otimes \gamma_2$$

is self-adjoint and

$$Q^2 = d\Gamma_{\rm b}(A_1) \otimes \gamma_1^2 + d\Gamma_{\rm b}(A_2) \otimes \gamma_2^2 = d\Gamma_{\rm b}(A_1 + A_2) \otimes I.$$

Since Γ is written as $\Gamma = I \otimes E_+ - I \otimes E_-$, it follows that

$$\{\Gamma, I \otimes \gamma_j\} = 0, \quad j = 1,2,$$

[5] Two self-adjoint operators on a Hilbert space are said to *strongly commute* if their spectral measures commute.

which imply (3.2). Hence Q is a Dirac type operator such that $Q^2 = \Delta(A_1 + A_2, 0)$. Clearly Q does not have the mapping property of $Q(A, B)$ in Thereom 3.3.

Remark. It is an open problem to characterize (or classify) Dirac type operators $Q(A, B)$ satisfying (3.17), but not having the mapping property described in the assumption of Theorem 3.3 in a more detailed way than (3.19) and (3.20).

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