On Strongly separable Frobenius extensions

Kozo Sugano

Series #325. December 1995

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#301 J. Zhai, Harmonic maps and Ginzburg-Landau type elliptic system, 20 pages. 1995.

#302 M.-H. Giga and Y. Giga, Geometric evolution by nonsmooth interfacial energy, 15 pages. 1995.

#303 S. Jimbo and J. Zhai, Ginzburg-Landau equation with magnetic effect: non-simply-connected domains, 21 pages. 1995.

#304 T. Ozawa, On the nonlinear Schrödinger equations of derivative type, 27 pages. 1995.

#305 N.H. Bingham and A. Inoue, Jordan's theorem for fourier and hankel transforms, 30 pages. 1995.

#306 T. Honda and T. Suwa, Residue formulas for singular foliations defined by meromorphic functions on surfaces, 19 pages. 1995.

#307 J. Yoshizaki, On the structure of the singular set of a complex analytic foliation, 25 pages. 1995.

#308 A. Arai, Representation of canonical commutation relations in a gauge theory, the Aharonov-Bohm effect, and Dirac Weyl operator, 17 pages. 1995.

#309 A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of the spin-boson Hamiltonian, 20 pages. 1995.

#310 S. Izumiya and T. Sano, Generic affine differential geometry of plane curves, 8 pages. 1995.

#311 N. Kawazumi, On the stable cohomology algebra of extended mapping class groups for surfaces, 13 pages. 1995.

#312 H.M.Ito and T. Mikami, Poissonian asymptotics of a randomly perturbed dynamical system: Flip-flop of the Stochastic Disk Dynamo, 20 pages. 1995.

#313 T. Nakazi, Slice maps and multipliers of invariant subspaces, 11 pages. 1995.

#314 T. Mikami, Weak convergence on the first exit time of randomly perturbed dynamical systems with a repulsive equilibrium point, 20 pages. 1995.

#315 A. Arai, Canonical commutation relations, the Weierstrass Zetafunction, and infinite dimensional Hilbert space representations of the quantum group $U_q\ (\mathfrak{sl}_2)$, 22 pages. 1995.

#316 Y. Shibukawa, Vertex-face correspondence in elliptic solutions of the Yang-Baxter equation, 8 pages. 1995.

#317 M.-H. Giga and Y. Giga, Consistency in evolutions by crystalline curvature, 16 pages. 1995.

#318 Wei-Zhi Sun, Shadows of moving surfaces, 19 pages. 1995.

#319 S. Izumiya and G.T. Kossioris, Bifurcations of shock waves for viscosity solutions of Hamilton-Jacobi equations of one space variable, 39 pages. 1995.

#320 T. Teruya, Normal intermediate subfactors, 44 pages. 1995.

#321 M. Ohnuma, Axisymmetric solutions and singular parabolic equations in the theory of viscosity solutions, 26 pages. 1995.

#322 T. Nakazi, An outer function and several important functions in two variables, 12 pages. 1995.

#323 N. Kawazumi, An infinitesimal approach to the stable cohomology of the moduli of Riemann surfaces, 22 pages. 1995.

#324 A. Arai, Factorization of self-adjoint operators by abstract Dirac operators and its application to second quantizations on Boson Fermion Fock spaces, 15 pages. 1995.

On Strongly separable Frobenius extensions

Kozo Sugano

(Received)

Introduction. In their paper [3] A. Mewborn and E. McMahon defined the strongly separable extension of non commutative ring, which is a generalization of the separable algebra over a commutative ring. Strongly separability is a weaker than H-separability, but stronger than separability. In [3] Mewborn and McMahon gave a few necessary and sufficient conditions for ring extensions to be strongly separable (See Theorem 3.5 [3]). Afterwords K. Yamashiro gave some new characterizations of strongly separability which are the complete improvements of Mewborn McMahon's Proposition 3.10 [3] (See Theorem 1.4 [10]). These characterizations of strongly separability are generarizations of the characterizations of H-separability. In the first section of this paper we will give a new proof of Mewborn-McMahon and Yamashiro's theorem concerning with the charecterizations of strongly separability.

H-separable extension has a closed relation with Frobenius extension. For example each projective H-separable extension of a (not necessarily artinian) simple ring is an Frobenius extension, and every H-separable extension of full linear ring is a free Frobenius extension (See Theorem 4 [7]). Recently in [8] the author gave some necessary and sufficient conditions for the Frobenius extension to be H-separable. The aim of this paper is to generalize the results obtained in [8] to the case of strongly separable extension.

Let A be a ring with its center C, B a subring of A and D the centralizer of B in A. In the case where D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$, A is strongly separable over B if and only if there exists $\Sigma x_k \otimes y_k$ in $(A \otimes_B A)^A$ such that $d = \Sigma x_k d d_i y_k e_i = \Sigma d_i x_k e_i d y_k$ holds for each d in D. Furthermore if these conditions are satisfied, A is an H-separable Frobenius extension of $B^*=V_A(V_A(B))$ (Theorem 1). Since the strongly separable extension is a generalization of separable algebra, the following conjecture naturally comes out;

'Is A an H-separable extension of B^*, and B^* a separable extension of B, in the case where A is a strongly separable extension of B'

Theorem 1 of this paper gave a partial answer to this problem. In the second theorem we will show that, in the case where A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h^*\}$, A is a strongly separable over B, if and only if D is a Frobenius C-algebra with a Frobenius homomorphism h, where h is defined by $h(d)=\Sigma x_k d y_k$ for each $d \in D$.

§1. Throughout this paper we will use the same notation as the author's previous paper [8]. In particular A will always be a ring with identity 1, B a subring of A containing 1, C the center of A and D = $V_A(B)$, the centralizer of B in A. For an A-A-module M we write $M^A = \{m \mid m \in M$ and $am = ma$ for every $a \in A\}$. We hve an issomorphism of $\mathrm{Hom}({}_AA_A, {}_AM_A)$ to M^A, via $f \rightarrow f(1)$ for each $f \in \mathrm{Hom}({}_AA_A, {}_AM_A)$.

Lemma 1.1 For an A-A-module M the following conditions are equivalent;

(i) M is isomorphic to a direct summand of a finite direct sum of copies of A as A-A-module

(ii) There exist finite f_i in $\mathrm{Hom}({}_AM_A, {}_AA_A)$ and m_i in M^A such that $m = \Sigma f_i(m)m_i$ holds for every m in M

(iii) M^A is C-finitely generated projective, and the map μ of $A \otimes_C M^A$ to M such that $\mu(a \otimes m) = am$ for $a \in A$ and $m \in M$ is an isomorphism.

Proof. (i)⇔(iii) is shown in Proposition 5.2 [2]. (i)⇔(ii) can be shown by the same method as the existance of the dual basis of a finitly generated projective module, using the isomorphism $\mathrm{Hom}({}_AA_A, {}_AM_A) \cong M^A$ So we will ommit it.

Following [2] we will give a definition;

Definition. A two-sided A-module M is centrally projective over A, if M satisfies the condition of Lemma 1.1.

Let R be an arbitrary ring, and M and N any left R-modules. Write $S = \mathrm{Hom}({}_RM, {}_RM)$ and $T = \mathrm{Hom}({}_RN, {}_RN)$. We regard M and N as left R-S, and R-T-modules, respectively, by writing f(x) for each f in S (or T) and x in M (or N). Now we have the following natural R-S-homomorphism

$$\tau : M \rightarrow \mathrm{Hom}({}_T\mathrm{Hom}({}_RM, {}_RN), {}_TN)$$

such that $\tau(m)(f) = f(m)$ for each f in $\mathrm{Hom}({}_RM, {}_RN)$ and m in M. Write $H = \mathrm{Hom}({}_RM, {}_RN)$. Let $m \in M$. Then we have $f(m) = 0$ for each f in H, if and only if $\tau(m)(f) = 0$ for each f in H, that is, $\tau(m) = 0$. This means that $\mathrm{Ker}\,\tau = \cap_{f \in H} \mathrm{Ker} f$ (= $\mathrm{Rej}({}_M(N))$)(See page 109 [1]).

Now let M be an A-A-module, and put $N = A$ and $R = A\otimes_C A^\circ$, where A° is the opposite ring of A. Then M and N are left R-modules. Under this situation let S and T be as above. Then $T \cong C$, and the above R-S-homomorphism τ becomes the following A-A-homomorphism

$$\tau : M \to \mathrm{Hom}(\mathrm{Hom}({}_AM_A, {}_AA_A)_C, A_C)$$

Lemma 1.2. If an A-A-module M is centrally projective over A, the map τ given above is an isomorphism.

Proof. By Lemma 1.1 there exist $f_i \in \mathrm{Hom}({}_AM_A, {}_AA_A)$ and $m_i \in M^A$ such that $m = \Sigma f_i(m)m_i$ for each m in M. Let $m \in \mathrm{Ker}\,\tau$. Then $\tau(m)(f_i) = f_i(m) = 0$ for each i, and we have $m = 0$. Thus we have that $\mathrm{Ker}\,\tau = 0$.

For each f in $\mathrm{Hom}({}_AM_A, {}_AA_A)$ we have obviously $f(m_i) \in C$ for each i and $f = \Sigma f(m_i)f_i$. Therefore for each $h \in \mathrm{Hom}(\mathrm{Hom}({}_AM_A, {}_AA_A)_C, A_C)$ we have

$$\tau(\Sigma h(f_i)m_i)(f) = f(\Sigma h(f_i)m_i) = \Sigma h(f_i)f(m_i) = h(\Sigma f_i f(m_i)) = h(f)$$

Thus we have $h = \tau(\Sigma h(f_i)m_i)$, and we see that τ is an epimorphism.

There always exists an A-A-homomorphism

$$\eta : A\otimes_B A \to \mathrm{Hom}({}_CD, {}_CA)$$

such that $\eta(a\otimes b)(d) = adb$ for any a, b in A and d in D.

Througout this paper η will always be the map defined in the above way, and K will stand for $\mathrm{Ker}\,\eta$.

Also there exists the well known isomorphism ν of $\mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A)$ to D such that $\nu(f) = f(1\otimes 1)$ for each $f \in \mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A)$. ν induces the isomorphism

$$\nu^* : \mathrm{Hom}({}_CD, {}_CA) \to \mathrm{Hom}({}_C\mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A), {}_CA)$$

such that $\nu^*(f) = f\nu$ for each f in $\mathrm{Hom}({}_CD, {}_CA)$, and direct caluculation shows that the composition $\nu^*\eta$ is exactly equal to the map τ of M to $\mathrm{Hom}(\mathrm{Hom}({}_AM_A, {}_AA_A)_C, A_C)$ indroduced above.

In [3] A. Mewborn and E. McMahon defined that A is a strongly separable extension of B in the case where η is an A-A-split epimorphism, and D is C-finitely generated projective.

Now we will give a new proof of Theoren 3.5 [3] and Theorem 1.4 [10]

Among five conditions in the next theorem, the equivalence of (i), (ii) and (v) was first proved by Mewborn-McMahon, and the equivalence of (i), (iii) and (iv) was proved by Yamashiro.

Theorem 1.1. The following conditions are equivalent

(i) A is a strongly separable extension of B

(ii) There exist finite d_i in D and $\Sigma x_{ij}\otimes y_{ij}$ in $(A\otimes_B A)^A$ such that $d = \Sigma d_i x_{ij} d y_{ij}$ holds for each d in D

(iii) For each A-A-module M there exists a C-submodule X such that $M^B = DM^A \oplus X$ and $X \subseteq \mathrm{Rej}_M(A)$

(iv) We have $(A\otimes_B A)^B = D(A\times_B A)^A \oplus X$ as C-module with $X \subseteq \mathrm{Ker}\,\eta$

(v) $A\otimes_B A = N \oplus L$ as A-A-module, where N is A-centrally projective, and $\mathrm{Hom}({}_A L_A, {}_A A_A) = 0$.

Proof. Assume (i). Then there exists an A-A-map ϕ of $\mathrm{Hom}({}_C D, {}_C A)$ to $A\times_B A$ such that $\eta\phi$ is equal to the identity map on $\mathrm{Hom}({}_C D, {}_C A)$. Then we have $A\otimes_B A = K \oplus N$, where $K = \mathrm{Ker}\,\eta$ and $N = \mathrm{Im}\,\phi = \mathrm{Hom}({}_C D, {}_C A)$. The latter is A-centrally projective, since D is C-finitely generated projective. Let f be any A-A-map of K to A. Then since K is an A-A-direct summand of $A\otimes_B A$, f is extended to an A-A-map f^* of $A\otimes_B A$ to A. But as is remarked above K is equal to $\mathrm{Rej}_{A\otimes_B A}(A)$. Therefore we have $f(K) = f^*(K) = 0$, and $\mathrm{Hom}({}_A K_A, {}_A A_A) = 0$. Thus we have (v). Next let $\{g_i, d_i\}$ be a dual basis of D over C, and $\phi(g_i) = \Sigma x_{ij}\otimes y_{ij}$ for each i. Then since $g_i \in \mathrm{Hom}({}_C D, {}_C C) = [\mathrm{Hom}({}_C D, {}_C A)]^A$, and ϕ is an A-A-map, we have $\Sigma x_{ij}\otimes y_{ij} \in (A\otimes_B A)^A$ for each i. Let ι be the inclusion map of D to A. Then $\iota(d) = d = \Sigma d_i g_i(d)$ for each d in D, which implies that $\iota = \Sigma d_i g_i$. Then we have $\iota = \eta\phi(\iota) = \eta(\Sigma d_i \phi(g_i)) = \eta(\Sigma d_i x_{ij}\otimes y_{ij})$, and $d = \iota(d) = \eta(\Sigma d_i x_{ij}\otimes y_{ij})(d) = \Sigma d_i x_{ij} d y_{ij}$ for each d in D. Thus we have also (ii). Next assume (ii). Let M be an A-A-module, and $X = \{m - \Sigma d_i x_{ij} m y_{ij} \mid m \in M^B\}$. Then X is a C-submodul of M, and we have $M^B = X + \Sigma d_i x_{ij} M^B y_{ij}$. For each m in M^B and each A-A-map f of M to A we have $f(m) \in D$ and

$$f(m - \Sigma d_i x_{ij} m y_{ij}) = f(m) - \Sigma d_i x_{ij} f(m) y_{ij} = f(m) - f(m) = 0.$$

Thus we have $f(X) = 0$, and $X \subseteq Rej_M(A)$. Since $\Sigma x_{ij} \otimes y_{ij} \in (A \otimes_B A)^A$, it is obvious that $\Sigma x_{ij} M^B y_{ij} \subseteq M^A$. Hence we have $M^B = DM^A + K$. We will show that the above sum is a direct sum. Let $\Sigma e_i n_i = m - \Sigma d_i x_{ij} m y_{ij}$ for some $e_i \in D$, $n_i \in M^A$ and $m \in M^B$. Then we have

$$\Sigma d_k x_{kl} \Sigma e_i n_i y_{kl} = \Sigma d_k x_{kl} m y_{kl} - \Sigma d_k x_{kl} d_i x_{ij} m y_{ij} y_{kl}$$

$$= \Sigma d_k x_{kl} m y_{kl} - \Sigma d_k x_{kl} d_i y_{kl} x_{ij} m y_{ij} = \Sigma d_k x_{kl} m y_{kl} - \Sigma d_i x_{ij} m y_{ij} = 0$$

But $\Sigma d_k x_{kl} e_i n_i y_{kl} = \Sigma d_k x_{kl} e_i y_{kl} n_i = \Sigma e_i n_i$. Hence $\Sigma e_i n_i = 0$, and we have $X \cap DM^A = 0$. Thus we have (iii). It is obvious that (iii) implies (iv). As for the prooves of (iv)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) see the prooves of (3)$\Rightarrow$(1) of Theorem 1.4 [10] and (2)$\Rightarrow$(1) of Theorem 3.5 [3], repectively. Lastly assume (v). Let ι be the A-A-split monomorphism of N to $A \otimes_B A$. Since $\mathrm{Hom}({}_A L_A, {}_A A_A) = 0$, ι induces the isomorphism ι^* of $\mathrm{Hom}({}_A A \otimes_B A_A, {}_A A_A)$ to $\mathrm{Hom}({}_A N_A, {}_A A_A)$. On the other hand since N is A-centrally projective, we have the A-A-isomomorphism τ_N of N to $\mathrm{Hom}({}_C\mathrm{Hom}({}_A N_A, {}_A A_A), {}_C A)$ defined in the same way as Lemma 1.2. Then we have the following commutative diagram of A-A-maps

$$\begin{array}{ccc}
N & \xrightarrow{\qquad\qquad} & A \otimes_B A \\
\tau_N \downarrow & \iota & \tau \downarrow \\
\mathrm{Hom}({}_C\mathrm{Hom}({}_A N_A, {}_A A_A), {}_C A) & \longrightarrow & \mathrm{Hom}({}_C\mathrm{Hom}({}_A A \otimes_B A_A, {}_A A_A), {}_C A) \\
 & \iota^{**} & \nu^* \downarrow \\
 & & \mathrm{Hom}({}_C D, {}_C A)
\end{array}$$

where τ and ν are the same as defined above, and ι^{**} and ν^* are the maps induced canonically by ι and μ, respectively. Then since ι split as A-A-map, and ι^{**}, τ_N and ν^* are A-A-isomorphisms, we can see that τ and η $(= \nu^* \tau)$ splits as A-A-map.

We have also the following left A-A-homomorphism

$$\eta_l : D \otimes_C A \to \mathrm{Hom}({}_B A, {}_B A)$$

such that $\eta_l(d \otimes a)(x) = dxa$ for $x, a \in A$ and $d \in D$. The map η_r of $A \otimes_C D$ to $\mathrm{Hom}(A_B, A_B)$ is similarly defined.

Now we will give a new characterization of strongly separability which is a generalization of Lemma 2 [8]

Theorem 1.2. Let A be left B-finitely generated projective. Then the following three conditions are equivalent;

(i) A is a strongly separable extension of B

(ii) $\mathrm{Hom}({}_BA, {}_BA) = N \oplus X$ as A-A-module with N A-centrally projective and $X^A = 0$.

(iii) η_1 is an A-A-split monomorphism, and D is C-finitely generated projective.

Proof. By Proposition 3.9 [3] and the definition of the strongly separable extension (i) implies (iii). Now assume (iii). Then we have $\mathrm{Hom}({}_BA, {}_BA) \cong D\otimes_C A \oplus X$ as A-A-modume. Since D is C-finitly generated projective, $D\otimes_C A$ is A-centrally projective, and we have $(D\otimes_C A)^A = D\otimes_C C \cong D$ by Lemma 3 [5]. Then we have

$$D \cong \mathrm{Hom}({}_BA_A, {}_BA_A) = [\mathrm{Hom}({}_BA, {}_BA)]^A \cong (D\otimes_C A)^A \oplus X^A \cong D \oplus X^A$$

which implies that $X^A = 0$. Thus we have (ii). Next assume (ii). Since $X^A = 0$, we have $D \cong [\mathrm{Hom}({}_BA, {}_BA)]^A = X^A \oplus N^A = N^A$. We will denote this isomorphism by μ. For each $d\in D$ $\mu(d)$ is a B-A-endomorphism of A given by $\mu(d)(a) = da$ for $a\in A$. We have also a natural A-A-isomorphism

$$\sigma : N \rightarrow \mathrm{Hom}({}_A\mathrm{Hom}(N_A, A_A), {}_AA)$$

such that $\sigma(x)(f) = f(x)$ for $x\in N$ and $f\in \mathrm{Hom}(N_A, A_A)$, since N is right A-finitly generated projective. Then since σ is an A-A-isomorphism, σ induces a C-isomorphism

$$\sigma : N^A \rightarrow \mathrm{Hom}({}_A\mathrm{Hom}(N_A, A_A)_A, {}_AA_A)$$

Furthermore since A is left B-finitely generated projective, we have a natural isomorphism

$$\rho : A\otimes_B A \rightarrow \mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A)$$

such that $\rho(a\otimes b)(f) = af(b)$ for $a, b\in A$ and $f\in \mathrm{Hom}({}_BA, {}_BA)$. Now by assumption we have $\mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A) = \mathrm{Hom}(X_A, A_A) \oplus \mathrm{Hom}(N_A, A_A)$ as

A-A-module. Thus we have $A\otimes_B A \cong X^* \oplus N^*$, where $X^* = \mathrm{Hom}(X_A, A_A)$ and $N^* = \mathrm{Hom}(N_A, A_A)$ as A-A-module. Since N is A-centrally projective, it is obvious that N^* is also A-centrally projective. Therefore we need only to show that $\mathrm{Hom}({}_AX^*{}_A, {}_AA_A) = 0$. Let ι be the A-A-split monomorphism of N to $\mathrm{Hom}({}_BA, {}_BA)$, and write $\iota^* = \mathrm{Hom}(\iota, A)$, $\iota^{**} = \mathrm{Hom}(\iota^*, A)$ and $\rho^* = \mathrm{Hom}(\rho, A)$. Now consider the composition of all those maps

$$D \to_{\mu} N^{\wedge} \to_{\sigma} \mathrm{Hom}({}_AN^*{}_A, {}_AA_A) \to_{\iota**} \mathrm{Hom}({}_A\mathrm{Hom}(\mathrm{Hom}({}_BA, {}_BA)_A, A_A)_A, {}_AA_A)$$
$$\to_{\rho*} \mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A) \to D$$

Then we have

$$(\rho^* \iota^{**} \sigma \mu(d))(1\otimes 1) = (\iota^{**} \sigma \mu(d))(\rho(1\otimes 1))$$
$$= (\sigma \mu(d))(\iota^*(\rho(1\otimes 1))) = (\sigma \mu(d))(\rho(1\otimes 1)\iota) = \rho(1\otimes 1)(\iota \mu(d))$$
$$= \rho(1\otimes 1)(\mu(d)) = 1\mu(d)(1) = d$$

for each d in D. Hence The composition of all of the above maps is the idendity map on D. This means that $\mathrm{Hom}({}_AX^*{}_A, {}_AA_A) = 0$.

2. Strongly separable Frobenius extensions.

In this section we will give the generalization of the results obtained in [8]. The next theorem is a generalization of Theorem 2 [8].

Theorem 2.1. Suppose that D is a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$. Then A is a strongly separable extension of B if and only if there exists $\Sigma x_k\otimes y_k$ in $(A\otimes_B A)^{\wedge}$ such that $d = \Sigma x_k d d_i y_k e_i = \Sigma d_i x_k e_i d y_k$ hold for each d in D.

If these conditions are satisfied, then A is an H-separable Frobenius extension of $B^* = V_A(V_A(B))$ with a Frobenius system $\{x_k, y_k, h^*\}$, where h^* is defined by $h^*(x) = \Sigma d_i x e_i$ for each x in A.

Proof. Suppose that A is a strngly separable extension of B. then there exists an A-A-maps of $\mathrm{Hom}({}_CD, {}_CA)$ to $A\otimes_B A$ such that $\eta\phi$ is the identity map on $\mathrm{Hom}({}_CD, {}_CA)$, and we have $A\otimes_B A = K \oplus N$, where $K = \mathrm{Ker}\,\eta$ and $N = \mathrm{Im}\,\phi$. Since the restriction of η on N is an A-A-isomorphism, η induces a C-isomorphism of $N^{\wedge}$ to $[\mathrm{Hom}({}_CD, {}_CA)]^{\wedge} = \mathrm{Hom}({}_CD, {}_CC)$. Hence there exists $\Sigma x_k\otimes y_k$ in $N^{\wedge}$ $(\subseteq (A\otimes_B A)^{\wedge})$ such that $\eta(\Sigma x_k\otimes y_k) = h$. Then we have

$d = \Sigma d_i h(e_i d) = \Sigma d_i x_k e_i d y_k$ and $d = \Sigma h(dd_i)e_i = \Sigma x_k dd_i y_k e_i$ for each d in D. Coversely assume that there exists $\Sigma x_k \otimes y_k$ in $(A \otimes_B A)^\wedge$ which satisfies the above condition. Then also $\Sigma x_k e_i \otimes y_k$ belongs to $(A \otimes_B A)^\wedge$ for each i, and we have $d = \Sigma d_i x_k e_i d y_k$ for each d in D. Then A is a strongly separable extension of B by Theorem 3.5 [3]. Thus we have proved the equivalence. Next define h^* as in the theorem. Since $\Sigma x_k D y_k \subseteq C$, we have $\Sigma x_k h^*(y_k x) = \Sigma x_k d_i y_k x e_i = x \Sigma x_k d_i y_k e_i = x1 = x$ for each x in A. Similarly we have $\Sigma h^*(x x_k) y_k = x$ for each x in A. Furthermore since $\Sigma d_i \otimes e_i \in (D \otimes_C D)^D$, we have $h^*(A) = \Sigma d_i A e_i \subseteq V_A(D) = B^*$. Hence h^* is a B^*-B^*-map of A to B^*. Then $\{x_k, y_k, h^*\}$ is a Frobenius system of A over B^*. On the other hand since A is a strongly separable extension of B, η is an A-A-split epimorphism. Then since $B^* \supseteq B$ and $V_A(B^*) = D$, η induces an epimorphism η^* of $A \otimes_{B^*} A$ to $\mathrm{Hom}({}_C D, {}_C A)$. Now let $\Sigma a_j \otimes b_j \in \mathrm{Ker}\, \eta^*$. Then since $\Sigma a_j d_i b_j = 0$ for each i, we have in $A \otimes_{B^*} A$

$$\Sigma a_i \otimes b_j = \Sigma a_j \otimes h^*(b_j x_k) y_k = \Sigma a_j h^*(b_j x_k) \otimes y_k$$
$$= \Sigma a_j d_i b_j x_k e_i \otimes y_k = \Sigma 0 \otimes y_k = 0$$

Thus $\mathrm{Ker}\, \eta^* = 0$, and we see that η^* is an isomorphism. Then since $V_A(B^*)$ (= D) is C-finitely generated projective, A is an H-separable extension of B^*.

The next theorem is a generalization of Theorem 1 [8]

Theorem 2.2. Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, h^*\}$, then the following conditions are equivalent

(i) A is a strongly separable extension of B

(ii) There exist finite d_i, e_i in D such that $\Sigma d_i x_k e_i d y_k = d$ holds for each d in D

(iii) D is a Frobenius C-algebra with a Frobenius homomorphism h which is defined by $h(d) = \Sigma x_k d y_k$ for each d in D

Proof. Since $(A \otimes_B A)^\wedge = \Sigma x_k D \otimes y_k$, the equivalence of (i) and (ii) is an immediate consequence of Theorem 3.5 [3]. Now assume (ii), and define h as in (iii). Then since $\Sigma x_k \otimes y_k \in (A \otimes_B A)^\wedge$ and $\Sigma x_k e_i \otimes y_k \in (A \otimes_B A)^\wedge$ for each i, if we define $f_i(d) = \Sigma x_k e_i d y_k$ (= $h(e_i d)$) for each d in D, f and each f_i belong to $\mathrm{Hom}({}_C D, {}_C C)$, and we have $d = \Sigma d_i x_k e_i d y_k = \Sigma d_i f_i(d)$ by assumption. Therefore $\{f_i, d_i\}$ forms a dual basis for D over C. Now write $D^* = \mathrm{Hom}({}_C D, {}_C C)$, and let θ be the map of D to D^* defined by $\theta(d) = hd$ for each d in D. Then since $d = \Sigma d_i h(e_i d)$ we have

$$f(d) = f(\Sigma d_i h(e_i d)) = \Sigma f(d_i) h(e_i d) = h(\Sigma f(d_i) e_i d)$$
$$= (h \Sigma f(d_i) e_i)(d)$$

and consequently $f = h \Sigma f(d_i) e_i$ for each f in D^*. Thus θ is an epimorphism. Moreover since D is C-finitely generated projective, D^* is also C-finitely generated projective. Hence θ splits as C-map, and we can write $D \cong D^* \oplus X$ where $X = \mathrm{Ker}\,\theta$. Furthermore for each maximal ideal m of C we have $D^* \otimes_C C/m \cong \mathrm{Hom}_{C/m}(D \otimes_C C/m, C/m) \cong D \otimes_C D/m$ as vector space over C/m (See e.g., Corollary 2.5 [1]). Then we have $X/mX = 0$ for each maximal ideal m of C, which implies that $X = 0$ by generalized Nakayama's Lemma. Therefore θ is a C-isomorphism. It is obvious that θ is a right D-map. Thus we have (iii). Next assume (iii). Then there exist a_j, b_j in D such that $\{a_j, b_j, h\}$ forms a Frobenius system of D over C, and we have $d = \Sigma a_j h(b_j d) = \Sigma a_j x_k b_j d y_k$ for each $d \in D$. But we have also $\Sigma x_k b_j \otimes y_k \in (A \otimes_B A)^A$ for each j, since $\Sigma x_k \otimes y_k \in (A \otimes_B A)^A$. Then we have (i).

By the above theorem we can obtained a new characterization of projecive separable algebra.

Corollary 2.1. Let A be a Frobenius algebra over a commutative ring R with $\{x_k, y_k, h\}$ a Frobenius system, and define $h^*(x) = \Sigma x_k x y_k$ for each x in A. Then A is a separable R-algebra if and only if h^* is also a Frobenius homomorphism of A over R.

References

[1] F. DeMeyer and E. Ingraham; Separable Algebras Over Commutative Rings, Lecture Notes in Math., 181 (1971), Springer.

[2] K. Hirata; Separable extensions and centralizers of rings, Nagoya Math. J., 35 (1969), 31-45.

[3] A. Mewborn and E. McMahon; Separable extensions of noncommutative rings, Hokaido Math. J., 13 (1984), 74-88.

[4] K. Sugano; Note on semisimple extensions and separable extensions Osaka J. Math., 4 (1967), 265-270

[5] K. Sugano; Separable extensions and Frobenius extensions; Osaka J. Math., 7 (1970), 291-299.

[6] K. Sugano; On H-separable extensions of two-sided simple rings, Hokaido Math. J., 11 (1982), 246-252.

[7] K. Sugano; On flat H-separable extensions and Gabriel topology, Hokkaido Math. J., 15 (1986), 149-155.

[8] K. Sugano; Note on H-separable Frobenius extensions, Hokkaido Math. J., 24 (1995), 105-111.

[9] K. Yamashiro; On strongly separable extensions, Hokkaido Math. J., 19 (1990), 281-287.

Department of mathematics
Hokkaido University
Sapporo 060, Japan

Added in proof. The all part of section 1 of this paper, except for the proof of the equivalence of (ii), (iii) and (iv), have been introduced by the same author at the summer seminar which was held at Zhongshan University, Guangzhou China, in July 1993.