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Kimitoshi Tsutaya

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Local Regularity of Non-Resonant Nonlinear Wave Equations

Dedicated to Professor Rentaro Agemi on his sixtieth birthday

Kimitoshi Tsutaya *

Department of Mathematics
Brown University
Providence, R.I. 02912, U.S.A.

1 Introduction

In this paper we study the problem of minimal regularity of initial conditions required to ensure well-posedness (local existence, uniqueness and continuous dependence on the data) for systems of nonlinear wave equations with quadratic nonlinearity. We consider the Cauchy problem for systems of wave equations in three space dimensions :

$$
(1.1)\qquad \begin{cases} \Box_1 u \equiv (\partial_t^2 - C_1^2\Delta)u = F(u,v,\partial u,\partial v), & t>0,\ x\in R^3, \\ \Box_2 v \equiv (\partial_t^2 - C_2^2\Delta)v = G(u,v,\partial u,\partial v), & t>0,\ x\in R^3, \\ u(0,x)=u_0(x), \quad \partial_t u(0,x) = u_1(x), & x\in R^3, \\ v(0,x)=v_0(x), \quad \partial_t v(0,x) = v_1(x), & x\in R^3, \end{cases}
$$

where u, v are scalar functions of $t > 0$ and $x \in R^3$, the $C_j,\ j = 1,2$ are positive constants, and

$$
\partial = (\partial_t, \nabla) = (\partial_0, \partial_1, \partial_2, \partial_3), \qquad \partial_j = \partial/\partial x_j \ \text{ for } \ j = 1,2,3.
$$

The constants C_1 and C_2 are the propagation speeds. The functions F and G are assume to be quadratic and

$$
F(u,v,p,q),\ G(u,v,p,q) \in C^\infty(R\times R\times R^4\times R^4).
$$

*Permanent address : Department of Mathematics, Hokkaido University, Sapporo 060, Japan

First let us consider the case where the nonlinear terms F and G are independent of u and v, i.e., F and G have the form

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v, \quad \alpha, \beta = 0,1,2,3. \tag{1.2}$$

The proof of the classical local existence theorem relies on energy estimates and the Sobolev imbedding theorem. In R^{1+n}, this requires the initial conditions

$$u_0, v_0 \in H^s(R^n) \quad \text{and} \quad u_1, v_1 \in H^{s-1}(R^n) \tag{1.3}$$

with $s > n/2 + 1$ for (1.2). Recently, Klainerman and Machedon [2] proved that if $C_1 = C_2 = 1$, and F, G have the null forms,

$$Q_W(u,v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v, \tag{1.4a}$$

$$Q_R(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v, \quad 0 \leq \alpha, \beta \leq 3, \tag{1.4b}$$

then the Cauchy problem is well-posed for $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$. This remarkable result is based on new space-time estimates for the null forms (1.4a), (1.4b). Moreover, in [3] they improved minimal regularity assumptions for well-posedness to be $u_0, v_0 \in H^s$, $u_1, v_1 \in H^{s-1}$ with $s > 3/2$ if $C_1 = C_2 = 1$, and both F and G have the null form Q_W in (1.4a).

For general quadratically nonlinear terms of the form (1.2) in three space dimensions, the lower bound for the Sobolev exponent s in (1.3) can be lowered to $s > 2$. This result † of Ponce and Sideris [7] was proved by using a space-time estimate called the Strichartz estimate. In particular, in the spherically symmetric case, the Cauchy problem is well-posed for $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$ and for general nonlinear terms of the form (1.2) ‡. This was shown by Klainerman and Machedon [2].

On the other hand, in [5] Lindblad gave a sharp counterexample so that the Cauchy problem for the scalar wave equation

$$\Box u \equiv (\partial_t^2 - \Delta)u = (\partial_t u)^2$$

† The result generalizes to systems (1.1) with $C_1 \neq C_2$.

‡ This result is true not only for quadratically nonlinear terms of the form (1.2) but also for all nonlinear terms of the form $F(u, v, \partial u, \partial v)$ which are quadratic in $\partial u, \partial v$.

is ill-posed for the exponent $s < 2$. He also showed in [5] that in general the problems for the scalar wave equations

$$\begin{aligned} \Box u &= u\partial_t u, \\ \Box u &= u^2 \end{aligned}$$

are ill-posed for the exponents $s < 1$, $s \leq 0$, respectively. See also [6].

One aim of this paper is to show that the same type of estimates as those of Klainerman and Machedon [2] hold for $C_2 > C_1$ and for the forms $Q_{\alpha\beta}(u,v)$ in (1.2) and

$$Q_\alpha(u,v) = \partial_\alpha u \cdot v, \quad \alpha = 0,1,2,3. \tag{1.5}$$

Theorem 1 *Consider the solutions u, v of a system of inhomogeneous wave equations*

$$\begin{cases} \Box_1 u \equiv (\partial_t^2 - C_1^2\Delta)u = F(t,x), & t > 0,\ x \in R^3, \\ \Box_2 v \equiv (\partial_t^2 - C_2^2\Delta)v = G(t,x), & t > 0,\ x \in R^3, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in R^3, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), & x \in R^3. \end{cases} \tag{1.6}$$

Suppose $C_2 > C_1$.

(i) Let $u_0, v_0 \in H^1(R^3)$, $u_1, v_1 \in L^2(R^3)$ and $F, G \in L^1(R_+, L^2(R^3))$. Then, for the form (1.5), we have

$$\begin{aligned} &\int_0^T \int_{R^3} |Q_\alpha(u,v)|^2 dxdt \\ \leq\ & C\left(\|\nabla u_0\|_{L^2(R^3)} + \|u_1\|_{L^2(R^3)} + \int_0^T \|F(t)\|_{L^2(R^3)}dt\right)^2 \\ &\times \left(\|\nabla v_0\|_{L^2(R^3)} + \|v_1\|_{L^2(R^3)} + \int_0^T \|G(t)\|_{L^2(R^3)}dt\right)^2 \end{aligned}$$

for all $T > 0$.

(ii) Let $v_0 \in H^2(R^3)$, $u_0, v_1 \in H^1(R^3)$, $u_1 \in L^2(R^3)$, $F \in L^1(R_+, L^2(R^3))$ and $G \in L^1(R_+, H^1(R^3))$. Then, for the form (1.2), we have

$$\begin{aligned} &\int_0^T \int_{R^3} |Q_{\alpha\beta}(u,v)|^2 dxdt \\ \leq\ & C\left(\|\nabla u_0\|_{L^2(R^3)} + \|u_1\|_{L^2(R^3)} + \int_0^T \|F(t)\|_{L^2(R^3)}dt\right)^2 \\ &\times \left(\|\nabla^2 v_0\|_{L^2(R^3)} + \|\nabla v_1\|_{L^2(R^3)} + \int_0^T \|\nabla G(t)\|_{L^2(R^3)}dt\right)^2 \end{aligned}$$

for all $T > 0$.

Similar estimates hold for other dimensions $1+n$. Let $\dot{H}^s(R^n)$ denote the homogeneous Sobolev space and $\omega_0 = (-\Delta)^{1/2}$. The following corollary follows easily from the proof of Theorem 1.

Corollary 1 *Let u, v be the solutions of (1.6) in R^{1+n}. Suppose $C_2 > C_1$.*

(i) Let $u_0 \in \dot{H}^1(R^n)$, $u_1 \in L^2(R^n)$, $v_0 \in \dot{H}^{(n-1)/2}(R^n)$, $v_1 \in \dot{H}^{(n-3)/2}(R^n)$, $F \in L^1(R_+, L^2(R^n))$ and $G \in L^1(R_+, \dot{H}^{(n-3)/2}(R^n))$. Then, for Q_α in (1.5), we have

$$
\begin{aligned}
&\int_0^T \int_{R^3} |Q_\alpha(u,v)|^2 dx dt \\
\leq\ & C\left(\|\nabla u_0\|_{L^2(R^n)} + \|u_1\|_{L^2(R^n)} + \int_0^T \|F(t)\|_{L^2(R^n)} dt\right)^2 \\
&\times \left(\|\omega_0^{(n-1)/2} v_0\|_{L^2(R^n)} + \|\omega_0^{(n-3)/2} v_1\|_{L^2(R^n)} + \int_0^T \|\omega_0^{(n-3)/2} G(t)\|_{L^2(R^n)} dt\right)^2
\end{aligned}
$$

for all $T > 0$.

(ii) Let $u_0 \in \dot{H}^1(R^n)$, $u_1 \in L^2(R^n)$, $v_0 \in \dot{H}^{(n+1)/2}(R^n)$, $v_1 \in \dot{H}^{(n-1)/2}(R^n)$, $F \in L^1(R_+, L^2(R^n))$ and $G \in L^1(R_+, \dot{H}^{(n-1)/2}(R^n))$. Then, for $Q_{\alpha\beta}$ in (1.2), we have

$$
\begin{aligned}
&\int_0^T \int_{R^3} |Q_{\alpha\beta}(u,v)|^2 dx dt \\
\leq\ & C\left(\|\nabla u_0\|_{L^2(R^n)} + \|u_1\|_{L^2(R^n)} + \int_0^T \|F(t)\|_{L^2(R^n)} dt\right)^2 \\
&\times \left(\|\omega_0^{(n+1)/2} v_0\|_{L^2(R^n)} + \|\omega_0^{(n-1)/2} v_1\|_{L^2(R^n)} + \int_0^T \|\omega_0^{(n-1)/2} G(t)\|_{L^2(R^n)} dt\right)^2
\end{aligned}
$$

for all $T > 0$.

Remark 1 To show the above estimates, we need to solve some quadratic equations (see Section 2) while the corresponding equations in [2] are linear.

Even if $C_2 > C_1$, however, it seems impossible to construct local solutions for the form (1.2) and arbitrary initial data $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$ except spherically symmetric case. If we could show that

$$
\int_0^T \int_{R^3} |\partial Q_{\alpha\beta}(u,v)|^2 dx dt \tag{1.7}
$$

$$\leq C\left(\|\nabla^2 u_0\|_{L^2(R^3)} + \|\nabla u_1\|_{L^2(R^3)} + \int_0^T \|\nabla F(t)\|_{L^2(R^3)}dt\right)^2$$
$$\times\left(\|\nabla^2 v_0\|_{L^2(R^3)} + \|\nabla v_1\|_{L^2(R^3)} + \int_0^T \|\nabla G(t)\|_{L^2(R^3)}dt\right)^2$$

for all $T > 0$ and arbitrary $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$, $F, G \in L^1(R_+, H^1(R^3))$, then the well-posedness would be true for $u_0, v_0 \in H^2$, $u_1, v_1 \in H^1$. As in Klainerman and Machedon [2], the above inequality is true if we replace $Q_{\alpha\beta}(u,v)$ by Q_W or Q_R in (1.4) and set $C_1 = C_2 = 1$. This result is based on the symmetry between u and v [§]. As mentioned above, by using this space-time estimate they proved that the Cauchy problem for the null forms (1.4) is well-posed for $u_0, v_0 \in H^2$, $u_1, v_1 \in H^1$ and $C_1 = C_2 = 1$.

But in our problem, since the propagation speeds C_1 and C_2 are different, the symmetry between u and v fails, and (1.7) thus seems wrong for $Q_{\alpha\beta}$ and for general initial data.

Next let us consider the case where F and G have the form $\partial_\alpha u \cdot v$ or $u\partial_\beta v$. In this case, the classical local existence theorem requires the initial conditions (1.3) with $s > n/2$ in R^{1+n}. If we suppose $C_1 = C_2$ and non-spherically symmetric case, then it seems impossible to us to construct local solutions for $u_0, v_0 \in H^1$, $u_1, v_1 \in L^2$ from the results of Lindblad [5, 6] as above. However, in the case $C_2 > C_1$ the problem is well-posed for $u_0, v_0 \in H^1$, $u_1, v_1 \in L^2$ if F and G have the form (1.5). The following is the main result of this paper.

Theorem 2 *Assume the form (1.5) for the nonlinear terms F and G in (1.1) with $C_2 > C_1$. Then, for arbitrary initial data $u_0, v_0 \in H^1(R^3)$, $u_1, v_1 \in L^2(R^3)$ there exist a $T > 0$ depending only on C_1, C_2 and $\|u_0\|_{H^1(R^3)} + \|v_0\|_{H^1(R^3)} + \|u_1\|_{L^2(R^3)} + \|v_1\|_{L^2(R^3)}$ such that (1.1) has unique local solutions $(u(t), v(t))$ on $[0,T]$ satisfying*

$$u,\ v \in \bigcap_{j=0}^{1} C^j([0,T]; H^{1-j})$$

and

$$\int_0^T \|\partial_\alpha u \cdot v(t)\|^2_{L^2(R^3)}dt < \infty, \qquad \alpha = 0, 1, 2, 3.$$

[§] The following estimate holds for the solutions u, v of (1.1) with $C_2 = C_1 = 1$, $F = G = 0$, $u_0 = v_0 = 0$ and for $Q = Q_W$ or Q_R : $\|Q(u,v)\|_{L^2(R^{1+3})} \leq C\min\{\|u_1\|_{L^2(R^3)}\|\nabla v_1\|_{L^2(R^3)}, \|\nabla u_1\|_{L^2(R^3)}\|v_1\|_{L^2(R^3)}\}$.

Remark 2 Notice the asymmetry in (1.5). Let $Q^*_\alpha(u,v) = u\partial_\alpha v, \quad \alpha = 0,1,2,3.$ From the proof of Theorem 1, it is conjectured that even in the case $C_2 > C_1$ the result of Theorem 2 is wrong for general initial conditions if the nonlinear terms F and G have this form.

We remark that in the case where F and G in (1.1) with $C_2 > C_1$ have the form $Q_Z(u,v) = uv$, the Cauchy problem is well-posed for $u_0 \in L^2(R^3), u_1 \in \dot{H}^{-1}(R^3), v_0 \in H^1(R^3)$ and $v_1 \in L^2(R^3)$. This is proved by using the following corollary of Theorem 1.

Corollary 2 *Let u,v be the solutions of (1.6). Suppose $C_2 > C_1$. Then, for the form $Q_Z(u,v) = uv$, we have*

$$\begin{aligned}
&\int_0^T \int_{R^3} |Q_Z(u,v)|^2 dx dt \\
\leq\ & C\left(\|u_0\|_{L^2(R^3)} + \|\omega_0^{-1}u_1\|_{L^2(R^3)} + \int_0^T \|\omega_0^{-1}F(t)\|_{L^2(R^3)}dt\right)^2 \\
&\times\left(\|\nabla v_0\|_{L^2(R^3)} + \|v_1\|_{L^2(R^3)} + \int_0^T \|G(t)\|_{L^2(R^3)}dt\right)^2
\end{aligned}$$

for all $T > 0$ and $v_0 \in H^1(R^3), u_0, v_1 \in L^2(R^3), u_1 \in \dot{H}^{-1}(R^3), F \in L^1(R_+, \dot{H}^{-1}(R^3)), G \in L^1(R_+, L^2(R^3))$.

The motivation for studying the problem of minimizing regularity conditions for (1.1) with $C_2 > C_1$ has something to do with the global existence results. Let us recall the global existence result for nonlinear wave equations satisfying the null conditions.

It is well-known that in three space dimensions the scalar wave equations with quadratic nonlinearity have no global solutions in general even for small initial data. However, the system of wave equations with the same propagation speeds whose nonlinear terms satisfy the null conditions has global solutions for small initial data (see Klainerman [1]). This result is due to good decay estimates. The nonlinear terms satisfying the null conditions provide us a good decay property, which one can show by using the generators of the Lorentz group. Also, the Cauchy problem for the system with the nonlinear terms satisfying the null conditions is known to be well-posed under lower regularity assumptions by [2, 3].

On the other hand, the quadratic nonlinear terms of the coupled system (1.1) with $C_1 \neq C_2$ have also a good decay property, because the propagation speeds are different. Using this property, Kovalyov [4] proved the existence of global solutions of the system (1.1) with $C_1 \neq C_2$, $F, G = \partial_\alpha u \partial_\beta v$ for small initial data. For details, see [4]. Then, the following natural question arises : can we improve the regularity requirements on the initial data to ensure well-posedness for systems of quadratic nonlinear wave equations with *distinct* propagation speeds ? Our study began with this question. Although we have succeeded minimizing the regularity assumptions for some nonlinearity as above, we do not have any result which indicates a relation between decay estimates and minimal regularity assumptions.

Our plan in the present paper is as follows. In Section 2 we prove Theorem 1 following the way in [2]. In Section 3 we describe the proof of Theorem 2 by the standard iteration argument.

We conclude this section by giving several notations. For $\xi, \eta \in R^3$ we put $\xi \cdot \eta = \sum_{j=1}^{3} \xi_j \eta_j$. Let $\mathcal{S}$ be the Schwartz space on R^3. For $f \in \mathcal{S}(R^3)$ we define the Fourier transform $\hat{f}(\xi)$ of f with respect to the space variables by

$$\hat{f}(\xi) = \int_{R^3} e^{-ix\cdot\xi} f(x)dx.$$

We also denote the space-time Fourier transform of F by

$$\mathcal{F}(F)(\tau, \xi) = \tilde{F}(\tau, \xi) = \int\int_{R^{1+3}} e^{-i(t\tau + x\cdot\xi)} F(t, x)dxdt.$$

For a multi-index $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, we put

$$\partial^\lambda = \partial_t^{\lambda_0} \partial_1^{\lambda_1} \cdots \partial_n^{\lambda_n}.$$

In the course of calculations below, the various constants are simply denoted by C.

2 Proof of Theorem 1

We consider the following inhomogeneous wave equations with different propagation speeds

$$
(2.1) \qquad \begin{cases} \Box_1 u \equiv (\partial_t^2 - C_1^2 \Delta) u = F(t,x), & t>0,\ x \in R^3, \\ \Box_2 v \equiv (\partial_t^2 - C_2^2 \Delta) v = G(t,x), & t>0,\ x \in R^3, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in R^3, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), & x \in R^3. \end{cases}
$$

We first prove the following theorem.

Theorem 2.1 *Let u, v be the solutions of (2.1) with $C_2 > C_1$, $F = G = 0$, $u_0 = v_0 = 0$. Then, for the forms $Q = Q_\alpha$, $Q_{\alpha\beta}$ defined by (1.5), (1.2), we have*

$$
\int_R \int_{R^3} |Q(u,v)|^2 dx dt \le \begin{cases} C \|u_1\|^2_{L^2(R^3)} \|v_1\|^2_{L^2(R^3)} & \text{if } Q = Q_\alpha, \\ C \|u_1\|^2_{L^2(R^3)} \|\nabla v_1\|^2_{L^2(R^3)} & \text{if } Q = Q_{\alpha\beta} \end{cases}
$$

for all $u_1 \in L^2(R^3), v_1 \in L^2(R^3)$ (if $Q = Q_\alpha$), $v_1 \in H^1(R^3)$ (if $Q = Q_{\alpha\beta}$).

Proof) We first prove the above theorem with the data $u_1 = f$, $v_1 = g$ in the Schwartz space $\mathcal{S}(R^3)$, whose Fourier transforms $\hat{f}, \hat{g}$ belong to $C_0^\infty(R^3)$ with $0 \notin \{\text{supp } \hat{f}\} \cup \{\text{supp } \hat{g}\}$.

Let u, v be the solutions of the homogeneous wave equations

$$
(2.2) \qquad \begin{cases} \Box_1 u = 0, & t>0,\ x \in R^3, \\ \Box_2 v = 0, & t>0,\ x \in R^3, \\ u(0,x) = 0, \quad \partial_t u(0,x) = f(x), & x \in R^3, \\ v(0,x) = 0, \quad \partial_t v(0,x) = g(x), & x \in R^3. \end{cases}
$$

We write

$$
u = \frac{1}{2i}(u_+ - u_-), \qquad v = \frac{1}{2i}(v_+ - v_-),
$$

where

$$
(2.3) \qquad u_\pm(t,x) = (2\pi)^{-3} \int_{R^3} \frac{e^{\pm i C_1 t |\xi|}}{C_1 |\xi|} \hat{f}(\xi) e^{ix\cdot\xi} d\xi,
$$

$$
(2.4) \qquad v_\pm(t,x) = (2\pi)^{-3} \int_{R^3} \frac{e^{\pm i C_2 t |\xi|}}{C_2 |\xi|} \hat{g}(\xi) e^{ix\cdot\xi} d\xi.
$$

By repeating the argument of Klainerman and Machedon [2], $\mathcal{F}(\partial^\alpha u_\pm) * \mathcal{F}(\partial^\beta v_\pm)$ turns out to be well-defined in the sense of distributions, and

$$\mathcal{F}(\partial^\alpha u_\pm) * \mathcal{F}(\partial^\beta v_\pm) = \mathcal{F}(\partial^\alpha u_\pm \partial^\beta v_\pm) \in L^2(R^{1+3}). \tag{2.5}$$

We now compute explicitly the functions $\mathcal{F}(u_\pm v_\pm)$ and $\mathcal{F}(Q(u_\pm, v_\pm))$ using (2.5).

Lemma 2.2 *Assume that f, g belong to the Schwartz space $\mathcal{S}(R^3)$, whose Fourier transforms $\hat{f}, \hat{g}$ are C_0^∞ functions and vanish in a neighborhood of the origin. Let $C_2 > C_1$. Then, the functions $\mathcal{F}(u_+v_+)$ and $\mathcal{F}(u_-v_-)$ are supported in the interiors of the future and past light cones, $\{\tau \geq C_1|\xi|\}$ and $\{-\tau \geq C_1|\xi|\}$, respectively.*

Moreover, the functions $\mathcal{F}(u_+v_+)$ and $\mathcal{F}(u_-v_-)$ are expressed in the following forms :

$$\mathcal{F}(u_+v_+)(\tau, \xi) = C \int_{S^2} \frac{1}{\sqrt{D}} \alpha_- \hat{f}(\xi - \alpha_- \omega) \hat{g}(\alpha_- \omega) d\omega$$

and

$$\mathcal{F}(u_-v_-)(\tau, \xi) = -C \int_{S^2} \frac{1}{\sqrt{D}} \alpha_+ \hat{f}(\xi - \alpha_+ \omega) \hat{g}(\alpha_+ \omega) d\omega,$$

where

$$\begin{aligned} D = D(\tau, \xi, \omega) &= (C_2\tau - C_1^2 \xi \cdot \omega)^2 - (C_2^2 - C_1^2)(\tau^2 - C_1^2|\xi|^2), && (2.6) \\ \alpha_\pm &= \frac{C_2\tau - C_1^2\xi \cdot \omega \pm \sqrt{D}}{C_2^2 - C_1^2}, && (2.7) \end{aligned}$$

and C is a constant depending only on C_1 and C_2.

Remark 2.1 Note that $D(\tau, \xi, \omega) = 0$ if and only if $\tau = C_2 \xi \cdot \omega$ and $|\xi| = \xi \cdot \omega$, since a simple calculation gives us

$$\begin{aligned} D(\tau, \xi, \omega) &= (C_2\tau - C_1^2 \xi \cdot \omega)^2 - (C_2^2 - C_1^2)(\tau^2 - C_1^2|\xi|^2) \\ &= C_1^2((C_2^2 - C_1^2)(|\xi|^2 - (\xi \cdot \omega)^2) + (\tau - C_2\xi \cdot \omega)^2). \end{aligned} \tag{2.8}$$

Proof) From (2.3) and (2.4), we have

$$\widetilde{u_\pm}(\tau, \xi) = \int_R \frac{e^{\pm i C_1 t |\xi|}}{C_1|\xi|} \hat{f}(\xi) e^{-it\tau} dt$$

$$\begin{aligned} &= 2\pi \frac{\hat{f}(\xi)}{C_1|\xi|}\delta(\tau \mp C_1|\xi|) \\ &= 4\pi \hat{f}(\xi)\chi_\pm(\tau)\delta(\tau^2 - C_1^2|\xi|^2), \\ \widetilde{v_\pm}(\tau,\xi) &= 2\pi \frac{\hat{g}(\xi)}{C_2|\xi|}\delta(\tau \mp C_2|\xi|), \end{aligned}$$

where δ denotes the Dirac delta function, and χ_+, χ_- are the characteristic functions of the intervals $[0,\infty), (-\infty, 0]$, respectively. We put

$$\begin{aligned} \mu_\pm &= \frac{\hat{f}(\xi)}{C_1|\xi|}\delta(\tau \mp C_1|\xi|), \\ \nu_\pm &= \frac{\hat{g}(\xi)}{C_2|\xi|}\delta(\tau \mp C_2|\xi|). \end{aligned}$$

We first compute $\mathcal{F}(u_+v_+)$. By (2.5), we have

$$\mathcal{F}(u_+v_+) = 4\pi^2 \mu_+ * \nu_+$$

and

$$\begin{aligned} (2.9) \qquad \mu_+ * \nu_+(\tau,\xi) &= 2\int_{R^3}\int_{-\infty}^{\infty} \chi_+(\tau-\tau')\hat{f}(\xi-\xi')\delta((\tau-\tau')^2 - C_1^2|\xi-\xi'|^2) \\ &\quad \times \frac{\hat{g}(\xi')}{C_2|\xi'|}\delta(\tau' - C_2|\xi'|)d\tau' d\xi'. \end{aligned}$$

We see from the above integral and $C_2 > C_1$ that $\mu_+ * \nu_+$ is supported in the future of the origin $\{\tau \geq C_1|\xi|\}$. For $\tau' = C_2|\xi'|$, we have

$$\begin{aligned} \eta &\equiv (\tau-\tau')^2 - C_1^2|\xi-\xi'|^2 \\ &= \tau^2 - C_1^2|\xi|^2 - 2C_2\tau|\xi'| + (C_2^2 - C_1^2)|\xi'|^2 + 2C_1^2\xi\cdot\xi'. \end{aligned}$$

In terms of spherical coordinates $\xi' = \rho\omega$ with $|\omega| = 1$, η takes the form

$$\begin{aligned} \eta &= (C_2^2 - C_1^2)\rho^2 - 2(C_2\tau - C_1^2\xi\cdot\omega)\rho + \tau^2 - C_1^2|\xi|^2 \\ &= (C_2^2 - C_1^2)(\rho - \alpha_+)(\rho - \alpha_-), \end{aligned}$$

where $\alpha_\pm$ are defined by (2.7). Then, the integral (2.9) becomes

$$\begin{aligned} (2.10) \qquad &\mu_+ * \nu_+(\tau,\xi) \\ &= 2\int_{R^3}\chi_+(\tau - C_2|\xi'|)\hat{f}(\xi-\xi')\delta(\eta)\frac{\hat{g}(\xi')}{C_2|\xi'|}d\xi' \\ &= \frac{2}{C_2}\int_0^{\tau/C_2}\rho\int_{S^2}\hat{f}(\xi-\rho\omega)\delta((C_2^2-C_1^2)(\rho-\alpha_+)(\rho-\alpha_-))\hat{g}(\rho\omega)d\omega d\rho. \end{aligned}$$

We here need to check whether $\rho = \alpha_+, \alpha_-$ satisfy $\tau > C_2|\xi'| = C_2\rho$. Suppose $\tau \neq C_2|\xi|$ and $|\xi| > 0$. By (2.8), we have

$$
\begin{aligned}
(2.11) \qquad & C_2^2 D - C_1^4(\tau - C_2\xi\cdot\omega)^2 \\
= \;& C_1^2(C_2^2 - C_1^2)\big\{(\tau - C_2\xi\cdot\omega)^2 + C_2^2(|\xi|^2 - (\xi\cdot\omega)^2)\big\} \geq 0.
\end{aligned}
$$

Note that the last equality holds if and only if $\tau = C_2\xi\cdot\omega = C_2|\xi|$. Hence, it follows that

$$\tau - C_2\alpha_- = \frac{-C_1^2(\tau - C_2\xi\cdot\omega) + C_2\sqrt{D}}{C_2^2 - C_1^2} > 0,$$

and

$$\tau - C_2\alpha_+ = \frac{-C_1^2(\tau - C_2\xi\cdot\omega) - C_2\sqrt{D}}{C_2^2 - C_1^2} < 0.$$

Thus, when $\tau \neq C_2|\xi|$, only $\rho = \alpha_-$ satisfies $\tau > C_2\rho$. Moreover, since $\mu_+ * \nu_+$ is supported in $\{\tau \geq C_1|\xi|\}$ and $C_2 > C_1$, we see $\alpha_- \geq 0$. Hence, from (2.10) we obtain for $\tau \neq C_2|\xi|$

$$\mu_+ * \nu_+(\tau,\xi) = C\int_{S^2} \frac{1}{\sqrt{D}}\alpha_- \hat{f}(\xi - \alpha_-\omega)\hat{g}(\alpha_-\omega)d\omega.$$

Next suppose $\tau = C_2|\xi| > 0$. In this case, only when $|\xi| = \xi\cdot\omega$, we have $\alpha_+ = \alpha_- = |\xi|$. On the other hand, when $|\xi| \neq \xi\cdot\omega$, we see that only $\rho = \alpha_-$ satisfies $\tau > C_2\rho$ and $\rho \geq 0$ as before. Thus, from (2.10) we obtain for $\tau = C_2|\xi|$,

$$\mu_+ * \nu_+(\tau,\xi) = C\int_{S^2} \frac{1}{\sqrt{D}}\alpha_- \hat{f}(\xi - \alpha_-\omega)\hat{g}(\alpha_-\omega)d\omega.$$

For the case $|\xi| = 0$, the desired equality follows easily from the same argument as before.

It remains to prove the formula for $\mathcal{F}(u_-v_-)$. Since

$$
\begin{aligned}
\mu_- * \nu_-(\tau,\xi) \;=\; & 2\int_{R^3}\int_\tau^\infty \hat{f}(\xi - \xi')\delta((\tau - \tau')^2 - C_1^2|\xi - \xi'|^2) \\
& \times \frac{\hat{g}(\xi')}{C_2|\xi'|}\delta(\tau' + C_2|\xi'|)d\tau' d\xi',
\end{aligned}
$$

$\mu_- * \nu_-$ is supported in the past of the origin $\{-\tau \geq C_1|\xi|,\ \tau \leq 0\}$. Proceeding as before, we have

$$\mu_- * \nu_-(\tau,\xi) = C\int_{S^2} \frac{1}{\sqrt{D}}\overline{\alpha}_- \hat{f}(\xi - \overline{\alpha}_-\omega)\hat{g}(\overline{\alpha}_-\omega)d\omega,$$

where

$$\overline{\alpha}_- = \frac{-C_2\tau - C_1^2\xi\cdot\omega - \sqrt{\overline{D}}}{C_2^2 - C_1^2},$$
$$\overline{D} = (C_2\tau + C_1^2\xi\cdot\omega)^2 - (C_2^2 - C_1^2)(\tau^2 - C_1^2|\xi|^2).$$

Therefore, making the change of variables $\omega \to -\omega$, we obtain the desired formula. ■

Lemma 2.3 *Let f, g satisfy the conditions of the previous lemma. Then, the functions $\mathcal{F}(u_-v_+)$ and $\mathcal{F}(u_+v_-)$ are supported in the regions $\{\tau \geq -C_1|\xi|\}$ and $\{\tau \leq C_1|\xi|\}$, respectively.*

Moreover, the functions $\mathcal{F}(u_-v_+)$ and $\mathcal{F}(u_+v_-)$ are given by the formulas

$$\widetilde{u_-v_+}(\tau,\xi) = C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_+\hat{f}(\xi-\alpha_+\omega)\hat{g}(\alpha_+\omega)d\omega$$

and

$$\widetilde{u_+v_-}(\tau,\xi) = -C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_-\hat{f}(\xi-\alpha_-\omega)\hat{g}(\alpha_-\omega)d\omega,$$

where D and $\alpha_\pm$ are defined by (2.6) and (2.7), respectively, and C is a constant depending only on C_1 and C_2.

Remark 2.2 If $C_1 = C_2$, then the supports of $\widetilde{u_-v_+}$ and $\widetilde{u_+v_-}$ are contained in the region $\{|\xi| \geq C_1|\tau|\}$.

Proof) From (2.5), we have

$$\mathcal{F}(u_-v_+) = 4\pi^2\mu_- * \nu_+$$

and

$$(2.12)\qquad \mu_- * \nu_+(\tau,\xi) = 2\int_{R^3}\int_\tau^\infty \hat{f}(\xi-\xi')\delta((\tau-\tau')^2 - C_1^2|\xi-\xi'|^2) \times\frac{\hat{g}(\xi')}{C_2|\xi'|}\delta(\tau' - C_2|\xi'|)d\tau'd\xi'.$$

From the above integral and $C_2 > C_1$, we see that $\mu_- * \nu_+$ is supported in the region $\{\tau \geq -C_1|\xi|\}$. For $\tau' = C_2|\xi'|$, we have

$$\eta \equiv (\tau-\tau')^2 - C_1^2|\xi-\xi'|^2$$
$$= (C_2^2 - C_1^2)(\rho - \alpha_+)(\rho - \alpha_-)$$

and

$$\mu_- * \nu_+(\tau,\xi)$$
$$= \frac{2}{C_2}\int_0^\infty \chi_-(\tau - C_2\rho)\rho\int_{S^2}\hat{f}(\xi-\rho\omega)\delta((C_2^2-C_1^2)(\rho-\alpha_+)(\rho-\alpha_-))\hat{g}(\rho\omega)d\omega d\rho.$$

Proceeding as in the proof of Lemma 2.2 we see that only $\rho = \alpha_+$ satisfies $\tau < C_2\rho$ and $\rho \geq 0$. Thus, we obtain

$$\mu_- * \nu_+(\tau,\xi) = C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_+\hat{f}(\xi-\alpha_+\omega)\hat{g}(\alpha_+\omega)d\omega.$$

It remains to prove the formula for $\mathcal{F}(u_+v_-)$. Proceeding as in the proof of Lemma 2.2 for $\mathcal{F}(u_-v_-)$, we obtain the desired formula. We have completed the proof of Lemma 2.3. ■

The following lemma is proved in the same manner as above.

Lemma 2.4 *Let f, g satisfy the conditions of Lemma 2.2. Suppose $Q = Q_\alpha$, $\alpha = 0,1,2,3$, or $Q_{\alpha\beta}$, $\alpha,\beta = 0,1,2,3$. Then, the functions $\mathcal{F}(Q(u_+,v_+)), \mathcal{F}(Q(u_-,v_-)), \mathcal{F}(Q(u_-,v_+))$ and $\mathcal{F}(Q(u_+,v_-))$ are expressed as follows.*

i) For $Q = Q_\alpha$, $\alpha = 0,1,2,3$,

$$\begin{aligned}\mathcal{F}(Q_\alpha(u_+,v_\pm)) &= \pm C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_- K_{1\alpha_-}(\tau,\xi.\omega)\hat{f}(\xi-\alpha_-\omega)\hat{g}(\alpha_-\omega)d\omega,\\ \mathcal{F}(Q_\alpha(u_-,v_\pm)) &= \pm C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_+ K_{1\alpha_+}(\tau,\xi.\omega)\hat{f}(\xi-\alpha_+\omega)\hat{g}(\alpha_+\omega)d\omega,\end{aligned}$$

where

(2.13)
$$K_{1\sigma}(\tau,\xi,\omega) = \begin{cases}\tau - C_2\sigma & (\alpha = 0),\\ \xi_j - \sigma\omega_j & (\alpha = j = 1,2,3).\end{cases}$$

(ii) For $Q = Q_{\alpha\beta}$, $\alpha,\beta = 0,1,2,3$,

$$\begin{aligned}\mathcal{F}(Q_{\alpha\beta}(u_+,v_\pm)) &= \pm C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_-^2 K_{2\alpha_-}(\tau,\xi.\omega)\hat{f}(\xi-\alpha_-\omega)\hat{g}(\alpha_-\omega)d\omega,\\ \mathcal{F}(Q_{\alpha\beta}(u_-,v_\pm)) &= \pm C\int_{S^2}\frac{1}{\sqrt{D}}\alpha_+^2 K_{2\alpha_+}(\tau,\xi.\omega)\hat{f}(\xi-\alpha_+\omega)\hat{g}(\alpha_+\omega)d\omega,\end{aligned}$$

where

(2.14)
$$K_{2\sigma}(\tau,\xi,\omega) = \begin{cases}\tau - C_2\sigma & (\alpha = \beta = 0),\\ (\tau - C_2\sigma)\omega_j & (\alpha = 0,\ \beta = j = 1,2,3),\\ \xi_j - \sigma\omega_j & (\alpha = j = 1,2,3,\ \beta = 0),\\ (\xi_i - \sigma\omega_i)\omega_j & (\alpha = i,\ \beta = j,\ i,j = 1,2,3).\end{cases}$$

We show the following lemma using the above lemma.

Lemma 2.5 *Let u, v be the solutions of (2.2) with $C_2 > C_1$ and with f, g satisfying the conditions of Lemma 2.2. Then, for the forms (1.5), (1.2), we have*

$$
\begin{aligned}
&\|Q(u_+,v_+)\|_{L^2(R^{1+3})} + \|Q(u_-,v_-)\|_{L^2(R^{1+3})} \\
&\quad + \|Q(u_-,v_+)\|_{L^2(R^{1+3})} + \|Q(u_+,v_-)\|_{L^2(R^{1+3})} \\
\leq &\begin{cases} C\|f\|_{L^2(R^3)}\|g\|_{L^2(R^3)} & \text{if } Q = Q_\alpha,\ \alpha = 0,1,2,3, \\ C\|f\|_{L^2(R^3)}\|\nabla g\|_{L^2(R^3)} & \text{if } Q = Q_{\alpha\beta},\ \alpha,\beta = 0,1,2,3, \end{cases}
\end{aligned}
$$

where a constant C is independent of f and g.

Proof) We first prove the lemma for $Q(u_+, v_+)$. From Lemma 2.4 and from the Schwarz inequality on S^2, we have

$$
\begin{aligned}
&\|Q(u_+,v_+)\|^2_{L^2(R^{1+3})} \\
= &(2\pi)^{-4}\int_{R^3}\int_{C_1|\xi|}^{\infty}|\tilde{Q}(\tau,\xi)|^2 d\tau d\xi \\
\leq &\begin{cases} C\int_{R^3}\int_{C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K_{1\alpha_-}^2\alpha_-^2|\hat{f}(\xi-\alpha_-\omega)|^2|\hat{g}(\alpha_-\omega)|^2 d\omega d\tau d\xi \\ \qquad \text{if } Q = Q_\alpha,\ \alpha = 0,1,2,3, \\ C\int_{R^3}\int_{C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K_{2\alpha_-}^2\alpha_-^4|\hat{f}(\xi-\alpha_-\omega)|^2|\hat{g}(\alpha_-\omega)|^2 d\omega d\tau d\xi \\ \qquad \text{if } Q = Q_{\alpha\beta},\ \alpha,\beta = 0,1,2,3, \end{cases}
\end{aligned} \tag{2.15}
$$

where $K_{1\alpha_-}$ and $K_{2\alpha_-}$ are defined by (2.13) and (2.14), respectively. We change variables $\tau \to \alpha_-$ keeping the other variables ξ, ω fixed. From (2.11) we have

$$\frac{d\alpha_-}{d\tau} = \frac{C_2\sqrt{D} - C_1^2(\tau - C_2\xi\cdot\omega)}{(C_2^2 - C_1^2)\sqrt{D}} > 0.$$

Hence, $\alpha_- = \alpha_-(\tau)$ is an increasing smooth function with a smooth inverse function. Moreover, we see that when $\tau = C_1|\xi|$,

$$\alpha_- = \frac{C_1(C_2|\xi| - C_1\xi\cdot\omega) - C_1|C_2|\xi| - C_1(\xi\cdot\omega)|}{C_2^2 - C_1^2} = 0.$$

In fact, since $C_2 > C_1$, we have $C_2|\xi| - C_1(\xi\cdot\omega) \geq 0$. Thus, from (2.15) we have

$$\|Q(u_+,v_+)\|^2_{L^2(R^{1+3})} \tag{2.16}$$

$$\leq \begin{cases} C\int_{R^3}\int_0^\infty\int_{S^2}\frac{1}{D}K_{1\alpha_-}^2\frac{d\tau}{d\alpha_-}\alpha_-^2|\hat{f}(\xi-\alpha_-\omega)|^2|\hat{g}(\alpha_-\omega)|^2 d\omega d\alpha_- d\xi \\ \hfill \text{if } Q=Q_\alpha, \\ C\int_{R^3}\int_0^\infty\int_{S^2}\frac{1}{D}K_{2\alpha_-}^2\frac{d\tau}{d\alpha_-}\alpha_-^4|\hat{f}(\xi-\alpha_-\omega)|^2|\hat{g}(\alpha_-\omega)|^2 d\omega d\alpha_- d\xi \\ \hfill \text{if } Q=Q_{\alpha\beta}. \end{cases}$$

We here claim that $\frac{1}{D}K_{1\alpha_-}^2\frac{d\tau}{d\alpha_-}$ and $\frac{1}{D}K_{2\alpha_-}^2\frac{d\tau}{d\alpha_-}$ are bounded. In fact, if $Q=Q_\alpha$, then for $\alpha=0$, we have by (2.8),

$$\begin{aligned} \frac{1}{D}(\tau-C_2\alpha_-)^2\frac{d\tau}{d\alpha_-} &= \frac{1}{D}\left(\frac{-C_1^2(\tau-C_2\xi\cdot\omega)+C_2\sqrt{D}}{C_2^2-C_1^2}\right)^2\frac{(C_2^2-C_1^2)\sqrt{D}}{C_2\sqrt{D}-C_1^2(\tau-C_2\xi\cdot\omega)} \\ &= \frac{C_2\sqrt{D}-C_1^2(\tau-C_2\xi\cdot\omega)}{\sqrt{D}(C_2^2-C_1^2)} \\ &\leq \frac{C_2\sqrt{D}+C_1^2|\tau-C_2\xi\cdot\omega|}{(C_2^2-C_1^2)\sqrt{D}} \\ &\leq \frac{1}{C_2-C_1}. \end{aligned}$$

For $\alpha=j=1,2,3$, we obtain

$$\frac{1}{D}(\xi_j-\alpha_-\omega_j)^2\frac{d\tau}{d\alpha_-}\leq\frac{1}{C_1^2(C_2-C_1)},$$

since

$$(\xi_j-\alpha_-\omega_j)^2\leq|\xi-\alpha_-\omega|^2=\frac{1}{C_1^2}(\tau-C_2\alpha_-)^2.$$

We can check the last equality using (2.6) – (2.8). Similarly, we obtain the same estimates for $\frac{1}{D}K_{2\alpha_-}^2\frac{d\tau}{d\alpha_-}$ as those for $\frac{1}{D}K_{1\alpha_-}^2\frac{d\tau}{d\alpha_-}$. Thus, $\frac{1}{D}K_{j\alpha_-}^2\frac{d\tau}{d\alpha_-}$, $j=1,2$ are bounded. Therefore, from (2.16), applying the Fubini theorem, the Planchrel theorem and the Young inequality, we obtain

$$\|Q(u_+,v_+)\|_{L^2(R^{1+3})}\leq\begin{cases} C\|f\|_{L^2(R^3)}\|g\|_{L^2(R^3)} & \text{if } Q=Q_\alpha, \\ C\|f\|_{L^2(R^3)}\|\nabla g\|_{L^2(R^3)} & \text{if } Q=Q_{\alpha\beta}. \end{cases} \tag{2.17}$$

For $Q(u_-,v_-)$, as before we have

$$\begin{aligned} &\|Q(u_-,v_-)\|_{L^2(R^{1+3})}^2 \\ &\leq \begin{cases} C\int_{R^3}\int_{-\infty}^{-C_1|\xi|}\int_{S^2}\frac{1}{D}K_{1\alpha_+}^2\alpha_+^2|\hat{f}(\xi-\alpha_+\omega)|^2|\hat{g}(\alpha_+\omega)|^2 d\omega d\tau d\xi \\ \hfill \text{if } Q=Q_\alpha, \\ C\int_{R^3}\int_{-\infty}^{-C_1|\xi|}\int_{S^2}\frac{1}{D}K_{2\alpha_+}^2\alpha_+^4|\hat{f}(\xi-\alpha_+\omega)|^2|\hat{g}(\alpha_+\omega)|^2 d\omega d\tau d\xi \\ \hfill \text{if } Q=Q_{\alpha\beta}. \end{cases} \end{aligned} \tag{2.18}$$

Making the change of variables $\tau \to -\tau$ and $\omega \to -\omega$ yields

$$\begin{gathered}\alpha_+(\tau,\xi,\omega) \to -\alpha_-(\tau,\xi,\omega),\\ D(\tau,\xi,\omega) \to D(\tau,\xi,\omega),\\ K^2_{j\alpha_+}(\tau,\xi,\omega) \to K^2_{j\alpha_-}(\tau,\xi,\omega), \qquad j=1,2.\end{gathered}$$

Thus, it follows from (2.18) that

$$\|Q(u_-,v_-)\|^2_{L^2(R^{1+3})} \le \begin{cases} C\int_{R^3}\int_{C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K^2_{1\alpha_-}\alpha_-^2|\hat f(\xi-\alpha_-\omega)|^2|\hat g(\alpha_-\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_\alpha,\\ C\int_{R^3}\int_{C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K^2_{2\alpha_-}\alpha_-^4|\hat f(\xi-\alpha_-\omega)|^2|\hat g(\alpha_-\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_{\alpha\beta}.\end{cases}$$

This is the same as (2.15). Therefore, in the same way as before, we obtain the desired results.

We next estimate $Q(u_+,v_-)$. As before, we have

$$\|Q(u_+,v_-)\|^2_{L^2(R^{1+3})} \le \begin{cases} C\int_{R^3}\int_{-\infty}^{C_1|\xi|}\int_{S^2}\frac{1}{D}K^2_{1\alpha_-}\alpha_-^2|\hat f(\xi-\alpha_-\omega)|^2|\hat g(\alpha_-\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_\alpha,\\ C\int_{R^3}\int_{-\infty}^{C_1|\xi|}\int_{S^2}\frac{1}{D}K^2_{2\alpha_-}\alpha_-^4|\hat f(\xi-\alpha_-\omega)|^2|\hat g(\alpha_-\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_{\alpha\beta}.\end{cases} \tag{2.19}$$

Changing variables $\tau \to -\tau$ and $\omega \to -\omega$ yields

$$\begin{gathered}\alpha_-(\tau,\xi,\omega) \to -\alpha_+(\tau,\xi,\omega),\\ D(\tau,\xi,\omega) \to D(\tau,\xi,\omega),\\ K^2_{j\alpha_-}(\tau,\xi,\omega) \to K^2_{j\alpha_+}(\tau,\xi,\omega), \qquad j=1,2.\end{gathered}$$

Thus, it follows from (2.19) that

$$\|Q(u_+,v_-)\|^2_{L^2(R^{1+3})} \le \begin{cases} C\int_{R^3}\int_{-C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K^2_{1\alpha_+}\alpha_+^2|\hat f(\xi-\alpha_+\omega)|^2|\hat g(\alpha_+\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_\alpha,\\ C\int_{R^3}\int_{-C_1|\xi|}^{\infty}\int_{S^2}\frac{1}{D}K^2_{2\alpha_+}\alpha_+^4|\hat f(\xi-\alpha_+\omega)|^2|\hat g(\alpha_+\omega)|^2 d\omega d\tau d\xi & \text{if } Q=Q_{\alpha\beta}.\end{cases} \tag{2.20}$$

Let ξ, ω be fixed. We change variables $\tau \to \alpha_+$. From (2.11) we have

$$\frac{d\alpha_+}{d\tau} = \frac{C_2\sqrt{D} + C_1^2(\tau - C_2\xi\cdot\omega)}{(C_2^2 - C_1^2)\sqrt{D}} > 0.$$

Hence, $\alpha_+ = \alpha_+(\tau)$ is an increasing smooth function with a smooth inverse function. Moreover, if $\tau = -C_1|\xi|$, then $\alpha_+ = 0$. Thus, we have from (2.20),

$$(2.21) \qquad \|Q(u_+, v_-)\|^2_{L^2(R^{1+3})}$$
$$\leq \begin{cases} C\int_{R^3}\int_0^\infty\int_{S^2}\frac{1}{D}K^2_{1\alpha_+}\frac{d\tau}{d\alpha_+}\alpha_+^2|\hat{f}(\xi-\alpha_+\omega)|^2|\hat{g}(\alpha_+\omega)|^2 d\omega d\alpha_+ d\xi \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } Q = Q_\alpha, \\ C\int_{R^3}\int_0^\infty\int_{S^2}\frac{1}{D}K^2_{2\alpha_+}\frac{d\tau}{d\alpha_+}\alpha_+^4|\hat{f}(\xi-\alpha_+\omega)|^2|\hat{g}(\alpha_+\omega)|^2 d\omega d\alpha_+ d\xi \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } Q = Q_{\alpha\beta}. \end{cases}$$

As before, we can check that $\dfrac{1}{D}K^2_{j\alpha_+}\dfrac{d\tau}{d\alpha_+}$, $j = 1, 2$ are bounded, using $(\tau - C_2\alpha_+)^2 = C_1^2|\xi - \alpha_+\omega|^2$. Therefore, we obtain the same estimates as (2.17).

Finally, we can similarly obtain the desired results for $\|Q(u_-, v_+)\|^2_{L^2(R^{1+3})}$. This completes the proof of Lemma 2.5. ■

Theorem 2.1 follows from Lemma 2.5 and a density argument. See [2]. ■

Proof of Theorem 1:

Theorem 1 follows by repeating the same reasoning as in [2] and using the Duhamel principle. ■

Similarly, Corollary 2 is proved by Lemmas 2.2 and 2.3.

3 Existence and Uniqueness

In this section we describe the proof of Theorem 2. We consider the Cauchy problem for a system of the type

$$(3.1) \qquad \begin{cases} \Box_1 u = Q_\alpha(u, v) = \partial_\alpha u \cdot v, & t > 0,\ x \in R^3, \\ \Box_2 v = Q_\alpha(u, v) = \partial_\alpha u \cdot v, & t > 0,\ x \in R^3, \\ u(0, x) = f_0(x), \quad \partial_t u(0, x) = f_1(x), & x \in R^3, \\ v(0, x) = g_0(x), \quad \partial_t v(0, x) = g_1(x), & x \in R^3, \end{cases}$$

where $C_2 > C_1 > 0$, $f_0, g_0 \in H^1(R^3)$, $f_1, g_1 \in L^2(R^3)$. The proof is based on the result of Theorem 1.

Existence : We first prove the existence part of the theorem. The following proposition holds.

Proposition 3.1 *There exist positive constants ε and C depending only on $\|f_0\|_{H^1(R^3)}$, $\|f_1\|_{L^2(R^3)}$, $\|g_0\|_{H^1(R^3)}$, $\|g_1\|_{L^2(R^3)}$ and C_1, C_2 such that the problem(3.1) has solutions $(u(t), v(t))$ on $[0, \varepsilon]$ satisfying*

$$u, v \in \bigcap_{j=0}^{1} C^j([0,\varepsilon]; H^{1-j}(R^3)),$$

$$\text{(3.2a)} \quad E_\varepsilon(u,v) \equiv \sup_{0\le t\le\varepsilon} E(u,v)(t) \equiv \sup_{0\le t\le\varepsilon} \sum_{0\le|\alpha|\le1} (\|\partial^\alpha u(t)\|_{L^2(R^3)} + \|\partial^\alpha v(t)\|_{L^2(R^3)}) \le C,$$

and

$$\text{(3.2b)} \qquad \int_0^\varepsilon \int_{R^3} |Q_\alpha(u,v)|^2 dxdt \le C.$$

Proof) We use an interation argument to prove the proposition. Let us define sequences of functions $\{u_n\}, \{v_n\}$ by

$$\text{(3.3)} \qquad \begin{cases} \Box_1 u_n = Q_\alpha(u_{n-1}, v_{n-1}), & t > 0,\ x \in R^3, \\ \Box_2 v_n = Q_\alpha(u_{n-1}, v_{n-1}), & t > 0,\ x \in R^3, \\ u_n(0,x) = f_0(x), \quad \partial_t u_n(0,x) = f_1(x), & x \in R^3, \\ v_n(0,x) = g_0(x), \quad \partial_t v_n(0,x) = g_1(x), & x \in R^3 \end{cases}$$

for $n \ge 1$ and $u_0 = v_0 = 0$. It suffices to show that there exist $\varepsilon \le 1$ and $M \ge 1$ such that

$$\text{(3.4a)} \qquad E(u_n, v_n)(t) \le \left(1 + \frac{1}{C_1} + \frac{1}{C_2}\right) M \quad \text{for } 0 \le t \le \varepsilon \text{ and all } n,$$

$$\text{(3.4b)} \qquad E(u_n - u_{n-1}, v_n - v_{n-1})(t) \le \frac{M}{2^{n-1}} \quad \text{for } 0 \le t \le \varepsilon \text{ and all } n.$$

To prove (3.4) we first show the following inequalities by induction on n :

$$\text{(3.5)} \qquad \int_0^\varepsilon \|Q_\alpha(u_n, v_n - v_{n-1})(s)\|^2_{L^2(R^3)} ds \le \frac{C_0 M^4}{2^{2(n-1)}},$$

$$\text{(3.6)} \qquad \int_0^\varepsilon \|Q_\alpha(u_n - u_{n-1}, v_n)(s)\|^2_{L^2(R^3)} ds \le \frac{C_0 M^4}{2^{2(n-1)}},$$

$$\text{(3.7)} \qquad \int_0^\varepsilon \|Q_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)} ds \le C_0 M^4,$$

where C_0 is the constant in the estimate of Theorem 1(i).

Let $M \geq 2(\|f_0\|_{H^1} + \|f_1\|_{L^2} + \|g_0\|_{H^1} + \|g_1\|_{L^2})$ and $M \geq 1$. For $n = 1$, (3.5)–(3.7) hold by Theorem 1.

Assume that (3.5)–(3.7) are true for some $n \geq 1$. Then, for $n+1$, we have from Theorem 1,

$$\begin{aligned}
\int_0^\varepsilon \|Q_\alpha(u_{n+1}, v_{n+1})(s)\|^2_{L^2(R^3)} ds &\leq C_0 \left(\frac{M}{2} + \int_0^\varepsilon \|Q_\alpha(u_n, v_n)(s)\|_{L^2(R^3)} ds\right)^4 \\
&\leq C_0 \left(\frac{M}{2} + \varepsilon^{1/2}\Big(\int_0^\varepsilon \|Q_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)} ds\Big)^{1/2}\right)^4 \\
&\leq C_0 M^4 \Big(\frac{1}{2} + \varepsilon^{1/2} C_0^{1/2} M\Big)^4,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\varepsilon \|Q_\alpha(u_{n+1}, v_{n+1} - v_n)(s)\|^2_{L^2(R^3)} ds \\
&\leq C_0 \left(\frac{M}{2} + \int_0^\varepsilon \|Q_\alpha(u_n, v_n)(s)\|_{L^2(R^3)} ds\right)^2 \left(\int_0^\varepsilon \|Q_\alpha(u_n, v_n) - Q_\alpha(u_{n-1}, v_{n-1})\|_{L^2(R^3)} ds\right)^2 \\
&\leq C_0 \left(\frac{M}{2} + \varepsilon^{1/2}\Big(\int_0^\varepsilon \|Q_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)} ds\Big)^{1/2}\right)^2 \\
&\quad \times \left(\varepsilon^{1/2}\Big(\int_0^\varepsilon \|Q_\alpha(u_n, v_n - v_{n-1})(s)\|^2_{L^2(R^3)} ds\Big)^{1/2}\right. \\
&\qquad \left. + \varepsilon^{1/2}\Big(\int_0^\varepsilon \|Q_\alpha(u_n - u_{n-1}, v_{n-1})(s)\|^2_{L^2(R^3)} ds\Big)^{1/2}\right)^2 \\
&\leq C_0 \Big(\frac{M}{2} + \varepsilon^{1/2} C_0^{1/2} M^2\Big)^2 4\varepsilon C_0 \frac{M^4}{2^{2(n-1)}}.
\end{aligned}$$

If we choose ε so small that

$$\varepsilon^{1/2} C_0^{1/2} M \leq \frac{1}{2}, \tag{3.8}$$

$$\Big(\frac{M}{2} + \varepsilon^{1/2} C_0^{1/2} M^2\Big)^2 4\varepsilon C_0 \leq \frac{1}{4}, \tag{3.9}$$

then we obtain

$$\begin{aligned}
\int_0^\varepsilon \|Q_\alpha(u_{n+1}, v_{n+1})(s)\|^2_{L^2(R^3)} ds &\leq C_0 M^4, \\
\int_0^\varepsilon \|Q_\alpha(u_{n+1}, v_{n+1} - v_n)(s)\|^2_{L^2(R^3)} ds &\leq \frac{C_0 M^4}{2^{2n}}.
\end{aligned}$$

Therefore, (3.5) and (3.7) hold for $n \geq 1$. In the same way, inequality (3.6) is proved.

Now we are in a position to prove (3.4). We rewrite (3.3) as the following integral equations.

$$(3.10)\qquad \begin{cases} u_n(t) &= \cos C_1\omega_0 t\cdot f_0 + \dfrac{\sin C_1\omega_0 t}{C_1\omega_0} f_1 \\ & \quad + \displaystyle\int_0^t \frac{\sin C_1\omega_0(t-s)}{C_1\omega_0} Q_\alpha(u_{n-1}, v_{n-1})(s)ds, \\ v_n(t) &= \cos C_2\omega_0 t\cdot g_0 + \dfrac{\sin C_2\omega_0 t}{C_2\omega_0} g_1 \\ & \quad + \displaystyle\int_0^t \frac{\sin C_2\omega_0(t-s)}{C_2\omega_0} Q_\alpha(u_{n-1}, v_{n-1})(s)ds, \end{cases}$$

where $\omega_0 = (-\Delta)^{1/2}$. From the Planchrel theorem, we have

$$(3.11)\qquad \begin{aligned} & \|u_n(t)\|_{H^1(R^3)} + \|v_n(t)\|_{H^1(R^3)} \\ \le\ & \|f_0\|_{H^1(R^3)} + \left(t + \frac{1}{C_1}\right)\|f_1\|_{L^2(R^3)} \\ & + \|g_0\|_{H^1(R^3)} + \left(t + \frac{1}{C_2}\right)\|g_1\|_{L^2(R^3)} \\ & + 2\int_0^t (t-s)\|Q_\alpha(u_{n-1}, v_{n-1})(s)\|_{L^2(R^3)}ds \\ & + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_0^t \|Q_\alpha(u_{n-1}, v_{n-1})(s)\|_{L^2(R^3)}ds. \end{aligned}$$

When $n = 1$, from (3.11) we have

$$E(u_1, v_1)(t) \le \left(1 + \frac{1}{C_1} + \frac{1}{C_2} + \varepsilon\right)\frac{M}{2} \equiv (C_{12} + \varepsilon)\frac{M}{2},$$

where $C_{12} = 1 + 1/C_1 + 1/C_2$. When $n \ge 2$, by (3.5)–(3.7) we have for $0 \le t \le \varepsilon$,

$$\begin{aligned} & E(u_n, v_n)(t) \\ \le\ & (C_{12} + \varepsilon)\frac{M}{2} + \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2}\left(\int_0^\varepsilon \|Q_\alpha(u_{n-1}, v_{n-1})(s)\|^2_{L^2(R^3)}ds\right)^{1/2} \\ \le\ & (C_{12} + \varepsilon)\frac{M}{2} + \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2}C_0^{1/2}M^2 \\ =\ & M\left(\frac{C_{12} + \varepsilon}{2} + \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2}C_0^{1/2}M\right) \end{aligned}$$

and

$$\begin{aligned} & E(u_n - u_{n-1}, v_n - v_{n-1})(t) \\ \le\ & \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\left(\int_0^\varepsilon \|Q_\alpha(u_{n-1}, v_{n-1}) - Q_\alpha(u_{n-2}, v_{n-2})(s)\|_{L^2(R^3)}ds\right) \end{aligned}$$

$$
\begin{aligned}
&\le \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right) \\
&\quad \times \Big(\int_0^\varepsilon (\|Q_\alpha(u_{n-1}, v_{n-1} - v_{n-2})(s)\|_{L^2(R^3)} + \|Q_\alpha(u_{n-1} - u_{n-2}, v_{n-2})(s)\|_{L^2(R^3)})ds\Big) \\
&\le \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2} \\
&\quad \times \Big\{\Big(\int_0^\varepsilon \|Q_\alpha(u_{n-1}, v_{n-1} - v_{n-2})(s)\|^2_{L^2(R^3)} ds\Big)^{1/2} \\
&\qquad + \Big(\int_0^\varepsilon \|Q_\alpha(u_{n-1} - u_{n-2}, v_{n-2})(s)\|^2_{L^2(R^3)} ds\Big)^{1/2}\Big\} \\
&\le \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2} 2\frac{C_0^{1/2} M^2}{2^{n-2}}.
\end{aligned}
$$

If we choose ε so small that

$$
\varepsilon + \frac{1}{2} \le C_{12} \qquad \text{and} \qquad \left(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\right)\varepsilon^{1/2} 2 C_0^{1/2} M \le \frac{1}{2},
$$

then, we obtain (3.4). Inequalities (3.4) imply that there exist $(u, v) \in (\cap_{j=0}^1 C^j([0,\varepsilon]; H^{1-j})) \times (\cap_{j=0}^1 C^j([0,\varepsilon]; H^{1-j}))$ such that (u_n, v_n) converge to (u, v) as $n \to \infty$. Clearly (u, v) satisfy the integral equations (3.10). This completes the proof of Proposition 3.1. ∎

Uniqueness : We next prove the uniqueness part of the theorem.

Proposition 3.2 *Let (u, v) and (u', v') be the solutions of (3.1) satisfying (3.2a) and (3.2b) with u, u', v, v' and having the same initial data $f_0, g_0 \in H^1$, $f_1, g_1 \in L^2$. Then, $u = u'$ and $v = v'$.*

Proof) In view of (3.10) and the Planchrel theorem, we have

$$
\begin{aligned}
(3.12) \qquad & E(u - u', v - v')(t) \\
&\le 2\int_0^t (t - s)(\|Q_\alpha(u, v - v')(s)\|_{L^2(R^3)} + \|Q_\alpha(u - u', v')(s)\|_{L^2(R^3)})ds \\
&\quad + \left(\frac{1}{C_1} + \frac{1}{C_2}\right) \int_0^t (\|Q_\alpha(u, v - v')(s)\|_{L^2(R^3)} + \|Q_\alpha(u - u'. v')(s)\|_{L^2(R^3)})ds.
\end{aligned}
$$

Furthermore, Theorem 1 yields

$$
\int_0^t (\|Q_\alpha(u, v - v')(s)\|_{L^2(R^3)} + \|Q_\alpha(u - u', v')(s)\|_{L^2(R^3)})ds
$$

$$
\begin{aligned}
&\leq t^{1/2}\left\{\left(\int_0^t \|Q_\alpha(u,v-v')(s)\|^2_{L^2(R^3)}ds\right)^{1/2}+\left(\int_0^t \|Q_\alpha(u-u',v')(s)\|^2_{L^2(R^3)}ds\right)^{1/2}\right\}\\
&\leq Ct^{1/2}\left(\|f_0\|_{H^1}+\|f_1\|_{L^2}+\int_0^t \|Q_\alpha(u,v)\|_{L^2}ds\right)\\
&\quad\times\left(\int_0^t (\|Q_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|Q_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\right)\\
&\quad+Ct^{1/2}\left(\|g_0\|_{H^1}+\|g_1\|_{L^2}+\int_0^t \|Q_\alpha(u',v')\|_{L^2}ds\right)\\
&\qquad\times\left(\int_0^t (\|Q_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|Q_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\right)\\
&\leq Ct^{1/2}\left(\int_0^t (\|Q_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|Q_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\right)
\end{aligned}
$$

for $0 \leq t \leq \varepsilon$. Here, since ε in Proposition 3.1 is sufficiently small, we obtain

$$
\int_0^t (\|Q_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|Q_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds \equiv 0
$$

for all $0 \leq t \leq \varepsilon$. Thus, (3.12) implies

$$
E(u-u',v-v')(t) \equiv 0 \qquad \text{for all } 0 \leq t \leq \varepsilon.
$$

This completes the proof. ∎

Therefore, we have completed the proof of Theorem 2. ∎

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