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# SPACE-TIME ESTIMATES FOR NULL GAUGE FORMS AND NONLINEAR SCHRÖDINGER EQUATIONS


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**Abstract.** We consider the Cauchy problem for the nonlinear Schrödinger equation in one space dimension with interaction satisfying null gauge condition. We prove the local well-posedness of the problem in the Sobolev space $H^{1/2}$. The method depends on the nonlinear gauge transformation and on sharp smoothing estimates for the null gauge form.



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## 1. Introduction

In this paper we consider the nonlinear Schrödinger equation of the form

$$i\partial_t u + \partial^2 u = i\lambda(\partial|u|^2)u + f(u) \tag{1.1}$$

where $u$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $\partial = \partial/\partial x$, $\lambda \in \mathbb{R}$, and $f(u)$ is a nonlinear interaction defined pointwisely by $f(u)(t, x) = f(u(t, x))$ with a complex valued function $f$. Equation (1.1) has both physical and mathematical importance in connection with the nonlinear self-modulation problem of the Benjamin-Ono equation [30], the Benjamin-Feir instability of the Stokes wave near the critical wavenumber [12],[13], theory of gauge transformation in the infinite integrable systems [2],[24],[25], and the null gauge condition proposed one of us [14],[31]. The Cauchy problem for (1.1) is studied in [10],[27],[29], where the Cauchy data is required to belong to the Sobolev spaces $H^s$ with $s \geqslant 1$ (See also [5],[6],[11],[18]). The purpose in this paper is to prove the local well-posedness of the Cauchy problem for (1.1) in the Sobolev space $H^{1/2}$. One of the difficulties for this end consists in making sense of the derivative term $(\partial|u|^2)u$. In order to state the main result precisely, we introduce the following notation and assumption. Let $U(t) = \exp(it\partial^2)$ be the free propagator, namely the unitary group which solves the free Schrödinger equation. We convert (1.1) into the integral equation

$$u(t) = U(t)u_0 - i\int_0^t U(t-\tau)(i\lambda(\partial|u|^2)u + f(u))(\tau)d\tau \tag{1.2}$$

where $u_0$ is the Cauchy data at $t = 0$. We consider the following assumption:

(H) $f$ is continuously differentiable in the real sense, $f(0) = 0$, and $f$ satisfies the estimate

$$|f'(z)| \equiv \max(|\partial f/\partial z|, |\partial f/\partial \bar{z}|) \leqslant C(1 + |z|^4) \tag{1.3}$$

for all $z \in \mathbb{C}$.

For a bounded time interval $I$ we define $X(I)$ the function space by

$$X(I) = \{u \in C(I; H^{1/2}(\mathbb{R})) \cap L^4(I; L^\infty(\mathbb{R})); u \in L^4(\mathbb{R}; L^\infty(I)), \partial u \in L^\infty(\mathbb{R}; L^2(I))\}$$

with norm

$$|||u||| = ||u; L_t^\infty H^{1/2}|| + ||u; L_t^4 L_x^\infty|| + ||u; L_x^4 L_t^\infty|| + ||\partial u; L_x^\infty L_t^2||.$$

**Theorem 1.** *Let $f$ satisfy (H). Then for any $\rho > 0$ there exists a positive constant $T(\rho)$ depending only on $\rho$ and $f$ with the following properties. For any $u_0 \in H^{1/2}(\mathbb{R})$ with $||u_0; H^{1/2}|| \leqslant \rho$ the equation (1.2) has a unique solution $u$ on the time interval $I \equiv [0, T(\rho)]$ such that $u \in X(I)$ and*

$$\partial |u|^2 \in L^2(I \times \mathbb{R}). \tag{1.4}$$

*Moreover, for any $T$ with $0 < T < T(\rho)$ there exists $\epsilon > 0$ such that the map $\tilde{u}_0 \mapsto \tilde{u}$ is Lipschitz from $\{\tilde{u}_0 \in H^{1/2}(\mathbb{R}); ||\tilde{u}_0 - u_0; H^{1/2}|| < \epsilon\}$ to $X([0,T])$.*

In the proof of Theorem 1 we make extensive use of the space-time estimates of the free propagator. We use smoothing estimates of Strichartz type, of Kato type, and of maximal functions to reproduce the norm $||u; L_t^\infty L_x^2|| + ||u; L_t^4 L_x^\infty||$, the seminorm $||(-\Delta)^{1/4} u; L_t^\infty L_x^2|| + ||\partial u; L_x^\infty L_t^2||$, and the norm $||u; L_x^4 L_t^\infty||$ in the contraction argument on (1.2). It seems however that in addition to the regularity given by $X(I)$ one needs additional information to give a meaning to the derivative term $(\partial |u|^2) u$, since its $L_x^1 L_t^2$ norm is out of control in terms of the norm in $X(I)$ as long as the attempt depends exclusively on the Hölder inequalities. As is indicated in the theorem, a feature of our proof consists in making an observation on the space-time behavior of the coupling $\partial |u|^2$ instead of dealing with $u$ and $\partial u$ separately. To be more specific, we consider the sesquilinear form

$$(\phi, \psi) \longmapsto \partial(U(t)\phi \cdot \overline{U(t)\psi}) \tag{1.5}$$

and derive sharp estimates for (1.5) in the space-time integrals. In fact, the estimates are stronger than that which could be derived from the Strichartz and Sobolev inequalities. We now state the basic new estimates.

**Theorem 2.** *(1) Let $u(t) = U(t)u_0$ and $v(t) = U(t)v_0$. Then*

$$||(-\Delta)^{1/4} u\bar{v}; L^2(\mathbb{R} \times \mathbb{R})|| = 2^{-1/2} ||u_0; L^2|| \, ||v_0; L^2||. \tag{1.6}$$

*(2) Let*

$$u(t) = U(t)u_0 - i \int_0^t U(t-\tau) F(\tau, \cdot) d\tau,$$
$$v(t) = U(t)v_0 - i \int_0^t U(t-\tau) G(\tau, \cdot) d\tau,$$

*where $F \in L^1(\mathbb{R}; L^2)$ and $G \in L^1(\mathbb{R}; L^2)$. Then*

$$||(-\Delta)^{1/4} u\bar{v}; L^2(\mathbb{R} \times \mathbb{R})|| \leqslant 2^{-1/2} (||u_0; L^2|| + ||F; L_t^1 L_x^2||)(||v_0; L^2|| + ||G; L_t^1 L_x^2||). \tag{1.7}$$

*(3) Let $u$ and $v$ be as in Part (2) with $u_0, v_0 \in H^{1/2}$, $F, G \in L_t^1 H^{1/2}$. Then*

$$\begin{aligned} &\|\partial(u\bar{v}); L^2(\mathbb{R}\times\mathbb{R})\| \\ &\leqslant 2^{-1/2}(\|u_0; H^{1/2}\| + \|F; L_t^1 H^{1/2}\|)(\|v_0; H^{1/2}\| + \|G; L_t^1 H^{1/2}\|) \end{aligned} \tag{1.8}$$

The result in Theorem 2 is reminiscent of the space-time estimates given by Klainerman and Machedon for the null forms of the wave equation [22],[23]. The null forms for the wave equation in $\mathbb{R}^{1+n}$ are

$$\partial_t u \partial_t v - \nabla u \cdot \nabla v, \tag{1.9}$$

$$\partial_t u \partial_j v - \partial_j u \partial_t v, \quad 1 \leqslant j \leqslant n, \tag{1.10}$$

$$\partial_j u \partial_k v - \partial_k u \partial_j v, \quad 1 \leqslant j < k \leqslant n. \tag{1.11}$$

The null forms (1.9)-(1.11) are directly related to the nonlinearities verifying the null condition proposed by Klainerman [21]. As for the Schrödinger equation the crucial factor in the nonlinearity which provides an analogous effect is given by $\partial|u|^2$. Roughly speaking, the nonlinearity involving $\partial|u|^2$ is called to satisfy the null gauge condition [31] and accordingly, we call

$$\partial(u\bar{v}) \tag{1.12}$$

or (1.5) the null gauge form for the Schrödinger equation in one space dimension. We remark that (1.9)-(1.11) are Lorentz invariant while (1.12) is gauge invariant. We also note that (1.9) comes out in connection with the exactly solvable equation

$$\Box u = (\partial_t u)^2 - (\nabla u)^2, \tag{1.13}$$

while the corresponding exactly solvable equation seems to be

$$i\partial_t u + \Delta u = (\nabla u)^2, \tag{1.14}$$

or rather, in the one dimensional case,

$$i\partial_t u + \partial^2 u = \partial(u^2) \tag{1.15}$$

(see [28]). The bilinear forms associated with the exactly solvable equations (1.14) and (1.15) are

$$\nabla u \cdot \nabla v, \tag{1.16}$$

$$\partial(uv), \tag{1.17}$$

respectively, though both (1.16) and (1.17) are not invariant under the gauge and Galilei tranformations.

We now describe the sketch of proof of Theorem 1, whose full detail will be given in Section 4. Let $u$ be a solution to (1.1) with sufficient regularity in space-time and integrability in space. We define

$$\phi = e^{-i(\lambda/2)\theta}u, \tag{1.18}$$

where

$$\theta(t,x) = \int_{-\infty}^{x} |u(t,y)|^2 dy. \tag{1.19}$$

By (1.1),

$$\partial_t|u|^2 = \partial(2\mathrm{Im}u\partial\overline{u} + \lambda|u|^4) + 2\mathrm{Im}\overline{u}f(u) \tag{1.20}$$

and therefore

$$i\partial_t\theta + \partial^2\theta = 2u\partial\overline{u} + i\lambda|u|^4 + 2i\mathrm{Im}\int_{-\infty}^{x} \overline{u}f(u)dy. \tag{1.21}$$

By (1.1), (1.18), and (1.21),

$$\begin{aligned} i\partial_t\phi + \partial^2\phi &= e^{-i(\lambda/2)\theta}(i\partial_t u + \partial^2 u - i\frac{\lambda}{2}(i\partial_t\theta + \partial^2\theta)u - i\lambda|u|^2\partial u - \frac{\lambda^2}{4}|u|^4 u) \\ &= \frac{\lambda^2}{4}|\phi|^4\phi + e^{-i(\lambda/2)\theta}f(e^{i(\lambda/2)\theta}\phi) \\ &+ \lambda\phi\mathrm{Im}\int_{-\infty}^{x} e^{-i(\lambda/2)\theta}\overline{\phi}f(e^{i(\lambda/2)\theta}\phi)dy. \end{aligned} \tag{1.22}$$

Since $|\phi| = |u|$, $\theta$ is written by $\phi$ and (1.21) yields the following equation of $\phi$:

$$\begin{aligned} i\partial_t\phi + \partial^2\phi = &\frac{\lambda^2}{4}|\phi|^4\phi + e^{-i(\lambda/2)\Theta}f(e^{i(\lambda/2)\Theta}\phi) \\ &+ \lambda\phi\mathrm{Im}\int_{-\infty}^{x} e^{-i(\lambda/2)\Theta}\overline{\phi}f(e^{i(\lambda/2)\Theta}\phi)dy \equiv F(\phi) \end{aligned} \tag{1.23}$$

where

$$\Theta(t,x) = \int_{-\infty}^{x} |\phi(t,y)|^2 dy. \tag{1.24}$$

Conversely, let $\phi$ satisfy (1.23). Then we define

$$u = e^{i(\lambda/2)\Theta}\phi. \tag{1.25}$$

By (1.23),

$$\partial_t|\phi|^2 = \partial(2\mathrm{Im}\phi\partial\overline{\phi}) + 2\mathrm{Im}\int_{-\infty}^{x} e^{-i(\lambda/2)\Theta}\overline{\phi}f(e^{i(\lambda/2)\Theta}\phi)dy \tag{1.26}$$

and therefore

$$i\partial_t\Theta + \partial^2\Theta = 2\phi\partial\overline{\phi} + 2i\mathrm{Im}\int_{-\infty}^{x} e^{-i(\lambda/2)\Theta}\overline{\phi}f(e^{i(\lambda/2)\Theta}\phi)dy. \tag{1.27}$$

By (1.23), (1.25), and (1.27), we reproduce (1.1) as

$$\begin{aligned} i\partial_t u + \partial^2 u &= e^{i(\lambda/2)\Theta}(i\partial_t\phi + \partial^2\phi + i\frac{\lambda}{2}(i\partial_t\Theta + \partial^2\Theta)\phi + i\lambda|\phi|^2\partial\phi - \frac{\lambda^2}{4}|\phi|^4\phi) \\ &= e^{i(\lambda/2)\Theta}(i\lambda\phi(\partial|\phi|^2) + e^{-i(\lambda/2)\Theta}f(e^{i(\lambda/2)\Theta}\phi)) \\ &= i\lambda(\partial|u|^2)u + f(u). \end{aligned}$$

On the basis of the formal calculations above, we solve the local Cauchy problem for (1.1) by making a reduction to (1.23). For given data $u_0$ to (1.1), we define the data $\phi_0$ by (1.18) at $t = 0$. We solve the local Cauchy problem for (1.23) with data $\phi_0$. Then we define $u$ by (1.25), which solves the Cauchy problem for (1.1) with data $u_0$. As compared to (1.1), the transformed equation (1.23) has the advantage that it involves no derivatives in the nonlinearity. We now state the basic existence and uniqueness result on the local Cauchy problem for (1.23) in $H^{1/2}(\mathbb{R})$. We consider the following assumption that is more general than the previous one (H).

(H)′ $f$ is continuously differentiable in the real sense, $f(0) = 0$, and for some $p$ with $1 < p < \infty$ $f$ satisfies the estimate

$$|f'(z)| \leqslant C(1 + |z|^{p-1}), z \in \mathbb{C}. \tag{1.28}$$

**Theorem 3.** *Let $f$ satisfy (H)′. Then for any $\rho > 0$ there exists a positive constant $T(\rho)$ depending only on $\rho$ and $f$ with the following properties. For any $\phi_0 \in H^{1/2}(\mathbb{R})$ with $\|\phi_0; H^{1/2}\| \leqslant \rho$, (1.23) has a unique solution $\phi \in X(I)$ with $I = [0, T(\rho)]$. Moreover, for any $T$ with $0 < T < T(\rho)$ there exists $\epsilon > 0$ such that the map $\tilde{\phi_0} \mapsto \tilde{\phi}$ is Lipschitz from $\{\tilde{\phi_0} \in H^{1/2}(\mathbb{R}); \|\tilde{\phi_0} - \phi_0; H^{1/2}\| < \epsilon\}$ to $X([0,T])$.*

**Remark 1.1.** If we consider the regular solution, the solvability of the Cauchy problem for (1.23) is equivalent to that of the Cauchy problem for (1.1). But this is not necessarily the case with the weak solution. It has to be proved that the solution of (1.23) given by Theorem 3 is transformed to a solution of (1.1) by the gauge transformation (1.25) under (H). This is an important part of the proof of Theorem 1.

**Remark 1.2.** If we assume the gauge invariance of $f$, i.e., $f(e^{i\theta}z) = e^{i\theta}f(z)$ for any $(\theta, z) \in \mathbb{R} \times \mathbb{C}$ and $f(\mathbb{R}) \subset \mathbb{R}$, then (1.23) becomes

$$i\partial_t\phi + \partial^2\phi = \frac{\lambda^2}{4}|\phi|^4\phi + f(\phi)$$

and the proof of Theorem 3 given below will be extremely simplified since the major technical obstacles are related to the phase factors $e^{\pm i(\lambda/2)\Theta}$ and the nonlocal effect given by the integral in $F(\phi)$.

In order to complete the proof of Theorem 1, we prove that the gauge transformation $\phi \mapsto u$ given by (1.25) leaves $X(I)$ invariant and that $\partial|u|^2 = \partial|\phi|^2 \in L^2(I \times \mathbb{R})$ when $p \leqslant 5$. We use Theorem 2 to prove the last statement and the restriction $p \leqslant 5$ comes out in the application of Theorem 2.

This paper is organized as follows. In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 3. In Section 4 we prove Theorem 1. In Section 5 we present an extension of Theorem 2 to higher dimensional cases.

We conclude the introduction by giving notations to be used throughout the paper. For any $p$ with $1 \leqslant p \leqslant \infty$ we denote by $p'$ the conjugate exponent defined by $1/p + 1/p' = 1$. For any interval $I$ and any Banach space $X$ we denote by $C(I;X)$ the space of strongly continuous functions from $I$ to $X$ and by $L^p(I;X)$ the space of measurable functions $u$ from $I$ to $X$ such that $||u(\cdot);X|| \in L^p(I)$. The abbreviation such as $L_t^q L_x^r = L_t^q(I;L_x^r) = L_t^q(I;L_x^r(\mathbb{R}))$ and $L_t^q L_x^r = L_t^q(\mathbb{R};L_x^r(I))$ will often be made when this could cause no confusion. For any $p$ with $1 < p < \infty$ and any $m \in \mathbb{R}$ we denote by $H_p^m$ and $\dot{H}_p^m$ the usual and homogeneous Sobolev spaces of $L^p$ style of order $m$. We put $H^m = H_2^m$ and $\dot{H}^m = \dot{H}_2^m$ for simplicity. We denote by $*$ the convolution with respect to space variables. Different positive constants might be denoted by the same letter $C$.

## 2. Estimates for the Null Gauge Form

In this section we prove Theorem 2. We prove the Fourier transform with respect to the space variable by

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-ix\xi)u(x)dx,$$

so that $U(t)u_0 = \mathcal{F}^{-1}\exp(-it\xi^2)\hat{u}_0$.

We denote by $(\tau, \xi)$ the dual variables of $(t, x)$ respectively.

*Proof of Theorem 2:* (1) We give two proofs.

(a) ***First Proof.*** We take the Fourier transform of $u\overline{v}$ with respect to space to obtain

$$\begin{aligned}\mathcal{F}(u\overline{v})(\xi) &= (2\pi)^{-1/2}(\mathcal{F}u) * (\mathcal{F}\overline{v})(\xi) \\ &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-it(\xi^2 - 2\xi\eta))\hat{u}_0(\xi - \eta)\hat{\overline{v}}_0(\eta)d\eta. \qquad (2.1)\end{aligned}$$

We take the Fourier transform of (2.1) with respect to time and interchange the integrals in space and time to obtain

$$
\begin{aligned}
&(2\pi)^{-1/2}\int_{-\infty}^{\infty}\exp(-it\tau)\mathcal{F}(u\overline{v})(\xi)dt \\
&=(2\pi)^{-1}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}\exp(-it(\tau+\xi^2-2\xi\eta))dt)\hat{u}_0(\xi-\eta)\hat{\overline{v}}_0(\eta)d\eta \\
&=\int_{-\infty}^{\infty}\delta(\tau+\xi^2-2\xi\eta)\hat{u}_0(\xi-\eta)\hat{\overline{v}}_0(\eta)d\eta \\
&=\frac{1}{2|\xi|}\int_{-\infty}^{\infty}\delta(\frac{\tau+\xi^2}{2\xi}-\eta)\hat{u}_0(\xi-\eta)\hat{\overline{v}}_0(\eta)d\eta \\
&=\frac{1}{2|\xi|}\hat{u}_0(\frac{1}{2}(\xi-\frac{\tau}{\xi}))\hat{\overline{v}}_0(\frac{1}{2}(\xi+\frac{\tau}{\xi})),
\end{aligned}
\tag{2.2}
$$

where $\delta$ denotes the Dirac measure and we have used the homogeneity of degree $-1$ of $\delta$. The calculation above is always justified by inserting a convergence factor ($\exp(-\epsilon t^2)$, $\epsilon > 0$, for instance) and passing to the limit. Application of the Parseval formula to (2.2) yields

$$
\begin{aligned}
&||(-\Delta)^{1/4}u\overline{v}; L^2(\mathbb{R}\times\mathbb{R})||^2 \\
&=\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|\xi|}|\hat{u}_0(\frac{1}{2}(\xi-\frac{\tau}{\xi}))\hat{\overline{v}}_0(\frac{1}{2}(\xi+\frac{\tau}{\xi}))|^2d\tau d\xi \\
&=\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{u}_0(\frac{1}{2}(\xi-s))\hat{\overline{v}}_0(\frac{1}{2}(\xi+s))|^2dsd\xi \\
&=\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{u}_0(p)\hat{\overline{v}}_0(q)|^2dpdq \\
&=\frac{1}{2}||\hat{u}_0; L^2||^2||\hat{\overline{v}}_0; L^2||^2=\frac{1}{2}||u_0; L^2||^2\ ||v_0; L^2||^2,
\end{aligned}
\tag{2.3}
$$

where we have made changes of variables $\tau\mapsto s=\tau/\xi$ and then $(s,\xi)\mapsto(p,q)=((\xi-s)/2,(\xi+s)/2)$.

(b) *Second Proof.* We multiply the Fourier representations of $u$ and $\overline{v}$ and make a change of variables $(\xi,\eta)\mapsto(p,q)=(\xi+\eta,-\xi+\eta)$ and then $q\mapsto\tau=pq$ to obtain

$$
\begin{aligned}
u\overline{v}(t,x) &= (2\pi)^{-1}\Big(\int_{-\infty}^{\infty} \exp(ix\xi - it\xi^2)\hat{u}_0(\xi)d\xi\Big)\overline{\Big(\int_{-\infty}^{\infty} \exp(ix\eta - it\eta^2)\hat{v}_0(\eta)d\eta\Big)} \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ix(\xi-\eta) - it(\xi^2-\eta^2))\hat{u}_0(\xi)\overline{\hat{v}_0(\eta)}d\xi d\eta \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ix(\xi+\eta) - it(\xi^2-\eta^2))\hat{u}_0(\xi)\hat{\overline{v}}_0(\eta)d\xi d\eta\ \cdot \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ixp + itpq)\hat{u}_0\Big(\frac{1}{2}(p-q)\Big)\hat{\overline{v}}_0\Big(\frac{1}{2}(p+q)\Big)\frac{1}{2}dpdq \\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(ixp + it\tau)\hat{u}_0\Big(\frac{1}{2}(p-\frac{\tau}{p})\Big)\hat{\overline{v}}_0\Big(\frac{1}{2}(p+\frac{\tau}{p})\Big)\frac{1}{2|p|}d\tau dp.
\end{aligned}
\tag{2.4}
$$

Since (2.2) and (2.4) are equivalent, the proof proceeds as in (2.3).

(2) By (1.6), a duality argument, the Schwarz inequality, and the unitarity of $U(t)$,

$$
\begin{aligned}
&\|(-\Delta)^{1/4}(U(t)u_0 \cdot \int_0^t \overline{U(t-\tau)G(\tau)}d\tau); L^2(\mathbb{R}\times\mathbb{R})\| \\
&= \sup\Big\{\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (-\Delta)^{1/4}(U(t)u_0 \cdot \int_0^t \overline{U(t-\tau)G(\tau)}d\tau)\cdot\psi(t,x)dtdx\Big|; \\
&\qquad\qquad \psi \in L^2(\mathbb{R}\times\mathbb{R}), \|\psi; L^2(\mathbb{R}\times\mathbb{R})\| = 1\Big\} \\
&\leqslant \sup\Big\{\int_{-\infty}^{\infty}\int_0^t \|(-\Delta)^{1/4}(U(t)u_0 \cdot \overline{U(t)U(-\tau)G(\tau)}); L^2_x\|\ \|\psi(t); L^2_x\|d\tau dt; \\
&\qquad\qquad \psi \in L^2(\mathbb{R}\times\mathbb{R}), \|\psi; L^2(\mathbb{R}\times\mathbb{R})\| = 1\Big\} \\
&\leqslant \sup\Big\{\int_{-\infty}^{\infty} \|(-\Delta)^{1/4}(U(\cdot)u_0 \cdot \overline{U(\cdot)U(-\tau)G(\tau)}; L^2(\mathbb{R}\times\mathbb{R})\|\ \|\psi; L^2(\mathbb{R}\times\mathbb{R})\|d\tau; \\
&\qquad\qquad \psi \in L^2(\mathbb{R}\times\mathbb{R}), \|\psi; L^2(\mathbb{R}\times\mathbb{R})\| = 1\Big\} \\
&\leqslant 2^{-1/2}\int_{-\infty}^{\infty} \|u_0; L^2_x\|\ \|U(-\tau)G(\tau); L^2_x\|d\tau \\
&= 2^{-1/2}\|u_0; L^2_x\|\ \|G; L^1_t L^2_x\|.
\end{aligned}
$$

Similarly,

$$||(-\Delta)^{1/4}(\overline{U(t)v_0} \cdot \int_0^t U(t-\tau)F(\tau)d\tau); L^2(\mathbb{R}\times\mathbb{R})||$$
$$\leqslant 2^{-1/2}||v_0; L_x^2|| \; ||F; L_t^1 L_x^2||,$$
$$||(-\Delta)^{1/4}(\int_0^t U(t-\tau)F(\tau)d\tau \cdot \int_0^t \overline{U(t-\tau)G(\tau)}d\tau); L^2(\mathbb{R}\times\mathbb{R})||$$
$$\leqslant 2^{-1/2}||F; L_t^1 L_x^2|| \; ||G; L_t^1 L_x^2||.$$

This proves part (2).

(3) In the same way as in the proof of part (1), we have

$$\begin{aligned}
&||\partial(U(\cdot)u_0 \cdot \overline{U(\cdot)v_0}); L^2(\mathbb{R}\times\mathbb{R})||^2 \\
&= ||(-\Delta)^{1/2}(U(\cdot)u_0 \cdot \overline{U(\cdot)v_0}); L^2(\mathbb{R}\times\mathbb{R})||^2 \\
&= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{u}_0(\frac{1}{2}(\xi - \frac{\tau}{\xi}))\hat{\bar{v}}_0(\frac{1}{2}(\xi + \frac{\tau}{\xi}))|^2 d\tau d\xi \\
&= \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\xi| \; |\hat{u}_0(\frac{1}{2}(\xi - s))\hat{\bar{v}}_0(\frac{1}{2}(\xi + s))|^2 ds d\xi \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |p+q| \; |\hat{u}_0(p)\hat{\bar{v}}_0(q)|^2 dp dq \\
&\leqslant \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+p^2)^{1/2}(1+q^2)^{1/2}|\hat{u}_0(p)\hat{\bar{v}}_0(q)|^2 dp dq \\
&= \frac{1}{2}||u_0; L_t^2 H^{1/2}||^2 \; ||v_0; L_t^2 H^{1/2}||^2.
\end{aligned}$$

Therefore part (3) follows by the same argument as in part (2). QED

## 3. Proof of Theorem 3

In this section we prove Theorem 3. Without loss of generality we assume $p > 3$. We decompose $f$ as $f = f_1 + f_2$ with $f_j \in C^1(\mathbb{C};\mathbb{C}), f_j(0) = 0$,

$$|f_1(z)| \leqslant C|z|, \; |f_1'(z)| \leqslant C,$$
$$|f_2(z)| \leqslant C|z|^p, \; |f_2'(z)| \leqslant C|z|^{p-1},$$

for all $z \in \mathbb{C}$. We decompose $F$ as $F = F_1 + F_2 + F_3 + F_4 + F_5$, where

$$
\begin{aligned}
F_1(\phi) &= \frac{\lambda^2}{4}|\phi|^4\phi, \\
F_2(\phi) &= e^{-i(\lambda/2)\Theta} f_1(e^{i(\lambda/2)\Theta}\phi), \\
F_3(\phi) &= e^{-i(\lambda/2)\Theta} f_2(e^{i(\lambda/2)\Theta}\phi), \\
F_4(\phi) &= \lambda\phi \mathrm{Im} \int_{-\infty}^{x} e^{-i(\lambda/2)\Theta}\overline{\phi} f_1(e^{i(\lambda/2)\Theta}\phi)dy \equiv \phi I_1(\phi), \\
F_5(\phi) &= \lambda\phi \mathrm{Im} \int_{-\infty}^{x} e^{-i(\lambda/2)\Theta}\overline{\phi} f_2(e^{i(\lambda/2)\Theta}\phi)dy \equiv \phi I_2(\phi).
\end{aligned}
$$

We make essential use of the following estimates of the free propagator [1],[3], [9],[15], [16],[17],[18],[20],[32].

**Lemma 3.1.** *(1) Let $q$ and $r$ satisfy $0 \leqslant 2/q = 1/2 - 1/r \leqslant 1/2$ and let $\phi \in L^2(\mathbb{R})$. Then $U(\cdot)\phi$ satisfies the estimate*

$$||U(\cdot)\phi; L_t^q(\mathbb{R}; L_x^r)|| \leqslant C||\phi; L_x^2||. \tag{3.1}$$

*(2) Let $(q_1, r_1)$ and $(q_2, r_2)$ satisfy $0 \leqslant 2/q_j = 1/2 - 1/r_j \leqslant 1/2, j = 1, 2$. Then for any interval $I \subset \mathbb{R}$ with $0 \in \overline{I}$, the operator $G$ defined by*

$$(Gv)(t) = \int_0^t U(t-\tau)v(\tau)d\tau \tag{3.2}$$

*satisfies the estimate*

$$||Gv; L_t^{q_1}(I; L_x^{r_1})|| \leqslant C||v; L_t^{q_2'}(I; L_x^{r_2'})||, \tag{3.3}$$

*where the constant $C$ is independent of $I$ and $p'$ is the exponent conjugate to $p$.*
*(3) $U$ satisfies the estimates*

$$
\begin{aligned}
||(-\Delta)^{1/4}U(\cdot)\phi; L_x^\infty(\mathbb{R}; L_t^2(\mathbb{R}))|| &\leqslant C||\phi; L_x^2||, && (3.4)\\
||U(\cdot)\phi; L_x^4(\mathbb{R}; L_t^\infty(\mathbb{R}))|| &\leqslant C||(-\Delta)^{1/8}\phi; L_x^2||. && (3.5)
\end{aligned}
$$

*(4) For any interval $I \subset \mathbb{R}$ with $0 \in \overline{I}$, the operator $G$ defined by (3.2) satisfies the estimates*

$$
\begin{aligned}
||(-\Delta)^{1/4}Gv; L_t^\infty(I; L_x^2)|| &\leqslant C||v; L_x^1(\mathbb{R}; L_t^2(I))||, && (3.6)\\
||Gv; L_x^4(\mathbb{R}; L_t^\infty))|| &\leqslant C|I|^{3/4}||(-\Delta)^{1/8}v; L_t^4(I; L_x^2)|| && (3.7)\\
||\partial Gv; L_x^\infty(\mathbb{R}; L_t^2(I))|| &\leqslant C||v; L_x^1(\mathbb{R}; L_t^2(I))||, && (3.8)\\
||\partial Gv; L_x^\infty(\mathbb{R}; L_t^2(I))|| &\leqslant C|I|^{1/2}||(-\Delta)^{1/8}v; L_t^2(I; L_x^2)||, && (3.9)
\end{aligned}
$$

*where $C$ is independent of $I$ and $|I|$ is the length of $I$.*

*Proof.* For (3.1) and (3.3), see [3],[9],[15],[16],[32]. For (3.5), (3.6), and (3.8), see [17],[18]. For (3.4), see [20]. For (3.9), see [1]. We prove (3.7). By the Hölder inequality and (3.5), we have

$$\begin{aligned}
||Gv; L_x^4(\mathbb{R}; L_t^\infty(I))|| &\leqslant || \int_I \sup_t |U(t-\tau)v(\tau)| d\tau; L_x^4|| \\
&\leqslant |I|^{3/4} || \int_I \sup_t |U(t-\tau)v(\tau)|^4 d\tau; L_x^1||^{1/4} \\
&\leqslant |I|^{3/4} \Big(\int_I ||U(\cdot)U(-\tau)v(\tau); L_x^4 L_t^\infty||^4 d\tau\Big)^{1/4} \\
&\leqslant C|I|^{3/4} \Big(\int_I ||(-\Delta)^{1/8}U(-\tau)v(\tau); L_x^2||^4 d\tau\Big)^{1/4} \\
&= C|I|^{3/4} ||(-\Delta)^{1/8} v; L_t^4(I; L_x^2)||.
\end{aligned}$$

QED

We now collect basic estimates on the nonlinear terms.

**Lemma 3.2.** *Let $F = F_1 + F_2 + F_3 + F_4 + F_5$ be as above. Let $I = [0, T]$ with $T > 0$. Then for $\phi, \psi \in X(I)$*

$$\max_{1\leqslant j\leqslant 5} ||F_j(\phi); L_t^\infty(I; L_x^2)|| \leqslant C(|||\phi||| + |||\phi|||^{p+2}), \tag{3.10}$$

$$\max_{1\leqslant j\leqslant 5} ||F_j(\phi) - F_j(\psi); L_t^\infty(I; L_x^2)|| \leqslant C(1 + |||\phi|||^{p+3} + |||\psi|||^{p+3})|||\phi - \psi|||, \tag{3.11}$$

$$\max_{j=2,4,5} ||(-\Delta)^{1/4} F_j(\phi); L_t^\infty(I; L_x^2)|| \leqslant C(|||\phi||| + |||\phi|||^{p+2}), \tag{3.12}$$

$$\begin{aligned}
&\max_{j=2,4,5} ||(-\Delta)^{1/4}(F_j(\phi) - F_j(\psi)); L_t^\infty(I; L_x^2)|| \\
&\qquad \leqslant C(1 + |||\phi|||^{p+3} + |||\psi|||^{p+3})|||\phi - \psi|||,
\end{aligned} \tag{3.13}$$

$$\max_{j=1,3} ||F_j(\phi); L_x^1(\mathbb{R}; L_t^2(I))|| \leqslant CT^{1/2}(|||\phi|||^5 + |||\phi|||^p), \tag{3.14}$$

$$\begin{aligned}
&\max_{j=1,3} ||F_j(\phi) - F_j(\psi); L_x^1(\mathbb{R}; L_t^2(I))|| \\
&\qquad \leqslant CT^{1/2}(|||\phi|||^4 + |||\psi|||^4 + |||\phi|||^{p+1} + |||\psi|||^{p+1})|||\phi - \psi|||,
\end{aligned} \tag{3.15}$$

$$\max_{1\leqslant j\leqslant 5} ||(-\Delta)^{1/8} F_j(\phi); L_t^\infty(I; L_x^2)|| \leqslant C(|||\phi||| + |||\phi|||^{p+2}), \tag{3.16}$$

$$\max_{1\leqslant j\leqslant 5} ||(-\Delta)^{1/8}(F_j(\phi) - F_j(\psi)); L_t^\infty(I; L_x^2)|| \leqslant C(1 + |||\phi|||^{p+3} + |||\psi|||^{p+3})|||\phi - \psi|||. \tag{3.17}$$

*Proof.* We first consider (3.10) and (3.11). By a simple calculation, we have

$$\sum_{j=1}^{5} |F_j(\phi)| \leqslant C(|\phi| + |\phi|^5 + |\phi|^p) + C(||\phi; L_x^2||^2 + ||\phi; L_x^{p+1}||^{p+1})|\phi|. \tag{3.18}$$

By the Sobolev embedding $H^{1/2} \hookrightarrow L^q$ with $2 \leqslant q < \infty$, (3.18) implies (3.10). Another simple calculation gives

$$\begin{aligned}
&\sum_{j=1}^{5} |F_j(\phi) - F_j(\psi)| \\
&\leqslant C(1 + |\phi|^4 + |\psi|^4 + |\phi|^{p-1} + |\psi|^{p-1})|\phi - \psi| \\
&+ C(||\phi; L_x^2||^2 + ||\psi; L_x^2||^2 + ||\phi; L_x^{p+1}||^{p+1} + ||\psi; L_x^{p+1}||^{p+1})|\phi - \psi| \\
&+ C(||\phi; L_x^2|| + ||\psi; L_x^2|| + ||\phi; L_x^2||^{p+2} + ||\psi; L_x^2||^{p+2} + ||\phi; L_x^{p+1}||^{p+2} + ||\psi; L_x^{p+1}||^{p+2}) \\
&\qquad \cdot ||\phi - \psi; L_x^2||(|\phi| + |\psi|) \\
&+ C(||\phi; L_x^2|| + ||\psi; L_x^2||)||\phi - \psi; L_x^2||(|\phi|^p + |\psi|^p) \\
&+ C(||\phi; L_x^{p+1}||^{p+1} + ||\psi; L_x^{p+1}||^{p+1})||\phi - \psi; L_x^{p+1}||(|\phi| + |\psi|),
\end{aligned} \tag{3.19}$$

where we have used the inequality

$$\begin{aligned}
&|\exp(\pm i\frac{\lambda}{2}\int_{-\infty}^{x} |\phi|^2 dy) - \exp(\pm i\frac{\lambda}{2}\int_{-\infty}^{x} |\psi|^2 dy)| \\
&\leqslant \frac{|\lambda|}{2}||\,|\phi|^2 - |\psi|^2; L_x^1|| \leqslant \frac{|\lambda|}{2}(||\phi; L_x^2|| + ||\psi; L_x^2||)||\phi - \psi; L_x^2||
\end{aligned}$$

and this is the reason why the degree of norms of functions on the RHS of (3.19) is greater than that of (3.18) by two. By the Sobolev embedding, (3.19) implies (3.11).

We next consider (3.12) and (3.13). We write

$$(-\Delta)^{1/4}F_2(\phi) = [(-\Delta)^{1/4}, e^{i(\lambda/2)\Theta}]f_1(e^{i(\lambda/2)\Theta}\phi) + e^{-i(\lambda/2)\Theta}(-\Delta)^{1/4}f_1(e^{i(\lambda/2)\Theta}\phi), \tag{3.20}$$

$$(-\Delta)^{1/4}F_4(\phi) = [(-\Delta)^{1/4}, I_1(\phi)]\phi + \phi(-\Delta)^{1/4}I_1(\phi), \tag{3.21}$$

$$(-\Delta)^{1/4}F_5(\phi) = [(-\Delta)^{1/4}, I_2(\phi)]\phi + \phi(-\Delta)^{1/4}I_2(\phi), \tag{3.22}$$

where $[\cdot,\cdot]$ denotes the commutator and the functions are also regarded as the corresponding multiplication operators. By the $L^2$ boundedness theorem of commutators [8],[26], the first term on the RHS of (3.20) is estimated by

$$\begin{aligned}
&\|[(-\Delta)^{1/4}, e^{-i(\lambda/2)\Theta}]f_1(e^{i(\lambda/2)\Theta}\phi); L_x^2\| \\
&\leqslant C\|(-\Delta)^{1/4}e^{-i(\lambda/2)\Theta}; BMO\| \, \|f_1(e^{i(\lambda/2)\Theta}\phi); L_x^2\| \\
&\leqslant C\|(-\Delta)^{1/2}e^{-i(\lambda/2)\Theta}; L_x^2\| \, \|f_1(e^{i(\lambda/2)\Theta}\phi); L_x^2\| \\
&\leqslant C\|\partial e^{-i(\lambda/2)\Theta}; L_x^2\| \, \|\phi; L_x^2\| \\
&\leqslant C\|\phi; L_x^4\|^2 \, \|\phi; L_x^2\| \leqslant C\|\phi; H^{1/2}\|^3,
\end{aligned} \tag{3.23}$$

where BMO is the space of functions of bounded mean oscillation of John and Nirenberg and we have used the embedding $\dot{H}^{1/2} \hookrightarrow BMO$. In the same way as above, the terms with commutators in (3.21) and (3.22) are estimated by

$$\begin{aligned}
\|[(-\Delta)^{1/4}, I_1(\phi)]\phi; L_x^2\| &\leqslant C\|\partial I_1(\phi); L_x^2\| \, \|\phi; L_x^2\| \\
&\leqslant C\|\phi; L_x^4\|^2 \, \|\phi; L_x^2\| \leqslant C\|\phi; H^{1/2}\|^3,
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\|[(-\Delta)^{1/4}, I_2(\phi)]\phi; L_x^2\| &\leqslant C\|\phi; L_x^{2(p+1)}\|^{p+1} \, \|\phi; L_x^2\| \\
&\leqslant C\|\phi; H^{1/2}\|^{p+2}.
\end{aligned} \tag{3.25}$$

By the chain rule for fractional derivatives with $f_1'(e^{i(\lambda/2)\Theta}\phi)$ in $L_x^\infty$ [7],[19] and the same commutator estimate as in (3.23), the second term on the RHS of (3.20) is estimated by

$$\begin{aligned}
&\|e^{-i(\lambda/2)\Theta}(-\Delta)^{1/4}f_1(e^{i(\lambda/2)\Theta}\phi); L_x^2\| \\
&\leqslant C\|(-\Delta)^{1/4}e^{i(\lambda/2)\Theta}\phi; L_x^2\| \\
&\leqslant C\|[(-\Delta)^{1/4}, e^{i(\lambda/2)\Theta}]\phi; L_x^2\| + C\|e^{i(\lambda/2)\Theta}(-\Delta)^{1/4}\phi; L_x^2\| \\
&\leqslant C\|\phi; H^{1/2}\|^3 + C\|\phi; H^{1/2}\|.
\end{aligned} \tag{3.26}$$

By the Hölder inequality, the embedding $\dot{H}^1_{4/3} \hookrightarrow \dot{H}^{1/2}_4$ and the $L^{4/3}$ boundedness of the Hilbert transform, the second term on the RHS of (3.21) is estimated by

$$\begin{aligned}
\|\phi(-\Delta)^{1/4}I_1(\phi); L_x^2\| &\leqslant \|\phi; L_x^4\| \, \|(-\Delta)^{1/4}I_1(\phi); L_x^4\| \\
&\leqslant C\|\phi; L_x^4\| \, \|(-\Delta)^{1/2}I_1(\phi); L_x^{4/3}\| \\
&\leqslant C\|\phi; L_x^4\| \, \|\partial I_1(\phi); L_x^{4/3}\| \\
&\leqslant C\|\phi; L_x^4\| \, \|\phi; L_x^{8/3}\|^2 \leqslant C\|\phi; H^{1/2}\|^3.
\end{aligned} \tag{3.27}$$

Similarly, the second term on the RHS of (3.22) is estimated by

$$\begin{aligned}
\|\phi(-\Delta)^{1/4}I_2(\phi); L_x^2\| &\leqslant C\|\phi; L_x^4\| \, \|\phi; L_x^{(4(p+1))/3}\|^{p+1} \\
&\leqslant C\|\phi; H^{1/2}\|^{p+2}.
\end{aligned} \tag{3.28}$$

The required estimate (3.12) follows from (3.20)-(3.28) and a similar estimate yields (3.13), where the degree of norms of functions on the RHS is greater than that of (3.12) by two because of the term $\partial(I_2(\phi) - I_2(\psi))$ in the estimates.

We next consider (3.14) and (3.15). By the Hölder inequality,

$$\begin{aligned}||F_3(\phi); L_x^1 L_t^2|| &\leqslant C|| \, ||\phi; L_t^\infty||^2 ||\phi^{p-2}; L_t^2||; L_x^1|| \\ &\leqslant C||\phi; L_x^4 L_x^\infty||^2 ||\phi^{p-2}; L_t^2 L_x^2|| \\ &\leqslant CT^{1/2}||\phi; L_x^4 L_t^\infty||^2 ||\phi; L_t^\infty L_x^{2(p-2)}||^{p-2} \\ &\leqslant CT^{1/2}||\phi; L_x^4 L_t^\infty||^2 ||\phi; L_t^\infty H^{1/2}||^{p-2}. \end{aligned} \tag{3.29}$$

By a similar estimate with necessary modification caused by phase factors,

$$\begin{aligned}&||F_3(\phi) - F_3(\psi); L_x^1 L_t^2|| \\ &\leqslant CT^{1/2}(||\phi; L_t^\infty H^{1/2}||^{p-1} + ||\psi; L_t^\infty H^{1/2}||^{p-1})(||\phi; L_x^4 L_t^\infty||^2 + ||\psi; L_x^4 L_t^\infty||^2) \\ &\qquad \cdot ||\phi - \psi; L_t^\infty L_x^2|| \\ &+ CT^{1/2}(||\phi; L_t^\infty H^{1/2}||^{p-2} + ||\psi; L_t^\infty H^{1/2}||^{p-2})(||\phi; L_x^4 L_t^\infty|| + ||\psi; L_x^4 L_t^\infty||) \\ &\qquad \cdot ||\phi - \psi; L_x^4 L_t^\infty||. \end{aligned} \tag{3.30}$$

A similar and simpler calculation gives

$$||F_1(\phi); L_x^1 L_t^2|| \leqslant CT^{1/2}||\phi; L_x^4 L_t^\infty||^2 ||\phi; L_t^\infty H^{1/2}||^3, \tag{3.31}$$

$$\begin{aligned}||F_1(\phi) - F_1(\psi); L_x^1 L_t^2|| &\leqslant CT^{1/2}(||\phi; L_t^\infty H^{1/2}||^{p-2} + ||\psi; L_t^\infty H^{1/2}||^{p-2}) \\ &\quad \cdot (||\phi; L_x^4 L_t^\infty|| + ||\psi; L_x^4 L_t^\infty||)||\phi - \psi; L_x^4 L_t^\infty||. \end{aligned} \tag{3.32}$$

The required estimates (3.14) and (3.15) follow from (3.29)-(3.32).

We finally consider (3.16) and (3.17). Since we already know (3.10)-(3.13), we have (3.16) and (3.17) for $F_j$ with $j = 2, 4, 5$ by using the interpolation inequality

$$||(-\Delta)^{1/8}u; L_x^2|| \leqslant ||(-\Delta)^{1/4}u; L_x^2||^{1/2}||u; L_x^2||^{1/2}. \tag{3.33}$$

It remains to prove (3.16) and (3.17) for $F_j$ with $j = 1, 3$. By the embedding $\dot{H}^{1/2}_{4/3} \hookrightarrow \dot{H}^{1/4}$, the chain rule of fractional derivatives with $f_2'(e^{i(\lambda/2)\Theta}\phi)$ in $L_x^4$ and $(-\Delta)^{1/4}e^{i(\lambda/2)\Theta}\phi$ in $L_x^2$, and (3.26), we obtain

$$\begin{aligned}||(-\Delta)^{1/8}F_3(\phi); L_t^\infty L_x^2|| &\leqslant C||\phi; L_t^\infty L_x^{4(p-1)}||^{p-1}||(-\Delta)^{1/4}e^{i(\lambda/2)\Theta}\phi; L_t^\infty L_x^2|| \\ &\leqslant C||\phi; L_t^\infty H^{1/2}||^p + C||\phi; L_t^\infty H^{1/2}||^{p+2}. \end{aligned} \tag{3.34}$$

By a similar estimate with necessary modification caused by phase factors,

$$
\begin{aligned}
&\|(-\Delta)^{1/8}(F_3(\phi)-F_3(\psi)); L_t^\infty L_x^2\| \\
&\leqslant C(1+\|\phi; L_t^\infty H^{1/2}\|^{p+3}+\|\psi; L_t^\infty H^{1/2}\|^{p+3})\|\phi-\psi; L_t^\infty H^{1/2}\|. && (3.35)
\end{aligned}
$$

A similar and simpler calculation implies (3.16) and (3.17) for $F_1$. QED

*Proof of Theorem 3.* For $\phi_0 \in H^{1/2}$ with $\|\phi_0; H^{1/2}\| \leqslant \rho$ and $\phi \in X([0,T])$ with $T>0$ we define $J(\phi)$ by

$$J(\phi) = U(\cdot)\phi_0 - iGF(\phi).$$

We prove that the map $\phi \mapsto J(\phi)$ is a contraction on a closed ball in $X([0,T])$ with $T>0$ sufficiently small. By (3.1), (3.3), (3.10), and (3.11), we have

$$
\begin{aligned}
&\|J(\phi); L_t^\infty L_x^2\| + \|J(\phi); L_t^4 L_x^\infty\| \leqslant C\rho + CT(|||\phi||| + |||\phi|||^{p+2}), && (3.36)\\
&\|J(\phi)-J(\psi); L_t^\infty L_x^2\| + \|J(\phi)-J(\psi); L_t^4 L_x^\infty\| \\
&\leqslant CT(1+|||\phi|||^{p+3}+|||\psi|||^{p+3})|||\phi-\psi|||. && (3.37)
\end{aligned}
$$

By (3.1), (3.3), (3.4), (3.12), (3.13), (3.14), and (3.15), we have

$$
\begin{aligned}
&\|(-\Delta)^{1/4}J(\phi); L_t^\infty L_x^2\| \\
&\leqslant C\rho + CT(|||\phi||| + |||\phi|||^{p+2}) + CT^{1/2}(|||\phi|||^5 + |||\phi|||^p), && (3.38)\\
&\|(-\Delta)^{1/4}(J(\phi)-J(\psi)); L_t^\infty L_x^2\| \\
&\leqslant CT(1+|||\phi|||^{p+3}+|||\psi|||^{p+3})|||\phi-\psi||| \\
&+CT^{1/2}(|||\phi|||^4+|||\psi|||^4+|||\phi|||^{p+1}+|||\psi|||^{p+1})|||\phi-\psi|||. && (3.39)
\end{aligned}
$$

By (3.5), (3.7), (3.16), and (3.17), we have

$$
\begin{aligned}
&\|J(\phi); L_x^4 L_t^\infty\| \\
&\leqslant C\rho + CT(|||\phi||| + |||\phi|||^{p+2}), && (3.40)\\
&\|J(\phi)-J(\psi); L_x^4 L_t^\infty\| \\
&\leqslant CT(1+|||\phi|||^{p+3}+|||\psi|||^{p+3})|||\phi-\psi|||. && (3.41)
\end{aligned}
$$

By (3.4), (3.8), (3.9), (3.14), (3.15), and (3.16) and (3.17) for $F_j$ with $j=2,4,5$ we have

$$
\begin{aligned}
&\|\partial J(\phi); L_t^\infty L_x^2\| \\
&\leqslant C\rho + CT^{1/2}(|||\phi|||^5 + |||\phi|||^p) + CT(|||\phi||| + |||\phi|||^{p+2}), && (3.42)\\
&\|\partial(J(\phi)-J(\psi)); L_t^\infty L_x^2\| \\
&\leqslant CT^{1/2}(|||\phi|||^4+|||\psi|||^4+|||\phi|||^{p+1}+|||\psi|||^{p+1})|||\phi-\psi||| \\
&+CT(1+|||\phi|||^{p+3}+|||\psi|||^{p+3})|||\phi-\psi|||. && (3.43)
\end{aligned}
$$

By (3.36), (3.38), (3.40), and (3.42), we have

$$|||J(\phi)||| \leqslant C\rho + CT^{1/2}(|||\phi|||^5 + |||\phi|||^p) + CT(|||\phi||| + |||\phi|||^{p+2}). \tag{3.44}$$

By (3.37), (3.39), (3.41), and (3.43), we have

$$\begin{aligned}
&|||J(\phi) - J(\psi)||| \\
&\leqslant CT^{1/2}(|||\phi|||^4 + |||\psi|||^4 + |||\phi|||^{p+1} + |||\psi|||^{p+1})|||\phi - \psi||| \\
&\quad + CT(1 + |||\phi|||^{p+3} + |||\psi|||^{p+3})|||\phi - \psi|||.
\end{aligned} \tag{3.45}$$

In view of (3.44) and (3.45), the contraction argument proves the existence of a unique fixed point of $J$ in $X([0,T])$ with $T > 0$ sufficiently small. The continuous dependence of solutions on the data follows similarly. QED

## 4. Proof of Theorem 1.

In this section we prove Theorem 1. We start with the following lemma.

**Lemma 4.1.** *Let $f$ satisfy (H). Let $\phi_0, \tilde{\phi}_0 \in H^{1/2}$ with $\max(||\phi_0; H^{1/2}||, ||\tilde{\phi}_0; H^{1/2}||) \leqslant \rho$ and let $\phi, \tilde{\phi} \in X(I)$ be the solutions to (1.23) with $(\phi(0), \tilde{\phi}(0)) = (\phi_0, \tilde{\phi}_0)$ given by Theorem 3. Let $u, \tilde{u}$ be given by*

$$u = e^{i(\lambda/2)\Theta}\phi, \tag{4.1}$$

$$\tilde{u} = e^{i(\lambda/2)\tilde{\Theta}}\tilde{\phi}, \tag{4.2}$$

*where*

$$\Theta(t,x) = \int_{-\infty}^{x} |\phi(t,y)|^2 dy,$$

$$\tilde{\Theta}(t,x) = \int_{-\infty}^{x} |\tilde{\phi}(t,y)|^2 dy.$$

*Then, $u, \tilde{u} \in X(I)$ and $\partial|u|^2, \partial|\tilde{u}|^2 \in L^2(I \times \mathbb{R})$. Moreover,*

$$|||u||| \leqslant C\rho|||\phi|||^2 + C(1+T)(|||\phi||| + |||\phi|||^9), \tag{4.3}$$

$$\begin{aligned}
|||u - \tilde{u}||| &\leqslant C\rho(|||\phi||| + |||\tilde{\phi}|||)|||\phi - \tilde{\phi}||| \\
&\quad + C(1+T)(|||\phi|||^2 + |||\tilde{\phi}|||^2 + |||\phi|||^8 + |||\tilde{\phi}|||^8)|||\phi - \tilde{\phi}|||,
\end{aligned} \tag{4.4}$$

$$||\partial|u|^2; L_t^2 L_x^2|| \leqslant C\rho^2 + C(1+T)^2(|||\phi|||^2 + |||\phi|||^{14}), \tag{4.5}$$

$$\begin{aligned}
||\partial|u|^2 - \partial|\tilde{u}|^2; L_t^2 L_x^2|| &\leqslant C\rho||\phi_0 - \tilde{\phi}_0; H^{1/2}|| \\
&\quad + C(1+T)^2(|||\phi||| + |||\tilde{\phi}||| + |||\phi|||^{15} + |||\tilde{\phi}|||^{15})|||\phi - \tilde{\phi}|||.
\end{aligned} \tag{4.6}$$

*Proof.* We first consider (4.5) and (4.6). By (4.1), Theorem 2, and Lemma 3.2 with $p=5$, we have

$$\begin{aligned}||\partial|u|^2; L_t^2L_x^2|| &= ||\partial|\phi|^2; L_t^2L_x^2|| \\ &\leqslant C||\phi_0; H^{1/2}||^2 + C||F(\phi); L_t^1H^{1/2}||^2 \\ &\leqslant C\rho^2 + C\max_{j=1,3}||(-\Delta)^{1/4}F_j(\phi); L_t^1L_x^2||^2 \\ &+ CT^2(|||\phi||| + |||\phi|||^7)^2. \end{aligned} \tag{4.7}$$

We estimate the second term on the RHS of the last inequality of (4.7) in the same way as in the proof of Lemma 3.2 and we have

$$\begin{aligned} &||(-\Delta)^{1/4}F_3(\phi); L_t^1L_x^2|| \\ &\leqslant ||[(-\Delta)^{1/4}, e^{i(\lambda/2)\Theta}]f_2(e^{-i(\lambda/2)\Theta}\phi); L_t^1L_x^2|| + ||(-\Delta)^{1/4}f_2(e^{i(\lambda/2)\Theta}\phi); L_t^1L_x^2|| \\ &\leqslant CT||\partial e^{i(\lambda/2)\Theta}; L_t^\infty L_x^2||\ ||f_2(e^{-i(\lambda/2)\Theta}\phi); L_t^\infty L_x^2|| \\ &+ C||\ ||f_2'(e^{i(\lambda/2)\Theta}\phi); L_x^\infty||\ ||(-\Delta)^{1/4}e^{i(\lambda/2)\Theta}\phi; L_x^2||; L_t^1|| \\ &\leqslant CT||\phi; L_t^\infty H^{1/2}||^7 + C||\phi; L_t^4L_t^\infty||^4(||\phi; L_t^\infty H^{1/2}|| + ||\phi; L_t^\infty H^{1/2}||^3) \\ &\leqslant CT|||\phi|||^7 + C(|||\phi|||^5 + |||\phi|||^7), \end{aligned} \tag{4.8}$$

$$||(-\Delta)^{1/4}F_1(\phi); L_t^1L_x^2|| \leqslant C|||\phi|||^5. \tag{4.9}$$

By (4.7), (4.8), and (4.9), we obtain (4.5). A similar calculation yields (4.6), since

$$\partial|u|^2 - \partial|\tilde{u}|^2 = \partial((\phi-\tilde{\phi})\overline{\phi}) + \partial(\tilde{\phi}\overline{(\phi-\tilde{\phi})}).$$

We now consider (4.5) and (4.6). By (4.1),

$$||u; L_t^\infty L_x^2|| + ||u; L_t^4L_x^\infty|| + ||u; L_x^4L_t^\infty|| = ||\phi; L_t^\infty L_x^2|| + ||\phi; L_t^4L_x^\infty|| + ||\phi; L_x^4L_t^\infty||. \tag{4.10}$$

As in the proof of Lemma 3.2, we have

$$||(-\Delta)^{1/4}u; L_t^\infty L_x^2|| \leqslant C||\phi; L_t^\infty H^{1/2}|| + C||\phi; L_t^\infty H^{1/2}||^3. \tag{4.11}$$

Since $\partial u = e^{i(\lambda/2)\Theta}(\partial\phi + i(\lambda/2)|\phi|^2\phi)$, we have

$$||\partial u; L_t^\infty L_x^2|| \leqslant ||\partial\phi; L_t^\infty L_x^2|| + C||\ |\phi|^2\phi; L_t^\infty L_x^2||. \tag{4.12}$$

By the Hölder and Gagliardo-Nirenberg inequalities, we obtain

$$\begin{aligned} ||\ |\phi|^2\phi; L_t^\infty L_x^2|| &\leqslant ||\ |\phi|^2\phi; L_t^2L_x^\infty|| \\ &\leqslant ||\ |\phi|^2; L_t^4L_x^\infty||\ ||\phi; L_t^4L_x^\infty|| \\ &\leqslant C||\ ||\partial|\phi|^2; L_x^2||^{1/2}||\ |\phi|^2; L_x^2||^{1/2}; L_t^4||\ ||\phi; L_t^4L_x^\infty|| \\ &\leqslant C||\partial|\phi|^2; L_t^2L_x^2||^{1/2}||\phi; L_t^\infty L_x^4||\ ||\phi; L_t^4L_x^\infty||. \end{aligned} \tag{4.13}$$

By (4.1), (4.5), (4.10), (4.11), (4.12), and (4.13), we have (4.3). A similar calculation yields (4.4), since

$$|\partial u - \partial \tilde{u}| \leqslant (|\lambda|/2)(\|\phi; L_t^\infty L_x^2\| + \|\tilde{\phi}; L_t^\infty L_x^2\|)\|\phi - \tilde{\phi}; L_t^\infty L_x^2\| \, |\partial u|$$
$$+ |\partial\phi - \partial\tilde{\phi}| + |\lambda|(|\phi|^2 + |\tilde{\phi}|^2)|\phi - \tilde{\phi}|.$$

QED

*Proof of Theorem 1.* Let $u_0 \in H^{1/2}$ and let $\{u_0^{(n)}\} \subset H^2$ be a sequence in $H^2$ such that $u_0^{(n)} \to u_0$ in $H^{1/2}$ as $n \to \infty$. We define $\phi_0$ and $\phi_0^{(n)}$ by

$$\phi_0 = e^{-i(\lambda/2)\theta_0} u_0,$$
$$\phi_0^{(n)} = e^{-i(\lambda/2)\theta_0^{(n)}} u_0^{(n)},$$

where

$$\theta_0(x) = \int_{-\infty}^{x} |u_0(y)|^2 dy,$$
$$\theta_0^{(n)}(x) = \int_{-\infty}^{x} |u_0^{(n)}(y)|^2 dy.$$

As in the proof of Theorem 3, we see that $\phi_0 \in H^{1/2}, \phi_0^{(n)} \in H^2$, and $\phi_0^{(n)} \to \phi_0$ in $H^{1/2}$ as $n \to \infty$. Let $\rho = \sup_n \|\phi_0^{(n)}; H^{1/2}\|$. Let $\phi, \phi^{(n)} \in X(I)$ be the solutions to (1.23) with $(\phi(0), \phi^{(n)}(0)) = (\phi_0, \phi_0^{(n)})$ given by Theorem 3. As in the proof of Theorem 3, we see that $\phi^{(n)} \to \phi$ in $X(I)$ as $n \to \infty$. With $\phi$ and $\phi^{(n)}$, we define respectively $u$ and $u^{(n)}$ in the same way as in (4.1). Then Lemma 4.1 shows that $u, u^{(n)} \in X(I)$ with $\partial|u|^2, \partial|u^{(n)}|^2 \in L^2(I \times \mathbb{R})$ and that $u^{(n)} \to u$ in $X(I)$ and $\partial|u^{(n)}|^2 \to \partial|u|^2$ in $L^2(I \times \mathbb{R})$ as $n \to \infty$. In the same way as in the introduction, $u^{(n)}$ satisfies

$$i\partial_t u^{(n)} + \partial^2 u^{(n)} = i\lambda(\partial|u^{(n)}|^2)u^{(n)} + f(u^{(n)})$$

with $u^{(n)}(0) = u_0^{(n)}$ and therefore

$$u^{(n)} = U(\cdot)u_0^{(n)} - iG(i\lambda(\partial|u^{(n)}|^2)u^{(n)} + f(u^{(n)})). \tag{4.14}$$

Since $(\partial|u^{(n)}|^2)u^{(n)} \to (\partial|u|^2)u$ and $f(u^{(n)}) \to f(u)$ in $L_t^1(I, L_x^2)$ as $n \to \infty$, the RHS of (4.14) tends to

$$U(\cdot)u_0 - iG(i\lambda(\partial|u|^2)u + f(u))$$

in $L_t^\infty(I;L_x^2)\cap L_t^4(I;L_x^\infty)$ as $n\to\infty$. Letting $n\to\infty$ in (4.14), we obtain

$$u = U(\cdot)u_0 - iG(i\lambda(\partial|u|^2)u + f(u)), \tag{4.15}$$

where the RHS of (4.15) makes sense in $L_t^\infty(I;L_x^2)\cap L_t^4(I;L_x^\infty)$ and therefore in $X(I)$ since $u-U(\cdot)u_0\in X(I)$. This proves the essential part of the theorem and the other part follows in the same way as in the preceding argument. QED

## 5. Estimates for the Null Gauge Form in Higher Dimensions

In this section we make an extension of Theorem 2 to the case of space-time $\mathbb{R}^{1+n}$ with $n\geqslant 2$. Let $U(t)=\exp(it\Delta)$ be the free propagator with Laplacian $\Delta$ in $\mathbb{R}^n$.

**Theorem 4.** *Let $u(t)=U(t)u_0$ and $v(t)=U(t)v_0$ with $u_0,v_0\in L^2(\mathbb{R}^n)$. Then*

$$||(-\Delta)^{(2-n)/4}u\bar{v};L^2(\mathbb{R}\times\mathbb{R}^n)||\leqslant C_n||u_0;L^2||\;||v_0;L^2||, \tag{5.1}$$

*where $C_n=2^{-n/2}\pi^{(2-n)/4}\Gamma(n/2)^{-1/2}$ and $\Gamma$ denotes the gamma function.*

**Remark 5.1.** We have stated an extension of part (1) of Theorem 2 only since the corresponding statements to parts (2) and (3) are obvious.

**Remark 5.2.** When $n=1$, (5.1) reduces to a weak form of (1.6) since $C_1=2^{-1/2}$ . In this sense (5.1) is regarded as an extension of (1.6).

**Rermark 5.3.** If we disregard an explicit bound of the constant in (5.1), the estimate for $n\geqslant 2$ follows from the embedding $\dot{H}^{n/2-1}_{n/(n-1)}\hookrightarrow L^2$ and the Strichartz inequality $||U(\cdot)\phi;L_t^4(\mathbb{R};L_x^{2n/(n-1)})||\leqslant C||\phi;L_x^2||$. Below we give a direct proof of (5.1) with explicit constant $C_n$ and we emphasize here that the constant $C_n$ is sharp. Indeed, we take the Gaussian and put $u_0(x)=v_0(x)=e^{-\alpha|x|^2}$ with $\alpha>0$. Then (5.1) is realized as an equality since

$$\begin{aligned}
&|(U(t)u_0)(x)|^2=(1+16\alpha^2t^2)^{-n/2}\exp(-(2\alpha/(1+16\alpha^2t^2))|x|^2),\\
&(\mathcal{F}|U(t)u_0|^2)(\xi)=2^{-n}\alpha^{-n/2}\exp(-((1+16\alpha^2t^2)/8\alpha)|\xi|^2),\\
&||(-\Delta)^{(2-n)/4}|u|^2;L^2(\mathbb{R}\times\mathbb{R}^n)||=2^{-n}\pi^{(2+n)/4}\Gamma(n/2)^{-1/2}\alpha^{-n/2},\\
&||u_0;L_x^2||^2=2^{-n/2}\pi^{n/2}\alpha^{-n/2}.
\end{aligned}$$

*Proof of Theorem 4.* We follow the first proof of part (1) of Theorem 2. The novelty of the proof here consists in the technique from the Radon transform in the Fourier space. We take the Fourier transform with respect to space to obtain

$$\begin{aligned}\mathcal{F}(u\overline{v})(\xi) &= (2\pi)^{-n/2}(\mathcal{F}u) * (\mathcal{F}\overline{v})(\xi) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp(-it(|\xi|^2 - 2\xi \cdot \eta))\hat{u}_0(\xi - \eta)\hat{\overline{v}}_0(\eta)d\eta. \qquad (5.2)\end{aligned}$$

We introduce the polar coordinates in the Fourier space such that $\xi = \rho\omega$ with $\rho > 0$ and $\omega \in S^{n-1}$, the unit sphere, and we take the Fourier transform of (5.2) with respect to time to obtain

$$\begin{aligned}&(2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-it\tau)\mathcal{F}(u\overline{v})(\rho\omega)dt \\ &= (2\pi)^{-(n+1)/2} \int_{-\infty}^{\infty} \exp(-it\tau) \int_{\mathbb{R}^n} \exp(-it(\rho^2 - 2\rho\omega \cdot \eta))\hat{u}_0(\rho\omega - \eta)\hat{\overline{v}}_0(\eta)d\eta dt \\ &= (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} ( \int_{-\infty}^{\infty} \exp(-it(\tau + \rho^2 - 2\rho\omega \cdot \eta))dt)\hat{u}_0(\rho\omega - \eta)\hat{\overline{v}}_0(\eta)d\eta \\ &= (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^n} \delta(2\rho((\tau + \rho^2)/(2\rho) - \omega \cdot \eta))\hat{u}_0(\rho\omega - \eta)\hat{\overline{v}}_0(\eta)d\eta \\ &= (2\pi)^{-(n+1)/2}(2\rho)^{-1} \int_{\mathbb{R}^n} \delta((\tau + \rho^2)/(2\rho) - \omega \cdot \eta)\hat{u}_0(\rho\omega - \eta)\hat{\overline{v}}_0(\eta)d\eta \\ &= (2\pi)^{-(n-1)/2}(2\rho)^{-1} \int_{\omega\cdot\eta=(\tau+\rho^2)/(2\rho)} \hat{u}_0(\rho\omega - \eta)\hat{\overline{v}}_0(\eta)d\eta, \qquad (5.3)\end{aligned}$$

where the last $d\eta$ stands for the $(n-1)$-dimensional Lebesgue measure on the hyperplane

$\{\eta \in \mathbb{R}^n; \omega \cdot \eta = (\tau + \rho^2)/(2\rho)\}$. Application of the Parseval formula to (5.3) yields

$$
\begin{aligned}
&||(-\Delta)^{(2-n)/4} u\overline{v}; L^2(\mathbb{R} \times \mathbb{R}^n)||^2 \\
&= \int_{|\omega|=1} \int_0^\infty \int_{-\infty}^\infty |(2\pi)^{-1/2} \int_{-\infty}^\infty \exp(-it\tau) \rho^{(2-n)/2} \mathcal{F}(u\overline{v})(\rho\omega) dt|^2 d\tau \rho^{n-1} d\rho d\omega \\
&= (2\pi)^{1-n} \int_{|\omega|=1} \int_0^\infty \int_{-\infty}^\infty (4\rho)^{-1} | \int_{\omega\cdot\eta=(\tau+\rho^2)/(2\rho)} \hat{u}_0(\rho\omega - \eta) \hat{\overline{v}}_0(\eta) d\eta|^2 d\tau d\rho d\omega \\
&= 2^{-1-n} \pi^{1-n} \int_{|\omega|=1} \int_0^\infty \int_{-\infty}^\infty | \int_{\omega\cdot\eta=(\sigma+\rho)/2} \hat{u}_0(\rho\omega - \eta) \hat{\overline{v}}_0(\eta) d\eta|^2 d\sigma d\rho d\omega \\
&= 2^{-2-n} \pi^{1-n} \int_{|\omega|=1} \int_{-\infty}^\infty \int_{-\infty}^\infty | \int_{\omega\cdot\eta=(\sigma+\rho)/2} \hat{u}_0(\rho\omega - \eta) \hat{\overline{v}}_0(\eta) d\eta|^2 d\sigma d\rho d\omega,
\end{aligned} \tag{5.4}
$$

where we have made a change of variable $\tau \mapsto \sigma = \rho^{-1}\tau$. Another change of variables $(\sigma, \rho) \mapsto (r, p) = ((\sigma + \rho)/2, (\sigma - \rho)/2)$ of the last integrals imply that the RHS of the last equality of (5.4) is equal to

$$
2^{-1-n} \pi^{1-n} \int_{|\omega|=1} \int_{-\infty}^\infty \int_{-\infty}^\infty | \int_{\omega\cdot\eta=r} \hat{u}_0((r-p)\omega - \eta) \hat{\overline{v}}_0(\eta) d\eta|^2 dr\ dp\ d\omega. \tag{5.5}
$$

We estimate (5.5) by the Schwarz inequality and use the plane wave decomposition formula to obtain

$$
\begin{aligned}
&||(-\Delta)^{(2-n)/4} u\overline{v}; L^2(\mathbb{R} \times \mathbb{R}^n)||^2 \\
&\leqslant 2^{-1-n} \pi^{1-n} \int_{|\omega|=1} \int_{-\infty}^\infty \int_{-\infty}^\infty ( \int_{\omega\cdot\eta=r} |\hat{u}_0((r-p)\omega - \eta)|^2 d\eta)( \int_{\omega\cdot\eta=r} |\hat{\overline{v}}_0(\eta)|^2 d\eta) dr\ dp\ d\omega \\
&= 2^{-1-n} \pi^{1-n} \int_{|\omega|=1} \int_{-\infty}^\infty \int_{-\infty}^\infty ( \int_{\omega\cdot\eta'=-p} |\hat{u}_0(\eta')|^2 d\eta')( \int_{\omega\cdot\eta=r} |\hat{\overline{v}}_0(\eta)|^2 d\eta) dr\ dp\ d\omega \\
&= 2^{-1-n} \pi^{1-n} \int_{|\omega|=1} ||\hat{u}_0; L^2||^2\ ||\hat{\overline{v}}_0; L^2||^2 d\omega \\
&= 2^{-n} \pi^{1-n/2} \Gamma(n/2)^{-1} ||u_0; L^2||^2\ ||v_0; L^2||^2.
\end{aligned}
$$

QED

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