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Series #329. February 1996

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February 15, 1996

Abstract

It is proven that the spin 1/2 Schrödinger operator $\tilde{H}$ with a constant magnetic field is the square of a sum of mutually strongly anticommuting self-adjoint operators. As an application, by using this formula, the essential spectrum of $\tilde{H}$ with a vector potential in a class is identified. The class contains a vector potential to which Shigekawa's theorem (I. Shigekawa, *J. Funct. Anal.*, 101:255–285, 1991) cannot be applied.

1 Introduction

The spectral properties of the Schrödinger operators $\tilde{H}$ with magnetic fields for a *spin 1/2* particle were deeply studied by Shigekawa in [9]. The operator is given by

$$\tilde{H} = \sum_{j=1}^{d}(-i\partial_j - a_j(x))^2 + \sum_{j,k=1}^{d} \frac{i}{2} b_{jk}(x)\gamma^j\gamma^k$$

acting in $L^2(\mathbb{R}^d) \otimes \mathbb{C}^r$ where $r = 2^l$, $l = [d/2]$ with $[\cdot]$ the Gauss symbol, $\partial_j = \partial/\partial x^j$, $\mathbf{a}(x) = \sum_{j=1}^{d} a_j(x)dx^j$ is a real 1-form called a vector potential, $\mathbf{b} = \sum_{j<k} b_{jk}\, dx^j \wedge dx^k = d\mathbf{a}$ with $b_{jk} = \partial_j a_k - \partial_k a_j$ is called a magnetic field and γ^j's are $r \times r$-Hermitian matrices (so called the Dirac matrices) satisfying

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta^{jk} \tag{1}$$

where the δ^{jk}'s are the Kronecker delta. This $\tilde{H}$ is also represented as $\tilde{H} = \mathbb{D}^2$ where $\mathbb{D}$ is the Dirac operator defined by

$$\mathbb{D} = \sum_{j=1}^{d} \gamma^j(-i\partial_j - a_j(x)).$$

For comparison, we define the Schrödinger operator H with a magnetic field for a *spinless* particle by

$$H = \sum_{j=1}^{d} (-i\partial_j - a_j(x))^2$$

acting in $L^2(\mathbb{R}^d)$.

We are mainly interested in $\tilde{H}$ with asymptotically constant magnetic fields. Assume that all a_j is C^∞ and

$$b_{jk}(x) \to \Lambda_{jk} \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad j, k = 1, \ldots, d, \tag{2}$$

where $\Lambda = (\Lambda_{jk})$ is a real skew-symmetric matrix. We note that Λ has eigenvalues of the form $\pm i\lambda_1, \ldots, \pm i\lambda_n, 0, \ldots, 0$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Without loss of generality, we can take $\lambda_j > 0$.

First, we consider the 2-dimensional constant magnetic field case, where $b(x) = \lambda\, dx^1 \wedge dx^2$ with a positive constant λ. We can take $\mathbf{a}(x) = \lambda(-x^2 dx^1 + x^1 dx^2)/2$. Let $\gamma^j = \sigma^j$, $j = 1, 2$. Here, σ^j, $j = 1, 2, 3$, are the Pauli matrices as follows:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With

$$A = (-i\partial_1 - a_1) + (\partial_2 - ia_2)$$

acting in $L^2(\mathbb{R}^2)$, we find

$$A^*A = AA^* + 2\lambda, \tag{3}$$

$$H = \frac{1}{2}(AA^* + A^*A) \quad \text{and} \quad \tilde{H} = H + \lambda\sigma^3.$$

Theorem 1.1 *Let $d = 2$ and $\mathbf{b} = \lambda\, dx^1 \wedge dx^2$ with $\lambda > 0$. Then*

$$\sigma(H) = \sigma_{ess}(H) = \{(2n+1)\lambda;\ n \in \mathbb{Z}_+\},$$

$$\sigma(\tilde{H}) = \sigma_{ess}(\tilde{H}) = \{2n\lambda;\ n \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\sigma(\cdot)$ and $\sigma_{ess}(\cdot)$ denote spectrum and essential spectrum, respectively. Moreover,

$$\ker \tilde{H} \subset \ker(\sigma^3 + 1).$$

It is well known that by virtue of the relation (3) we can prove this theorem in the same way as in the harmonic oscillator case (see, e.g., [5, 10]): All the eigenvectors of H are created by repeatedly acting A and A^* on the eigenvectors with the lowest eigenvalue.

In the higher dimensional case, Shigekawa has proved the following theorem.

Theorem 1.2 (Shigekawa, [9]) *Assume the condition (2).*

(i) Assume that 0 *is an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_n, 0, \ldots, 0$, $(\lambda_j > 0)$ *be eigenvalues of* Λ. *Then*

$$\sigma_{ess}(H) = [\lambda_1 + \cdots + \lambda_n, \infty),$$

$$\sigma(\tilde{H}) = \sigma_{ess}(\tilde{H}) = [0, \infty).$$

(ii) Assume that 0 *is not an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_m$, $(\lambda_j > 0,\ m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma_{ess}(H) = \{\sum_{j=1}^{m}(2k_j + 1)\lambda_j;\ k_j \in \mathbb{Z}_+\},$$

$$\sigma_{ess}(\tilde{H}) = \{\sum_{j=1}^{m} 2k_j\lambda_j;\ k_j \in \mathbb{Z}_+\}.$$

Moreover 0 *is an isolated point spectrum of* $\tilde{H}$.

Shigekawa proved this theorem using relations between the essential spectrum of $\tilde{H}$ and H such as the following:

$$\sigma_{ess}(\tilde{H}) = \sigma_{ess}(H - \sum_{j=1}^{n} i\lambda_j \gamma^{2j-1}\gamma^{2j}),$$

$$\sigma_{ess}(\tilde{H}) = \bigcup_{\epsilon_1 \cdots \epsilon_m = \pm 1} \sigma_{ess}(H + \sum_{j=1}^{m} \epsilon_j \lambda_j),$$

$$\bigcup_{\epsilon_1 \cdots \epsilon_m = 1} \sigma_{ess}(H + \sum_{j=1}^{m} \epsilon_j \lambda_j) \setminus \{0\} = \bigcup_{\epsilon_1 \cdots \epsilon_m = -1} \sigma_{ess}(H + \sum_{j=1}^{m} \epsilon_j \lambda_j) \setminus \{0\}.$$

These equations are derived from the Weyl theorem, (1) and the fact that H and all $i\gamma^{2j-1}\gamma^{2j}$ mutually strongly commute. In particular, in the proof of the part (i) he constructed concrete orthonormal functions in $L^2(\mathbb{R}^d)$ in order to use the Weyl criterion in a slightly strengthened version (see, e.g., [4]): Suppose that A is self-adjoint and $A \geq 0$. If there exists an orthonormal sequence $\{\phi_k\}_{k \in \mathbb{N}} \subset D(A)$ such that $\|(A+1)^{-1}(A - \alpha)\phi_k\| \to 0$ as $k \to \infty$, then $\alpha \in \sigma_{ess}(A)$.

Comparing Shigekawa's proof and that Theorem 1.1 can be proved by creating all eigenvectors of H using the relation (3), we feel that the inner structure of $\tilde{H}$ is not sufficiently clear in the higher dimensional case: Why does whether 0 is an eigenvalue of Λ or not cause the difference of $\sigma_{ess}(\tilde{H})$ in each case? The aim of this paper is to clarify the inner structure of $\tilde{H}$ and to identify the spectrum of $\tilde{H}$.

In this paper, we investigate $\mathbb{D}$ instead of $\tilde{H}$ itself, since $\mathbb{D}$ has more rich structures inherited from the Clifford algebra generated by γ^i's than $\tilde{H}$. In particular, in the constant magnetic field case, it is proven that $\mathbb{D}$ is a sum of operators which mutually strongly anticommute. We remark that the anticommutativity of self-adjoint operators restricts strongly themselves. Hence this property is very useful (see, [11, 7, 1, 2], and references therein). Therefore, it is very interesting to investigate the properties of $\mathbb{D}$ which are derived from the anticommutativity.

The plan of this paper is the following. In Section 2, we consider the constant magnetic field case. We prove that $\mathbb{D}$ is a sum of mutually strongly anticommuting self-adjoint operators. Using this, we identify the spectrum and essential spectrum of $\mathbb{D}$ and $\tilde{H}$. In Section 3, we consider perturbations of $\mathbb{D}$ and $\tilde{H}$. We define a new class of vector potentials $\mathbf{a}$, each in which implies the same essential spectrum for $\tilde{H}$ as in the constant magnetic field case (Theorem 3.2). This class contains vector potentials to which Theorem 1.2 cannot be applied. (See Example 3.4 in Section 3).

2 Constant magnetic field case

In this section, we investigate the inner structure and the spectrum of $\mathbb{D}$ and $\tilde{H}$ with a constant magnetic field. We recall a definition of the anticommutativity of self-adjoint operators: Two (non-zero) self-adjoint operators A and B in a Hilbert space are said to *strongly anticommute* if

$$\exp(itA)B \subset B\exp(-itA)$$

for all $t \in \mathbb{R}$ (see, [11, 7, 1]).

First of all, we prove a proposition and a lemma.

Proposition 2.1 *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with a grading operator γ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, and A and γ strongly anticommute. Let B be a self-adjoint operator in a Hilbert space $\mathcal{K}$. Then $A \otimes 1$ and $\gamma \otimes B$ are self-adjoint in $\mathcal{H} \otimes \mathcal{K}$ and strongly anticommute.*

Proof: The self-adjointness is followed from general theory on tensor product of self-adjoint operators [8]. The strongly anticommutativity follows from an application of Corollary 4.5 in [7]. □

We shall use the following lemma to answer the question, "Why does whether 0 is an eigenvalue of Λ or not cause the difference of $\sigma_{ess}(\tilde{H})$ in each case?"

Lemma 2.2 *Let A and B be as in Proposition 2.1. Assume that there exists a unitary operator U on $\mathcal{K}$ such that $U^*BU = -B$,*

$$\sigma(B) = \mathbb{R}, \tag{4}$$

and

$$\sigma_p(B) = \emptyset, \tag{5}$$

where $\sigma_p(\cdot)$ denotes point spectrum. Let

$$T = A \otimes 1 + \gamma \otimes B \quad \text{with} \quad D(T) = D(A \otimes 1) \cap D(\gamma \otimes B),$$

where $D(T)$ denotes the domain of the operator T. Then T is self-adjoint, $\sigma_p(T) = \emptyset$ and

$$\sigma(T) = (-\infty, -\delta] \cup [\delta, \infty) \quad \text{with} \quad \delta = \inf\{|x|;\ x \in \sigma(A)\}. \tag{6}$$

Proof: The self-adjointness of T follows from Lemma 2.1 in [1] and Proposition 2.1. By Lemma 2.4 in [1],

$$T^2 = A^2 \otimes 1 + 1 \otimes B^2$$

holds as operator equality. Thus, we have $\sigma_p(T^2) = \emptyset$ by (5) and

$$\sigma(T^2) = \overline{\{a + b;\ a \in \sigma(A^2),\ b \in \sigma(B^2)\}} = [\inf \sigma(A^2), \infty)$$

by (4). Since $(\gamma \otimes U)^* T (\gamma \otimes U) = -T$ and $\gamma \otimes U$ is unitary, we have $\sigma(T) = \sigma(-T)$. Therefore, we obtain (6). $\square$

In this section, we deal with the constant magnetic field case. Hence, assume that

$$b_{jk}(x) = \Lambda_{jk} \quad \text{for} \quad j, k = 1, \ldots, d, \tag{7}$$

with a constant matrix $\Lambda = (\Lambda_{jk})$. By an orthogonal transformation, we assume that Λ is of the form

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & \mathbf{0} & \\ & & \ddots & & & & & \\ & & & 0 & \lambda_n & & & \\ & & & -\lambda_n & 0 & & & \\ & & & & & 0 & & \\ & \mathbf{0} & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix},$$

where $\lambda_j > 0$, $j = 1, \ldots, n$. Moreover, we can take a vector potential $\mathbf{a}$ as follows:

$$a_{2j-1}(x) = \frac{\lambda_j}{2} x^{2j}, \quad a_{2j}(x) = -\frac{\lambda_j}{2} x^{2j-1} \quad \text{for} \quad j = 1, \ldots, n, \tag{8}$$

$$a_j(x) = 0 \quad \text{for} \quad j = 2n+1, \ldots, d.$$

We prove that $\mathbb{D}$ is a sum of operators which mutually anticommute. Let

$$\hat{d}_j = \sigma^1(-i\partial_{2j-1} + a_{2j-1}) + \sigma^2(-i\partial_{2j} + a_{2j})$$

acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$ for $j = 1, \ldots, [d/2]$. Since a_{2j-1} and a_{2j} contain only the variables x^{2j-1} and x^{2j}, these operators are well-defined. Moreover, $\hat{d}_j$ are essentially self-adjoint on the domain $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$. We denote the closure of $\hat{d}_j$ by the same symbol. We can easily check the following proposition.

Proposition 2.3 *For each $j = 1, \ldots, [d/2]$, the operators σ^3 and $\hat{d}_j$ strongly anticommute.*

Using $\hat{d}_j$, we construct self-adjoint operators D_j whose sum is $\mathbb{D}$ in each cases where $d = 2m$ and $d = 2m+1$. First, consider the case where $d = 2m$. For $j = 1, \ldots, m$, define

$$D_j = \underbrace{1 \otimes \cdots \otimes 1}_{j-1 \text{ times}} \otimes \hat{d}_j \otimes \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{m-j \text{ times}}$$

acting in $\otimes^m L^2(\mathbb{R}^2; \mathbb{C}^2) \simeq L^2(\mathbb{R}^{2m}; \mathbb{C}^r)$, $r = 2^m$. In the case $d = 2m+1$, define

$$D_j = \underbrace{1 \otimes \cdots \otimes 1}_{j-1 \text{ times}} \otimes \hat{d}_j \otimes \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{m-j \text{ times}} \otimes 1$$

for $j = 1, \ldots, m$, and

$$D_{m+1} = \underbrace{\sigma^3 \otimes \cdots \otimes \sigma^3}_{m \text{ times}} \otimes (-i\partial_{2m+1})$$

acting in $\otimes^m L^2(\mathbb{R}^2; \mathbb{C}^2) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}^{2m+1}; \mathbb{C}^r)$, $r = 2^m$. Since $\hat{d}_j$ are self-adjoint, each D_j is self-adjoint. If $d = 2m$, we define a grading operator Γ_{2m} by

$$\Gamma_{2m} = \otimes^m \sigma^3$$

acting in $\otimes^m L^2(\mathbb{R}^2; \mathbb{C}^2)$. Then Γ_{2m} and D_j strongly anticommute for $j = 1, \ldots, m$. We remark that $D_{m+1} = \Gamma_{2m} \otimes (-i\partial_{2m+1})$.

Lemma 2.4 *The operators D_j mutually strongly anticommute.*

Proof: By Propositions 2.1 and 2.3, $\hat{d}_1 \otimes \sigma^3$ and $1 \otimes \hat{d}_2$ strongly anticommute. Since the other components in D_1 and D_2 strongly commute, we can prove that D_1 and D_2 strongly anticommute with limit argument. In the same way, we can see that all $D_1, D_2, \ldots, D_m$ strongly anticommute.

In the case $d = 2m+1$, let $A = D_j$, $\gamma = \Gamma_{2m}$, $\mathcal{H} = \otimes^m L^2(\mathbb{R}^2;\mathbb{C}^2)$, $B = -i\partial_{2m+1}$ and $\mathcal{K} = L^2(\mathbb{R})$. Then, by Proposition 2.1, we obtain the desired results. □

The followings are the main theorems in this paper.

Theorem 2.5 *Assume (7). Let* $k = [(d+1)/2]$. *Then*

$$\mathbb{D} = D_1 + D_2 + \cdots + D_k \quad \text{with} \quad D(\mathbb{D}) = \bigcap_{j=1}^{k} D(D_j), \tag{9}$$

$$\tilde{H} = \mathbb{D}^2 = D_1^2 + D_2^2 + \cdots + D_k^2 \quad \text{with} \quad D(\tilde{H}) = \bigcap_{j=1}^{k} D(D_j^2), \tag{10}$$

hold as operator equality.

Remark: In Theorem 2.5, we take a representation of Dirac matrices γ^j as follows: $\gamma^1 = \sigma^1 \otimes [\otimes^{m-1}\sigma^3]$, $\gamma^2 = \sigma^2 \otimes [\otimes^{m-1}\sigma^3]$, $\gamma^3 = 1 \otimes \sigma^1 \otimes [\otimes^{m-2}\sigma^3]$, $\gamma^4 = 1 \otimes \sigma^2 \otimes [\otimes^{m-2}\sigma^3]$, and so on.

Proof: By direct computations, (9) holds on $C_0^\infty(\mathbb{R}^d;\mathbb{C}^r)$, $r = 2^{[d/2]}$. Since $\mathbb{D}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d;\mathbb{C}^r)$, by Lemma 2.1 in [1] and Lemma 2.4 we obtain (9) as operator equality. Moreover, by Lemma 2.4 and Lemma 2.4 in [1] we obtain (10) as operator equality. □

By this theorem and Theorem 1.1, we can obtain the following spectral properties of $\mathbb{D}$ and $\tilde{H}$.

Theorem 2.6 *Assume (7).*

(i) Assume that 0 *is an eigenvalue of* Λ. *Then*

$$\sigma(\tilde{H}) = \sigma_{ess}(\tilde{H}) = [0,\infty), \quad \sigma_p(\tilde{H}) = \emptyset,$$
$$\sigma(\mathbb{D}) = \sigma_{ess}(\mathbb{D}) = \mathbb{R}, \quad \sigma_p(\mathbb{D}) = \emptyset.$$

(ii) Assume that 0 *is not an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_m$, $(\lambda_j > 0,\ m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma(\tilde{H}) = \sigma_{ess}(\tilde{H}) = \{\sum_{j=1}^{m} 2k_j\lambda_j;\ k_j \in \mathbb{Z}_+\}, \tag{11}$$

$$\sigma(\mathbb{D}) = \sigma_{ess}(\mathbb{D}) = \{\pm\sqrt{\alpha};\ \alpha \in \sigma(\tilde{H})\}. \tag{12}$$

Moreover, we have

$$\ker \mathbb{D} \subset \ker(\sigma^3 + 1) \otimes \cdots \otimes \ker(\sigma^3 + 1). \tag{13}$$

Proof: First, we prove the part (ii). By Theorem 2.5, we can rewrite $\tilde{H}$ as

$$\tilde{H} = \hat{d}_1^2 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-1 \text{ times}} + 1 \otimes \hat{d}_2^2 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-2 \text{ times}} + \cdots + \underbrace{1 \otimes \cdots \otimes 1}_{m-1 \text{ times}} \otimes \hat{d}_m^2.$$

Therefore, we have

$$\sigma(\tilde{H}) = \overline{\{\alpha_1 + \cdots + \alpha_m;\ \alpha_j \in \sigma(\hat{d}_j^2),\ j = 1, \ldots, m\}},$$

$$\sigma_{ess}(\tilde{H}) = \overline{\{\alpha_1 + \cdots + \alpha_m;\ \alpha_j \in \sigma_{ess}(\hat{d}_j^2),\ j = 1, \ldots, m\}}.$$

Since $\sigma(\hat{d}_j^2) = \sigma_{ess}(\hat{d}_j^2) = \{2n_j\lambda_j;\ n_j \in \mathbb{Z}_+\}$ by Theorem 1.1, we have (11). By the supersymmetry with the grading operator Γ_d, we obtain (12) (see, Proposition 2.5 in [9]). By the self-adjointness, we have

$$\ker \mathbb{D} = \ker \tilde{H} = \ker \hat{d}_1 \otimes \cdots \otimes \ker \hat{d}_m.$$

Thus, we obtain (13) by Theorem 1.1.

We prove the part (i). Decompose $\mathbb{D}$ into two operators as follows. Let A be the Dirac operator in the case where $d = 2n$, a vector potential $\mathbf{a} = \sum_{j=1}^{2n} a_j(x)dx^j$ with a_j in (8), and a grading operator $\gamma = \Gamma_{2n}$ on $\mathcal{H} = \otimes^n L^2(\mathbb{R}^2; \mathbb{C}^2)$. Let B be the $(d-2n)$-dimensional Dirac operator with $\mathbf{a} = 0$ in $\mathcal{K} = L^2(\mathbb{R}^{d-2n}; \mathbb{C}^r)$, $r = 2^{[(d-2n)/2]}$. Then, we have $\mathbb{D} = A \otimes 1 + \gamma \otimes B$. Moreover, let U be a unitary operator on $\mathcal{K}$ by

$$(Uf)(x) = f(-x) \quad \text{for} \quad f \in \mathcal{K},\ x \in \mathbb{R}^{d-2n}.$$

Then, the set $\{A, \gamma, \mathcal{H}, B, U, \mathcal{K}\}$ satisfies the assumptions in Lemma 2.2. With the part (ii), we obtain the desired results. □

In the rest of this section, we consider a *spinless* case. We can rewrite H as follows: Let

$$\hat{h}_j = (-i\partial_{2j-1} - a_{2j-1})^2 + (-i\partial_{2j} - a_{2j})^2$$

in $L^2(\mathbb{R}^2)$ for $j = 1, \ldots, m$. If $d = 2m+1$, let

$$\hat{h}_{m+1} = (-i\partial_{2m+1})^2$$

acting in $L^2(\mathbb{R})$. Then,

$$H = \overline{\sum_{j=1}^{k} [\otimes^{j-1} 1] \otimes \hat{h}_j \otimes [\otimes^{k-j} 1]},$$

in $L^2(\mathbb{R}^d)$, $k = [(d+1)/2]$. Thus, we can prove the following theorem.

Theorem 2.7 *Assume (7).*

(i) Assume that 0 is an eigenvalue of Λ. Let $\pm i\lambda_1, \ldots, \pm i\lambda_n$, $(\lambda_j > 0)$, be eigenvalues of Λ. Then

$$\sigma(H) = \sigma_{ess}(H) = [\sum_{j=1}^{n} \lambda_j, \infty), \quad \sigma_p(H) = \emptyset.$$

(ii) Assume that 0 is not an eigenvalue of Λ. Let $\pm i\lambda_1, \ldots, \pm i\lambda_m$, $(\lambda_j > 0$, $m = d/2)$ be eigenvalues of Λ. Then

$$\sigma(H) = \sigma_{ess}(H) = \{\sum_{j=1}^{m} (2k_j + 1)\lambda_j;\ k_j \in \mathbb{Z}_+\}.$$

Proof: This theorem follows from general theory of tensor product of self-adjoint operators and Theorem 1.1. □

We can find far more discussions on this H in [6].

3 Perturbation

In this section, we consider perturbations of $\mathbb{D}_0$ which is the Dirac operator with a constant magnetic field considered in the previous section. Though Shigekawa proved Theorem 1.2 under conditions on the asymptotic behavior of a magnetic field $\mathbf{b} = d\mathbf{a}$ as (2), we shall give assumptions on the asymptotic behavior of a vector potential $\mathbf{a}$, up to gauge transformation. One of the reasons is that we investigate $\mathbb{D}$ instead of $\tilde{H}$ itself and $\mathbb{D}$ contains explicitly $\mathbf{a}$ and no $\mathbf{b}$. Therefore, this seems natural at least from the mathematical point of view. We will give a theorem with assumptions on $\mathbf{b}$, too.

We start with the following abstract lemma.

Lemma 3.1 *Assume that $\mathbf{a}_0 = \sum_{j=1}^{d} a_{0j}\, dx^j$ and $\mathbf{a} = \sum_{j=1}^{d} a_j\, dx^j$ are real 1-forms such that a_{0j} and a_j are in C^∞, $\mathbf{b}_0 = d\mathbf{a}_0$ is a bounded 2-form and $\mathbf{a} \to 0$ as $|x| \to \infty$ (i.e., $a_j(x) \to 0$ as $|x| \to \infty$ for all j). Let*

$$\mathbb{D} = \sum_{j=1}^{d} \gamma^j(-i\partial_j + a_{0j}(x)) \quad \text{and} \quad \mathbb{A} = \sum_{j=1}^{d} \gamma^j a_j(x)$$

acting in $L^2(\mathbb{R}^d; \mathbb{C}^r)$, $r = 2^{[d/2]}$. Then $\mathbb{A}$ is $\mathbb{D}$-compact.

Proof: Let $\mathbf{b}_0 = d\mathbf{a}_0 = \sum_{j<k} b_{0jk}\, dx^j \wedge dx^k$ with $b_{0jk} = \partial_j a_{0k} - \partial_k a_{0j}$. Then

$$\mathbb{D}^2 = H + \sum_{j<k} i\, b_{0jk}\gamma^j\gamma^k \quad \text{with} \quad H = \sum_{j=1}^{d} (-i\partial_j + a_{0j})^2.$$

Since $a_j(x) \to 0$ as $|x| \to \infty$, a_j is $-\Delta = -(\sum_{j=1}^d \partial_j^2)$-compact. Thus, a_j is H-compact by Lemma 2.3 in [3]. Since $\sum_{j<k} i\, b_{0jk}\gamma^j\gamma^k$ is bounded, a_j is $\mathbb{D}^2$-compact and thus $\mathbb{A}$ is so. Since $\mathbb{A}$ is $\mathbb{D}$-bounded with $\mathbb{D}$-bound 0, $\mathbb{A}$ is $\mathbb{D}$-compact by Theorem 9.11 in [12]. □

The following is the main theorem in this section.

Theorem 3.2 *Assume that the given vector potential* $\mathbf{A}$ *can be rewritten as the sum of 1-forms* $\mathbf{a}_0$ *and* $\mathbf{a}$ *such that* $d\mathbf{a}_0$ *is a constant magnetic field and* $\mathbf{a}$ *tends to* 0 *as* $|x| \to \infty$. *Define* $\mathbb{D}$ *and* $\tilde{H}$ *as the Dirac and Schrödinger operators with* $\mathbf{A}$, *respectively. Put* Λ *for* $d\mathbf{a}_0$ *as same as (7).*

(i) Assume that 0 *is an eigenvalue of* Λ. *Then*

$$\sigma_{ess}(\tilde{H}) = [0, \infty), \quad \sigma_{ess}(\mathbb{D}) = \mathbb{R}.$$

(ii) Assume that 0 *is not an eigenvalue of* Λ. *Let* $\pm i\lambda_1, \ldots, \pm i\lambda_m$, $(\lambda_j > 0,\ m = d/2)$ *be eigenvalues of* Λ. *Then*

$$\sigma_{ess}(\tilde{H}) = \{\sum_{j=1}^m 2k_j\lambda_j;\ k_j \in \mathbb{Z}_+\},$$

$$\sigma_{ess}(\mathbb{D}) = \{\pm\sqrt{\alpha};\ \alpha \in \sigma(\tilde{H})\}.$$

Proof: Define $\mathbb{D}_0$ as the Dirac operator with $\mathbf{a}_0$. Since $a_j(x) \to 0$ as $|x| \to \infty$, $\mathbb{D} - \mathbb{D}_0$ is $\mathbb{D}_0$-compact by Lemma 3.1. Therefore, by Theorem 2.6, we obtain the desired results. □

The following theorem gives a condition on magnetic field $\mathbf{b}$ which implies the same essential spectra of $\mathbb{D}$ and $\tilde{H}$ as in Theorem 3.2.

Theorem 3.3 *Assume that* $\mathbf{b} = \sum_{j<k} b_{jk}\, dx^j \wedge dx^k$ *is a real* C^∞ *2-form such that for any* j *and* k,

$$|x|(b_{jk}(x) - \Lambda_{jk}) \to 0 \quad as \quad |x| \to \infty$$

with a constant matrix $\Lambda = (\Lambda_{jk})$. *Then, the statements (i) and (ii) in Theorem 3.2 hold.*

Proof: For the 2-form $\tilde{\mathbf{b}} = \sum_{j<k}(b_{jk} - \Lambda_{jk})dx^j \wedge dx^k$ we can choose a 1-form $\mathbf{a}$ such that $\tilde{\mathbf{b}} = d\mathbf{a}$ and $\mathbf{a} \to 0$ as $|x| \to \infty$ by taking

$$\mathbf{a}(x) = \sum_{j<k} \int_0^1 t(b_{jk} - \Lambda_{jk})(tx)dt\,(x^j dx^k - x^k dx^j)$$

as in the proof of Poincaré's Lemma. Since the 2-form $\sum_{j<k} \Lambda_{jk}\, dx^j \wedge dx^k$ is a constant magnetic field, by Theorem 3.2 we obtain the desired results. □

Of course, above Theorem 3.3 is weaker than Theorem 1.2. However, Theorem 3.2 is not weaker than Theorem 1.2 as we see in the following example.

Example 3.4 Let $d = 2$ and $\mathbf{a} = a_1\,dx^1 + a_2\,dx^2$ be a C^∞ 1-form such that

$$a_1(x) = \frac{\lambda}{2}x^2 + \frac{\sin|x^2|^r}{|x|} \quad \text{and} \quad a_2(x) = -\frac{\lambda}{2}x^1$$

near $|x| = \infty$ with a constant $r > 2$ and a positive constant λ. Then, $\mathbf{a}$ satisfies the assumptions in Theorem 3.2. Therefore, we have $\sigma_{ess}(\tilde{H}) = \{2n\lambda;\ n \in \mathbb{Z}_+\}$. However, $\mathbf{b} = d\mathbf{a}$ does not converge as $|x| \to \infty$. Therefore, we can not apply Theorem 1.2.

We remark on perturbations of the spinless Schrödinger operator H. Assume that magnetic field $\mathbf{b}$ satisfies the conditions in Theorem 3.3. Then, with the aid of general theory of perturbations of differential operators (see, e.g., [12]) and using the vector potential $\mathbf{a}$ in the proof of Theorem 3.3 we can prove that the perturbed H has the same essential spectrum of the unperturbed H as in Theorem 2.7. However, this result is evidently weaker than the Shigekawa's results in [9]. This difference is due to the difference between the unperturbed operators taken in each proof.

Acknowledgements

The author would like to thank Professor Asao Arai for helpful discussions and encouragements.

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