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The Spectra Of Toeplitz Operators With Unimodular Symbols

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Abstract. The spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ on the open unit disc D for a unimodular symbol ϕ is studied. Exactly, many sufficient conditions for $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$ are given. If ϕ is a unimodular function in $H^\infty + C$, then $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$.

§1. Introduction

Let L^p be the Lebesgue space on the unit circle ∂D and let H^p be the corresponding Hardy space for $0 < p \leq \infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi x = P(\phi x)$ for x in H^2, where P is the orthogonal projection of L^2 onto H^2.

In this paper we study the spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ. It is known that $\sigma(T_\phi)$ is always connected. This is a hard and deep result due to H.Widom (cf.[2, Corollary 7.46]). If ϕ is a continuous function on ∂D, $\sigma(T_\phi)$ consists of the range of ϕ together with those points not in the range of ϕ that have a nonzero index with respect to ϕ (cf.[2, Corollary 7.28]). If ϕ is a real-valued function in L^∞, $\sigma(T_\phi) =$ [ess inf ϕ, ess sup ϕ] (cf.[2, Theorem 7.20]) and if ϕ is a function in H^∞, $\sigma(T_\phi) =$ the closure of $\phi(D)$ (cf.[2, Theorem 7.21]). In particular, we are interested in the spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ when ϕ is a unimodular function in L^∞. M.Lee and D.Sarason [6], and R.G.Douglas and D.Sarason [3] have considered $\sigma(T_\phi)$ when ϕ is a quotient of two inner functions. Under some conditions, they showed that $\sigma(T_\phi) = \bar{D}$ [6]. In this paper, we consider such a problem when ϕ is an arbitrary unimodular function. Theorem 1 in [6] is a corollary of (2) of Theorem 2 in this paper. For a real-valued function s in L^∞, $\tilde{s}$ denotes the harmonic conjugate with $\tilde{s}(0) = 0$. Our main tool is the following theorem [1].

Widom and Devinatz's Theorem. Let ϕ be a unimodular function in L^∞. Then the following (1) $\sim$ (3) are equivalent.

(1) T_ϕ is invertible.

(2) ϕ has the form : $\phi = e^{it}$ where t is a real-valued function in L^1 such that $\inf\{\|t - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$.

(3) ϕ has the form : $\phi = g_1 g_2/|g_1 g_2|$ where both g_1 and g_1^{-1} are in H^∞, and both g_2 and g_2^{-1} are in $\bigcup_{p>1} H^p$ with $Re\ g_2$ bounded away from 0 on ∂D.

In this paper, we give sufficient conditions for $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$, using $\inf \|t - \tilde{s} - a\|_\infty$ in Section 1 and $g/|g|$ in Section 2. Throughout this paper, for a function space X on ∂D, we let $X_R = \{Ref \ ; \ f \in X\}$, where Ref is a real part of f. C denotes a set of all continuous functions on ∂D and so C_R is a set of all real valued continuous functions on ∂D.

§2. Sufficient conditions using $\inf \|t - \tilde{s} - a\|_\infty$.

Lemma 1. Let ϕ be unimodular in L^∞ and $\lambda = a + ib$ in D. Then $\lambda \notin \sigma(T_\phi)$ if and only if ϕ has the form $\phi = e^{it}$ where t is a real-valued function in L^1 such that

$$\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$$

and $v_\lambda = \arctan\{(a \sin t - b \cos t)/(1 - (a \cos t + b \sin t))\}$.

Proof. We will first show the 'if' part. There exists a function s_λ in L^∞ such that $(1 - \lambda\bar{\phi})/|1 - \lambda\bar{\phi}| = e^{is_\lambda}$ and $\|s_\lambda\|_\infty < \pi/2$ because $|\lambda| < 1$. Then

$$\frac{1 - (a \cos t + b \sin t)}{|1 - \lambda\bar{\phi}|} + i\frac{a \sin t - b \cos t}{|1 - \lambda\bar{\phi}|} = \cos s_\lambda + i \sin s_\lambda.$$

Since $|a \cos t + b \sin t| \le |\lambda| < 1$, $\|v_\lambda\|_\infty < \pi/2$. Hence $\|v_\lambda - s_\lambda\|_\infty < \pi$ and $\tan v_\lambda = \tan s_\lambda$ *a.e.* and so $v_\lambda = s_\lambda$ *a.e.*. Therefore

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \phi\frac{1 - \lambda\bar{\phi}}{|1 - \lambda\bar{\phi}|} = e^{it}e^{iv_\lambda}$$

and by Widom and Devinatz's Theorem in the Introduction $T_{\phi-\lambda}$ is invertible because $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$. Conversely if $\lambda \notin \sigma(T_\phi)$, by Widom and Devinatz's Theorem there exists a real-valued function t_λ such that $(\phi - \lambda)/|\phi - \lambda| = e^{it_\lambda}$ and $\inf\{\|t_\lambda - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$. As in the proof of the 'if' part, there exists s_λ such that $(1 - \lambda\bar{\phi})/|1 - \lambda\bar{\phi}| = e^{is_\lambda}$, Moreover $\phi = e^{it}$ and $s_\lambda = v_\lambda$ if $t = t_\lambda - s_\lambda$. This implies the 'only if' part.

Theorem 1. Let ϕ be a unimodular function in L^∞.

(1) If $\phi = e^{it}$ and t is a real-valued function in L^1 such that $\inf\{\|t - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} = 0$, then $\sigma(T_\phi) \subseteq \partial D$.

(2) If $\inf\{\|t - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} \ge \pi$ for any $t \in L^1_R$ with $\phi = e^{it}$, then $\sigma(T_\phi) = \bar{D}$.

(3) If $\sigma(T_\phi) = \bar{D}$, then $\inf\{\|t - a\|_\infty \ ; \ a \in R\} \ge \pi$ for any $t \in L^1_R$ with $\phi = e^{it}$.

Proof. (1) If $\lambda = a + ib \in D$ and $v_\lambda = \arctan\{(a \sin t - b \cos t)/1 - (a \cos t + b \sin t)\}$, then $\|v_\lambda\|_\infty < \pi/2$ and hence $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$ because $\inf\{\|t - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} = 0$. By Lemma 1, $\lambda \notin \sigma(T_\phi)$ and hence $\sigma(T_\phi) \subseteq \partial D$. (2) If $\lambda \in D$ and $\lambda \notin \sigma(T_\phi)$, then by Lemma 1 $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$. Since $\|v_\lambda\|_\infty < \pi/2$, $\inf\{\|t - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R \text{ and } a \in R\} < \pi$. This implies (2). (3) is a result of a theorem of A.Brown and R.R.Halmos (*cf.*[2, Corollary 7.19]).

Corollary 1. Suppose $\phi = e^{it}$ and t is a real-valued function which satisfies one of the following (i) $\sim$ (iii), then $\sigma(T_\phi) \subseteq \partial D$.

(i) $\quad t = \tilde{u} + v$ where $u \in L^\infty_R$ and $v \in C_R$.

(ii) $\quad t = \tilde{u} + v$ where $u \in L^\infty_R$ and v is in the norm closure of H^∞_R.

(iii) $\quad t = \tilde{u} + v$ where $u \in L^\infty_R$ and $v = s \circ q$ for $s \in C_R$ and an inner function q.

Proof. If $v \in C_R$, then v is in the norm closure of H^∞_R and so (i) is a result of (ii). If $v \in H^\infty_R$, then $v = \tilde{s} + a$ for $s \in H^\infty_R$ and $a \in R$, and hence a simple computation implies (ii). If s is a real-valued polynomial of z and $\bar{z}$, then $v = s \circ q$ belongs to H^∞_R for an inner function q. Thus (iii) is a result of (ii).

Corollary 2. Let Q_j be a non-constant inner function, $a_j \in D$ and $b_j \in D$ for $1 \leq j \leq \max(n, m)$. Suppose $\phi = \bar{q}_1 q_2$ where $q_1 = \prod_{j=1}^{n}(Q_j - a_j)/(1 - \bar{a}_j Q_j)$ and $q_2 = \prod_{j=1}^{m}(Q_j - b_j)/(1 - \bar{b}_j Q_j)$. Then $\sigma(T_\phi) \subseteq \partial D$ if and only if $n = m$.

Proof. If $n = m$, put $u = 2\sum_{j=1}^{n} \log|(1 - \bar{a}_j Q_j)/(1 - \bar{b}_j Q_j)|$, then $u \in L^\infty_R$ and $\phi = \bar{q}_1 q_2 = \alpha e^{i\tilde{u}}$ for some constant α. (1) of Theorem 1 implies the corollary. Suppose $\sigma(T_\phi) \subseteq \partial D$. If $n > m$, then $\phi = \bar{q}_1 q_2 = \phi_1 \phi_2$ where $\phi_1 = \prod_{j=m+1}^{n} (1 - \bar{a}_j Q_j / Q_j - a_j), \phi_2 = \alpha e^{i\tilde{u}}$, α is a constant and $u = 2\sum_{j=1}^{m} \log|(1 - \bar{a}_j Q_j)/(1 - \bar{b}_j Q_j)|$. Therefore $T_\phi = T_{\phi_1} T_{\phi_2}$, and both T_ϕ and T_{ϕ_2} are invertible. This contradicts that T_{ϕ_1} is not invertible.

§3. Sufficient conditions using $g/|g|$ for g in H^p

Theorem 2. Let ϕ be a unimodular function in L^∞.

(1) If $\phi = g/|g|$ where both g and g^{-1} are in H^∞, then $\sigma(T_\phi) \subseteq \partial D$.

(2) If $\phi \neq g/|g|$ for any g in $\bigcup_{p>1/2} H^p$ whose inverse is in $\bigcup_{p>1/2} H^p$, then $\sigma(T_\phi) = \bar{D}$.

Proof. (1) This is a corollary of (1) of Ccrollary 1. But we will give another proof. If $\phi = g/|g|$ where both g and g^{-1} are in H^∞, put $h = g^{1/2}$, then $\phi = h/\bar{h}$ and both h and h^{-1} are in H^∞. For any $\lambda \in D, \phi - \lambda = (1/\bar{h})(1 - \lambda\bar{h}/h)h$ and hence

$$T_{\phi-\lambda} = T_{(1/\bar{h})} T_{(1-\lambda\bar{h}/h)} T_h.$$

This implies that $T_{\phi-\lambda}$ is invertible by Widom and Devinatz's Theorem. (2) For any $\lambda \in D$, $1 - \lambda\bar{\phi} = \phi_0 \ell$ where $|\phi_0| = 1$ *a.e.*, and both ℓ and ℓ^{-1} are in H^∞. Hence

$$\phi - \lambda = \phi(1 - \lambda\bar{\phi}) = \phi\phi_0\ell \text{ and } \bar{\phi}_0 - \ell = \lambda\bar{\phi}_0\bar{\phi}.$$

Since $\|\bar{\phi}_0 - \ell\|_\infty = |\lambda| < 1$, by Widom and Devinatz's Theorem $T_{\bar{\phi}_0}$ is invertible and $\bar{\phi}_0 = h/|h|$ for some $h \in H^a$ and $a > 1$. If $T_{\phi-\lambda}$ is invertible, then $T_{\phi\phi_0}$ is invertible and hence $\phi\phi_0 = k/|k|$ for some $k \in H^b$ and $b > 1$. Thefore $\phi = \bar{\phi}_0\phi\phi_0 = hk/|hk|$ and both hk and $(hk)^{-1}$ belong to H^p for some $p > 1/2$. This implies (2).

Corollary 3. If $\phi = g/|g|$ where $g \in \bigcap_{p<\infty} H^p$ and $g^{-1} \notin \bigcap_{p>1/2} H^p$, then $\sigma(T_\phi) = \bar{D}$.

Proof. If $\phi = h/|h|$ for some h in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$, then $\phi = |k|/k$ with $k = 1/h$. Hence kg is nonegative *a.e.* on ∂D and $kg \in H^{1/2}$. By [7], $g = ch$ for some positive constant c and $g^{-1} \in \bigcap_{p>1/2} H^p$. Now (2) of Theorem 2 implies the corollary.

Corollary 4. Let Q_j be a non-constant inner function, $a_j \in D$ and $b_j \in D$ for $1 \le j \le \max(n,m)$. Suppose $\phi = \bar{q}_1 q_2$ where $q_1 = \prod_{j=1}^{n}(Q_j - a_j)/(1 - \bar{a}_j Q_j)$ and $q_2 = \prod_{j=1}^{m}(Q_j - b_j)/(1 - \bar{b}_j Q_j)$. Then $\sigma(T_\phi) = \bar{D}$ if and only if $n \neq m$.

Proof. By Corollary 2, it is enough to show the 'if' part. If $n > m$, then by the proof of Corollary 2 $\phi = \phi_1\phi_2$ and so $\phi = \phi_1(g/|g|)$ where both g and g^{-1} are in H^∞, and ϕ_1 is a nonconstant inner function. If $\phi = h/|h|$ for some h in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$, $\phi_1 g h^{-1}$ is a non-negative function in $H^{1/2}$. By [7], this contradicts that ϕ_1 is nonconstant. Thus (2) of Theorem 2 implies that $\sigma(T_\phi) = \bar{D}$. When $n < m$, by a similar method we can show that $\sigma(T_\phi) = \bar{D}$.

Now using (2) of Theorem 2, we will give a proof of Theorem 1 in [6]. For each inner function q, sing q denotes the subset of ∂D on which q can not be analytically extended.

Corollary 5 ([6]). If $\phi = \bar{q}_1 q_2$ where q_1 and q_2 are inner functions with sing $q_1 \neq$ sing q_2, then $\sigma(T_\phi) = \bar{D}$.

Proof. By (2) of Theorem 2, it is enough to show that $\phi = \bar{q}_1 q_2 \neq g/|g|$ for any

g in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$. We may assume that sing $q_1 \not\ni z_0 \in$ sing q_2. There exists a constant $\lambda \in D$ such that $q = (q_2 - \lambda)/(1 - \bar{\lambda} q_2)$ is a Blaschke product with sing $q =$ sing q_2 by [5, p176]. Then $\bar{q}_1 q_2 = \bar{q}_1 q k/|k|$ where $k = (1 - \bar{\lambda} q_2)^2$. Since both k and k^{-1} are in H^∞, we may assume that q_2 is a Blaschke product. If $\bar{q}_1 q_2 = f/|f|$ and $q_1 \bar{q}_2 = g/|g|$ where $fg = 1 \quad a.e.$, $f \in H^{1/2}$ and $g \in H^{1/2}$, then $\bar{q}_1 q_2 g \geq 0 \quad a.e.$ and $\bar{q}_2 q_1 f \geq 0 \quad a.e.$. Since $\bar{q}_1 q_2 g \geq 0 \quad a.e.$, $g \in H^{1/2}$ and $z_0 \notin$ sing q_1, by [4] there exists an open arc J such that $z_0 \in J$ and $q_2 g$ can be continued analytically from D across J. The zeros of q_2 cannot cluster at any point of J. This contradicts that $z_0 \in$ sing q_2. Thus $\bar{q}_1 q_2$ satisfies the condition of (2) of Theorem 2. and hence $\sigma(T_\phi) = \bar{D}$.

Corollary 6. Let q_1 and q_2 be inner functions, and χ_E be a characteristic function of a measurable set E in ∂D. If $\phi = \bar{q}_1 q_2 (2\chi_E - 1)$ and there exists an open arc J in E such that (sing $q_2) \cap J \neq \emptyset$ and (sing $q_1) \cap J = \emptyset$, or (sing $q_1) \cap J \neq \emptyset$ and (sing $q_2) \cap J = \emptyset$, then $\sigma(T_\phi) = \bar{D}$.

Proof. As in Corollary 5, we may assume that q_2 is a Blaschke product. If $\phi = \bar{q}_1 q_2 (2\chi_E - 1) = f/|f| = |g|/g$ where $fg = 1 \quad a.e.$, $f \in H^{1/2}$ and $g \in H^{1/2}$, then $\bar{q}_1 q_2 (2\chi_E - 1) g \geq 0 \quad a.e.$ and $\bar{q}_2 q_1 (2\chi_E - 1) f \geq 0 \quad a.e.$. If there exists an open arc J in E such that (sing $q_2) \cap J \neq \emptyset$ and (sing $q_1) \cap J = \emptyset$, then

$$\bar{q}_1 q_2 (2\chi_E - 1) g = \bar{q}_1 q_2 g \geq 0 \quad a.e. \text{ on } J.$$

Now as in Corollary 5, we can get a contradiction and hence $\sigma(T_\phi) = \bar{D}$.

Let $q_a = \exp\{-a(1+z)/(1-z)\}$ for $a > 0$ and suppose b is a Blaschke product with sing $b = \{1\}$. Put $\phi_a = \bar{q}_a b$. Theorem 4 in [6] shows that if ϕ_a belongs to $H^\infty + C$ for all $a > 0$, then $\sigma(T_{\phi_a}) = \bar{D}$. This is a corollary of Corollary 7.

Corollary 7. If ϕ_a belongs to $H^\infty + C$ for some $a > 0$, then $\sigma(T_{\phi_c}) = \bar{D}$ for $0 < c < a$. If T_{ϕ_a} is invertible or $\sigma(T_{\phi_a}) \subseteq \partial D$, then $\sigma(T_{\phi_c}) = \bar{D}$ for arbitrary $c > 0$ with $c \neq a$.

Proof. By Theorem 2 in [8], $\phi_a = q e^{i(u+\tilde{v})}$ where q is inner, and u and v are in C_R. For $0 < c < a$, $\phi_c = q_{a-c} q e^{i(u+\tilde{v})}$ and so by (2) of Theorem 2 $\sigma(T_{\phi_c}) = \bar{D}$. For if $q_{a-c} q e^{i(u+\tilde{v})} = g/|g|$ for some g in $\bigcup_{p>1/2} H^p$ with $h = g^{-1} \in \bigcup_{p>1/2} H^p$, then $h q_{a-c} q e^{i(u+\tilde{v})} \geq 0 \quad a.e.$ and so $hkq_{a-c}q \geq 0 \quad a.e.$ where $k = e^{-\tilde{u}+v+i(u+\tilde{v})}$. Since both k and k^{-1} belong to $\bigcap_{p<\infty} H^p$, $hkq_{a-c}q$ is a nonnegative function in $H^{1/2}$ and so by [7], $hkq_{a-c}q$ is constant. This contradicts that $q_{a-c}q$ is not constant. Therefore (2) of Theorem 2 shows that $\sigma(T_{\phi_c}) = \bar{D}$

for $0 < c < a$. If T_{ϕ_a} is invertible, it is known that q is constant. In fact, we can show it as in the above proof. If q is constant, then for $c > 0$ with $c \neq a$

$$\phi_c = \bar{q}_c b = q_{a-c}\bar{q}_a b = q_{a-c} e^{i(u+\tilde{v})}.$$

By the first part of this theorem, we may assume that $c > a$. However, in this case we can show it as in case $0 < c < a$.

§4. Remark

If $\sigma(T_\phi) \subseteq \partial D$, then $\sigma(T_\phi) = J$ for some closed arc J in ∂D because $\sigma(T_\phi)$ is connected by a theorem of H.Widom (*cf.*[2 , Corollary 7.46]). Then, if the essential range $R(\phi)$ of ϕ is disconnected, by a theorem of A.Brown and P.R.Halmos (*cf.*[2, Corollary 7.19]), then $\sigma(T_\phi) \nsubseteq \partial D$. Hence if $\sigma(T_\phi) \subseteq \partial D, R(\phi)$ is connected and so $R(\phi) = J = \sigma(T_\phi)$ by the theorem of A.Brown and P.R.Halmos. If $\phi = \alpha e^{it}$, $\inf\{\|t - \tilde{s}\|_\infty \,;\, s \in L^\infty_R\} = 0$ and $R(\phi) = \partial D$, then $\sigma(T_\phi) = \partial D$ by (1) of Theorem 1. For a unimodular function ϕ in C, by Theorem 1 it is easy to see that $\sigma(T_\phi) \subseteq \partial D$ if and only if $\phi = e^{iv}$ for some $v \in C_R$. For a unimodular function ϕ in $H^\infty + C$, by [8, Theorem 2] and Theorem 1 it is easy to see that $\sigma(T_\phi) \subseteq \partial D$ if and only if $\phi = e^{i(\tilde{u}+v)}$ for some $u, v \in C_R$. In fact, by [8, Theorem 2], Theorems 1 and 2, $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$ for a unimodular function ϕ in $H^\infty + C$.

In Corollary 3, we can not change the condition : $g^{-1} \notin \bigcup_{p>1/2} H^p$ to $g^{-1} \notin \bigcup_{p>1} H^p$ even if $g \in H^\infty$. For example, put $g = 1 + z$ then $\sigma(T_\phi) \neq \bar{D}$. If $\phi = (1+q)^\alpha/|1+q|^\alpha$ where q is a nonconstant inner function and $2 \leq \alpha < \infty$, then by Corollary 3 $\sigma(T_\phi) = \bar{D}$ because $(1+q)^\alpha \in H^\infty$ and $(1+q)^{-\alpha} \notin \bigcup_{p>1/2} H^p$. We can show a more general theorem than Corollary 6, that is, for a symbol $\phi = \bar{q}_1 q_2 \phi_0$ where ϕ_0 is a unimodular step function. Let ϕ be an arbitrary unimodular function in L^∞, then by [8] $\phi = \bar{q}_1 q_2 e^{i(u+\tilde{v})}$ where both q_1 and q_2 are Blaschke products and $u, v \in C_R$. If sing $q_1 \neq$ sing q_2, then by the proof of Corollary 5 it is easy to see that $\phi \neq g/|g|$ for any g in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$ Thus by Theorem 2 $\sigma(T_\phi) = \bar{D}$.

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