Title	A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field
Author(s)	Hiroshima, F.
Citation	Hokkaido University Preprint Series in Mathematics, 338, 1-34
Issue Date	1996-5-1
DOI	10.14943/83484
Doc URL	http://hdl.handle.net/2115/69088
Type	bulletin (article)
File Information	pre338.pdf

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Series #338. May 1996

HOKKAIDO UNIVERSITY

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A Scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field

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Abstract

This paper presents a scaling limit of Hamiltonians which describe interactions of N -nonrelativistic charged particles in a scalar potential and a quantized radiation field in the Coulomb gauge with the dipole approximation. The scaling limit defines effective potentials. In one-nonrelativistic particle case, the effective potentials have been known to be Gaussian transformations of the scalar potential [J.Math.Phys.34(1993)4478-4518]. However it is shown that the effective potentials in the case of N -nonrelativistic particles are not necessary to be Gaussian transformations of the scalar potential.

1 INTRODUCTION

The main problem in this paper is to consider a scaling limit of a model in quantum electrodynamics which describes an interaction of many nonrelativistic charged particles and a quantized radiation field in the Coulomb gauge with the dipole approximation. For our discussion we may limit ourselves to the case of a fixed number N of the particles, since N does not change in time. The model we consider is called “the Pauli-Fierz model”, which has been a subject of great interests and by which real physical phenomena of charged particles and a quantized radiation field such as “Lamb shift” can be interpreted. There has been a considerable amount of literature on the Pauli-Fierz model with one-nonrelativistic charged particle, e.g., [1,2] from points of view of physics and [3,4,5,6,7,8] mathematical points of view. In particular, the authors of [5,6] have studied a scaling limit of the Pauli-Fierz model with one-nonrelativistic charged particle. We may well extend the scaling limit of one-particle system to N -particle system.

The authors of [5,6] defined Hamiltonians of the Pauli-Fierz model as self-adjoint operators H_ρ with an ultraviolet cut-off function ρ acting in the tensor product of the Hilbert space $L^2(\mathbb{R}^d)$ and a Boson Fock space $\mathcal{F}(\mathcal{W})$ over $\mathcal{W} = \bigoplus_{r=1}^{d-1} L^2(\mathbb{R}^d)$. Introducing scalings with respect to parameters c (the speed of light), m (the mass of the particle) and e (the charge of the particle), the authors have shown the existence of the strong resolvent limits of the scaled self-adjoint operators $H_\rho^{REN}(\kappa) + V \otimes I$ with an infinite self-energy of the non-relativistic particle subtracted with a scalar potential V , (we call the limit “the scaling limit of $H_\rho + V \otimes I$ ”): In [6] we have proved the following:

Let V and ρ satisfy some conditions and Δ be the Laplacian in $L^2(\mathbb{R}^d)$. Then $H_\rho^{REN}(\kappa) + V \otimes I$ is self-adjoint and bounded from below uniformly in sufficiently large $\kappa > 0$ with

$$s - \lim_{\kappa \rightarrow \infty} (H_\rho^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{S} \left\{ \left(-\frac{1}{2m_\infty} \Delta + V_{eff} - z \right)^{-1} \otimes P_0 \right\} \mathcal{S}^{-1},$$

where $z \in \mathbb{C} \setminus \mathbb{R}$, m_∞ is a positive constant, $\mathcal{S}$ a unitary operator on $L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W})$, P_0 a

projection on $\mathcal{F}(\mathcal{W})$ and V_{eff} a multiplication operator defined by

$$V_{eff}(x) = (2\pi\alpha)^{-\frac{d}{2}} \int dy e^{-|x-y|^2/2\alpha} V(y),$$

where α is a positive constant. The multiplication operator V_{eff} is called “the effective potential”.

One of the strongest methods to analyze the scaling limits in [5,6] was to find Bogoliubov transformations $\mathcal{U}$, which implements a unitary equivalence between the Pauli-Fierz Hamiltonians H_ρ and decoupled Hamiltonians of the form

$$\widetilde{H} = -\frac{1}{2\bar{m}}\Delta \otimes I + I \otimes H_b + \text{constant},$$

where $\bar{m}$ is a positive constant and H_b is the free Hamiltonian of the quantized radiation field in $\mathcal{F}(\mathcal{W})$; the authors of [5,6] show equations of the following type:

$$(H_\rho^{REN} + V \otimes I - z)^{-1} = \mathcal{U} (\widetilde{H} + \mathcal{U}^{-1}(V \otimes I)\mathcal{U} - z)^{-1} \mathcal{U}^{-1}. \quad (1.1)$$

In this paper, the Pauli-Fierz Hamiltonian $H_{\vec{p}}$ with N -nonrelativistic charged particles in the Coulomb gauge with the dipole approximation are defined as operators acting in the Hilbert space $\underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W})$ by

$$\begin{aligned} H_{\vec{p}} &= \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-i\hbar D_\mu^j \otimes I - eI \otimes A_\mu(\rho_j) \right)^2 + I \otimes H_b \\ &= -\frac{\hbar^2}{2m} \Delta \otimes I + I \otimes H_b + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(2e\hbar i D_\mu^j \otimes A_\mu(\rho_j) + e^2 I \otimes A_\mu^2(\rho_j) \right), \end{aligned}$$

where D_μ^j is the differential operator with respect to the j -th variable in the μ -th direction, Δ the Laplacian in $L^2(\mathbb{R}^{dN})$, $\hbar$ the Planck constant divided 2π and $A_\mu(\rho_j)$ the quantized radiation field in the μ -th direction with an ultraviolet cut-off function ρ_j in the Coulomb gauge. Problems arising in the many particles system are as follows:

(i) Do there any Bogoliubov transformations such as (1.1) exist?

- (ii) What kind of scalar potentials V and sets of ultraviolet cut-off functions $(\rho_1, \dots, \rho_N)$ do a scaling limit of the Hamiltonian $H_{\vec{p}} + V \otimes I$ exist for? Furthermore, what kind of infinite self-energy should be subtracted from the original Hamiltonian $H_{\vec{p}} + V \otimes I$?
- (iii) If the scaling limit exists, what form does the effective potential have?

With this motivation, we continue here to analyze a scaling limit of the Pauli-Fierz model with N -nonrelativistic charged particles.

We introduce the same scaling as [6] as follows;

$$c(\kappa) = c\kappa, e(\kappa) = e\kappa^{-\frac{1}{2}}, m(\kappa) = m\kappa^{-2}. \quad (1.2)$$

Introducing a pseudo differential operator $E^{REN}(D, \kappa)$ in $L^2(\mathbb{R}^{dN})$ with a symbol $E^{REN}(p, \kappa)$ such that $E^{REN}(p, \kappa) \rightarrow \infty$ as $\kappa \rightarrow \infty$, we define a Hamiltonian $H_{\vec{p}}^{REN}(\kappa)$ by

$$H_{\vec{p}}^{REN}(\kappa) = -E^{REN}(D, \kappa) \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(\kappa 2e\hbar i D_{\mu}^j \otimes A_{\mu}(\rho_j) + e^2 I \otimes A_{\mu}^2(\rho_j) \right).$$

For sufficiently large $\kappa > 0$ and a scalar potential V with some conditions, we shall show that $H_{\vec{p}}^{REN}(\kappa) + V \otimes I$ is essentially self-adjoint on $D(-\Delta \otimes I) \cap D(I \otimes H_b)$ and bounded from below uniformly in sufficiently large $\kappa > 0$, and the existence of Bogoliubov transformations $\mathcal{U}(\kappa)$, which gives a unitary equivalence of $H_{\vec{p}}^{REN}(\kappa) + V \otimes I$ and a self-adjoint operator $\widetilde{H}_{\vec{p}}(\kappa) + C_{\kappa}(V)$ as follows;

$$(H_{\vec{p}}^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\kappa)(\widetilde{H}_{\vec{p}}(\kappa) + C_{\kappa}(V) - z)^{-1}\mathcal{U}^{-1}(\kappa),$$

where $\widetilde{H}_{\vec{p}}(\kappa) = \widetilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b$, $\widetilde{E}(D, \kappa)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$ and $C_{\kappa}(V) = \mathcal{U}^{-1}(\kappa)(V \otimes I)\mathcal{U}(\kappa)$ (Theorem 3.5). Then we see that $\mathcal{U}(\kappa) \rightarrow \mathcal{U}(\infty)$ as $\kappa \rightarrow \infty$ strongly (Theorem 3.4) and hence we get

$$s - \lim_{\kappa \rightarrow \infty} (H_{\vec{p}}^{REN}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^{\infty}(D) + V_{eff} - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty),$$

where $E^\infty(D)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$ and V_{eff} a multiplication operator. (Theorems 3.6, 3.7). In the case of one-particle system the effective potential V_{eff} is a Gaussian transformation of a given scalar potential V . However, we shall see that in the N -particle system, V_{eff} is not necessary to be a Gaussian transformation. Actually it is determined by a matrix $\tilde{\Delta}^\infty = (\tilde{\Delta}_{ij}^\infty)_{1 \leq i, j \leq N}$ defined by

$$\tilde{\Delta}_{ij}^\infty = \frac{1}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right) \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k) \hat{\rho}_j(k)}{\omega(k)^3}, \quad (1.3)$$

where $\omega(k) = |k|, k \in \mathbb{R}^d$. In the case where $\tilde{\Delta}^\infty$ is non-degenerate, the effective potential V_{eff} is Gaussian transformations of V .

The outline of this paper is as follows. In section 2, we define the Pauli-Fierz Hamiltonian with N -nonrelativistic charged particles in the Coulomb gauge with the dipole approximation and show its self-adjointness. Moreover we construct an exact solution to the Heisenberg equation from the point of view of the operator theory (Corollary 2.9). In section 3, when the scaling parameter $\kappa > 0$ is sufficiently large, we show that a Bogoliubov transformation can be constructed, and define a renormalized self-adjoint operator $H_{\vec{p}}^{REN}(\kappa)$ which is the original Hamiltonian $H_{\vec{p}}(\kappa)$ with an infinite self-energy of the nonrelativistic charged particles subtracted. We shall show the existence of the scaling limit of $H_{\vec{p}} + V \otimes I$ and give an explicit form of the effective potential. We give a typical example of a scalar potential and a set of ultraviolet cut-off functions. In section 4, we give a physical interpretation of the matrix $\tilde{\Delta}^\infty$.

The author would like to thank Professor A.Arai for helpful discussions.

2 THE PAULI-FIERZ MODEL AND EXACT SOLUTION

To begin with, let us introduce some preliminary notations. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We denote the inner product and the associated norm by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ respectively.

The inner product is linear in $\cdot$ and antilinear in $*$. The domain of an operator A in $\mathcal{H}$ is denoted by $D(A)$. A notation The Fourier transformation of a function f is denoted by $\hat{f}$ (resp. $\check{f}$) and $\bar{f}$ the complex conjugate of f . In this paper, summations over repeated Greek letters are understood. Let

$$\mathcal{W} \equiv \underbrace{L^2(\mathbb{R}^d) \oplus \dots \oplus L^2(\mathbb{R}^d)}_{d-1}.$$

We define the Boson Fock space over $\mathcal{W}$ by

$$\mathcal{F}(\mathcal{W}) \equiv \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{W} \equiv \bigoplus \mathcal{F}_n(\mathcal{W}),$$

where $\otimes_s^0 \mathcal{W} \equiv \mathbb{C}$ and $\otimes_s^n \mathcal{W}$ ($n \geq 1$) denotes the n -fold symmetric tensor product. Put

$$\mathcal{F}^N(\mathcal{W}) \equiv \bigoplus_{n=0}^N \mathcal{F}_n(\mathcal{W}) \bigoplus_{n \geq N+1} \{0\}.$$

Moreover we define the finite particle subspace of $\mathcal{F}(\mathcal{W})$ by

$$\mathcal{F}^\infty(\mathcal{W}) \equiv \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathcal{W}).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ ($f \in \mathcal{W}$) act on the finite particle subspace and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$\begin{aligned} [a(f), a^\dagger(g)] &= \langle \bar{f}, g \rangle_{\mathcal{W}}, \\ [a^\sharp(f), a^\sharp(g)] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$, $a^\sharp$ denotes either a or $a^\dagger$. Furthermore,

$$\langle a^\dagger(f)\Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, a(\bar{f})\Psi \rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).$$

We define polarization vectors e^r ($r = 1, \dots, d-1$) as measurable functions $e^r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that

$$e^r(k)e^s(k) = \delta_{rs}, \quad e^r(k)k = 0, \quad a.e. k \in \mathbb{R}^d.$$

In this paper, we fix polarization vectors e^r . The μ -th direction time-zero smeared radiation field in the Coulomb gauge with the dipole approximation is defined as operators acting in $\mathcal{F}(\mathcal{W})$ by

$$A_\mu(f) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left(\bigoplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \hat{f}}{\sqrt{c\omega}} \right) + a \left(\bigoplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_\mu^r \tilde{\hat{f}}}{\sqrt{c\omega}} \right) \right\}, \quad (2.1)$$

and the conjugate momentum

$$\Pi_\mu(f) = \frac{i}{\sqrt{2}} \left\{ a^\dagger \left(\bigoplus_{r=1}^{d-1} \sqrt{\hbar} \sqrt{c\omega} e_\mu^r \hat{f} \right) - a \left(\bigoplus_{r=1}^{d-1} \sqrt{\hbar} \sqrt{c\omega} e_\mu^r \tilde{\hat{f}} \right) \right\}, \quad (2.2)$$

where $\tilde{g}(k) = g(-k)$. Note that in the case where f is real-valued, $A_\mu(f)$ and $\Pi_\mu(f)$ are symmetric operators. Let $\Omega = (1, 0, 0, \dots) \in \mathcal{F}(\mathcal{W})$. It is well known that

$$\mathcal{L} \left\{ a^\dagger(f_1) \dots a^\dagger(f_n) \Omega, \Omega \mid f_j \in \mathcal{W}, j = 1, \dots, n, n \geq 1 \right\}$$

is dense in $\mathcal{F}(\mathcal{W})$. For a nonnegative self-adjoint operator $h : \mathcal{W} \rightarrow \mathcal{W}$, an operator $\Gamma(e^{-th})$ is defined by

$$\begin{aligned} \Gamma(e^{-th}) a^\dagger(f_1) \dots a^\dagger(f_n) \Omega &= a^\dagger(e^{-th} f_1) \dots a^\dagger(e^{-th} f_n) \Omega, \\ \Gamma(e^{-th}) \Omega &= \Omega. \end{aligned}$$

The operator $\Gamma(e^{-th})$ defines a unique strongly continuous one-parameter semigroup on $\mathcal{F}(\mathcal{W})$. Hence, by Stone's theorem, there exists a nonnegative self-adjoint operator $d\Gamma(h)$ in $\mathcal{F}(\mathcal{W})$ such that

$$\Gamma(e^{-th}) = e^{-td\Gamma(h)}.$$

The operator $d\Gamma(h)$ is called “the second quantization of h ”. Put $\tilde{\omega} = \underbrace{\omega \oplus \dots \oplus \omega}_{d-1}$. The free Hamiltonian H_b in $\mathcal{F}(\mathcal{W})$ is defined by

$$H_b \equiv \hbar c d\Gamma(\tilde{\omega}).$$

Let M_d be a Hilbert space defined by

$$M_d = \left\{ f \left| \int |f(k)|^2 \omega(k)^d dk < \infty \right. \right\},$$

with the inner product

$$\langle f, g \rangle_n = \int_{\mathbb{R}^d} \bar{f}(k) g(k) \omega(k)^n dk.$$

We have the following commutation relations on $\mathcal{F}^\infty(\mathcal{W})$,

$$\begin{aligned} [A_\mu(f), A_\nu(g)] &= 0, & \hat{f}, \hat{g} &\in M_{-1}, \\ [\Pi_\mu(f), \Pi_\nu(g)] &= 0, & \hat{f}, \hat{g} &\in M_1, \\ [A_\mu(f), \Pi_\nu(g)] &= i\hbar \left\langle d_{\mu\nu} \hat{f}, \hat{g} \right\rangle_{L^2(\mathbb{R}^d)}, & \hat{f}, \hat{g} &\in M_{-1} \cap M_0 \cap M_1, \end{aligned}$$

and on $D(H_b^{\frac{3}{2}})$,

$$\begin{aligned} [H_b, A_\mu(f)] &= -i\hbar \Pi_\mu(f), & \hat{f} &\in M_{-1} \cap M_1, \\ [H_b, \Pi_\mu(f)] &= i\hbar c^2 A_\mu(-\Delta f), & \hat{f} &\in M_3 \cap M_1, \end{aligned}$$

where Δ is the Laplacian in the L^2 -sense and $d_{\mu\nu}(k) = \sum_{r=1}^d e_\mu^r(k) e_\nu^r(k)$. The Pauli-Fierz Hamiltonian with N -nonrelativistic charged particles interacting with the quantized radiation field in the Coulomb gauge with the dipole approximation is defined by

$$H_{\vec{p}} \equiv H_{\rho_1, \dots, \rho_N} \equiv \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-i\hbar D_\mu^j \otimes I - eI \otimes A_\mu(\rho_j) \right)^2 + I \otimes H_b,$$

acting in

$$\underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W}) \cong \int_{\mathbb{R}^{dN}}^\oplus \mathcal{F}(\mathcal{W}) dx,$$

where D_μ^j is the L^2 -derivative with respect to the j -th variable in the μ -th direction, ρ_j 's serve as ultraviolet cut-off functions. We introduce a scaling with respect to the parameters c, e, m as (1.2). Throughout this paper, for objects $A = A(c, e, m)$ containing the parameters c, e, m , we denote the scaled object by $A(\kappa) \equiv A(c(\kappa), e(\kappa), m(\kappa))$. We define a class of sets of functions as follows:

Definition 2.1 $\vec{\rho} = (\rho_1, \dots, \rho_N)$ is in P if and only if

- (1) $\hat{\rho}_j, j = 1, \dots, N$ are rotation invariant, $\hat{\rho}_j(k) = \hat{\rho}_j(|k|)$, and real-valued,
- (2) $\hat{\rho}_j/\omega, \hat{\rho}_j/\sqrt{\omega}, \hat{\rho}_j, \sqrt{\omega}\hat{\rho}_j \in L^2(\mathbb{R}^d)$.

Moreover $\vec{\rho}$ is in $\tilde{P}$ if and only if in addition to (1) and (2) above

- (3) For all $j = 1, \dots, N$, $\hat{\rho}_j/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$ and there exist $0 < \alpha < 1$ and $1 \leq \epsilon$ such that $\hat{\rho}_i(\sqrt{s})\hat{\rho}_j(\sqrt{s})(\sqrt{s})^{d-2} \in Lip(\alpha) \cap L^\epsilon([0, \infty))$, where $Lip(\alpha)$ is the set of the Lipschitz continuous functions on $[0, \infty)$ with order α ,
- (4) $\sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{3}{2}}(k)| < \infty, \sup_k |\hat{\rho}_j(k)\omega^{\frac{d}{2}-\frac{1}{2}}(k)| < \infty, j = 1, \dots, N$.

Observe that Definition 2.1 (1) implies that ρ_j 's are real-valued functions. Hence $A_\mu(\rho_j)$'s are symmetric operators. Put

$$H_0 = -\frac{1}{2m}\hbar^2\Delta \otimes I + I \otimes H_b,$$

where Δ is the Laplacian in $L^2(\mathbb{R}^{dN})$. It is well known that H_0 is a nonnegative self-adjoint operator on $D(H_0) = D\left(-\frac{1}{2m}\hbar^2\Delta \otimes I\right) \cap D(I \otimes H_b)$.

Theorem 2.2 ([3,4]) For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\vec{\rho}}(\kappa)$ is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_0 and nonnegative.

Let $\mathbf{F} = F \otimes I$, where F denotes the Fourier transform in $L^2(\mathbb{R}^{dN})$. It is clear that operators $\mathbf{F}H_{\vec{\rho}}\mathbf{F}^{-1}$ can be decomposable as follows:

$$\mathbf{F}H_{\vec{\rho}}(\kappa)\mathbf{F}^{-1} = \int_{\mathbb{R}^{dN}}^{\oplus} H_{\vec{\rho}}(p, \kappa) dp,$$

where

$$H_{\vec{\rho}}(p, \kappa) = \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(\kappa \hbar p_\mu^j - e A_\mu(\rho_j) \right)^2 + \kappa H_b.$$

Theorem 2.3 ([3,4]) *For $\vec{p} \in P$ and $\kappa > 0$, the operator $H_{\vec{p}}(p, \kappa)$ is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of H_b and nonnegative.*

Following [3,4,6], we shall construct a Heisenberg field concretely. The Heisenberg field $A_\mu(f, t, \kappa)$ with the scaling parameter κ is defined by a solution to the Heisenberg equation:

$$\begin{aligned} \frac{d}{dt} A_\mu(f, t, \kappa) &= \frac{i}{\hbar} [H_{\vec{p}}(p, \kappa), A_\mu(f, t, \kappa)], \\ A_\mu(f, 0, \kappa) &= A_\mu(f, \kappa). \end{aligned}$$

In order to construct the Heisenberg field in a rigorous way, we shall prepare some technical lemmas. We define an $N \times N$ matrix-valued function $\mathbb{D}(z) = (D_{ij}(z))_{1 \leq i, j \leq N}$ by

$$D_{ij}(z) = m\delta_{ij} - \frac{e^2}{c^2} \frac{d-1}{d} \int_{\mathbb{R}^d} \frac{\hat{\rho}_i(k) \hat{\rho}_j(k)}{z - |k|^2} dk, \quad z \in \mathbb{C} \setminus [0, \infty).$$

Lemma 2.4 *Let $(\cdot, \cdot)$ denote the Euclidean inner product. Suppose $\vec{p} \in \tilde{P}$. Then the followings hold:*

- (1) *The functions $D_{ij}(z, \kappa)$, $1 \leq i, j \leq N$, $\kappa > 0$ are analytic in $\mathbb{C} \setminus [0, \infty)$.*
- (2) *For $s \in [0, \infty)$ and $\kappa > 0$, the pointwise limit $D_{\pm ij}(s, \kappa) \equiv \lim_{h \rightarrow 0} D_{ij}(s \pm ih, \kappa)$ exists and has the following form*

$$\begin{aligned} D_{\pm ij}(s, \kappa) &= \frac{m}{\kappa^2} \delta_{ij} - \frac{1}{\kappa^3} \left(\frac{e^2}{c^2} \frac{\mathbf{V}_d}{2} \frac{d-1}{d} \right) H_{ij}(s) \pm \frac{2\pi i}{\kappa^3} \left(\frac{e^2}{c^2} \frac{\mathbf{V}_d}{2} \frac{d-1}{d} \right) K_{ij}(s), \\ K_{ij}(s) &= \hat{\rho}_i(\sqrt{s}) \hat{\rho}_j(\sqrt{s}) s^{\frac{d}{2}-1}, \\ H_{ij}(s) &= \lim_{\epsilon \rightarrow 0+} \int_{|s-x| > \epsilon} \frac{K_{ij}(x)}{s-x} dx, \end{aligned}$$

where $\mathbf{V}_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ ($\Gamma(z)$ is the gamma function). The convergence is uniform in $s \in [0, \infty)$; for any $\delta > 0$, there exists $h_0 > 0$ independent of s, κ , such that for $0 < h \leq h_0$,

$$|D_{ij}(s \pm ih, \kappa) - D_{\pm ij}(s, \kappa)| \leq \frac{\delta}{\kappa^3}.$$

Moreover $H_{ij}(s)$ is Lipschitz continuous in $s \in [0, \infty)$ with the same order as that of K_{ij} and contained in $L^\epsilon(\mathbb{R}^d)$ with some $\epsilon \geq 1$.

(3) Let $\kappa > 0$ be sufficiently large. Put $\mathbb{D}_\pm(s, \kappa) = (D_{\pm ij}(s, \kappa))_{1 \leq i, j \leq N}$. Then there exists a positive constant $d_1(\kappa)$ such that for $(w_1, \dots, w_N) = \vec{w} \in \mathbb{C}^N$,

$$\inf_{s \in [0, \infty)} |(\mathbb{D}_\pm(s, \kappa)\vec{w}, \vec{w})| > d_1(\kappa)|\vec{w}|^2.$$

(4) Let $\kappa > 0$ be sufficiently large. Then there exists a positive constant $d_2(\kappa)$ such that for $\vec{w} \in \mathbb{C}^N$,

$$\inf_{z \in \mathbb{C} \setminus [0, \infty)} |(\mathbb{D}(z, \kappa)\vec{w}, \vec{w})| > d_2(\kappa)|\vec{w}|^2.$$

Proof: The statements (1) and (2) are fundamental facts([9]). We shall prove (3). From (2) it follows that

$$(\mathbb{D}_\pm(s, \kappa)\vec{w}, \vec{w}) = \frac{m}{\kappa^2} \left\{ |\vec{w}|^2 - \frac{1}{\kappa} \frac{\lambda}{m} (H(s)\vec{w}, \vec{w}) \right\} \pm 2\pi i \frac{\lambda}{\kappa^3} (K(s)\vec{w}, \vec{w}),$$

where $\lambda = \frac{e^2}{c^2} \frac{V_d}{2} \frac{d-1}{d}$, $H(s) = (H_{ij}(s))_{1 \leq i, j \leq d}$, $K(s) = (K_{ij}(s))_{1 \leq i, j \leq d}$. Since H_{ij} is a Lipschitz continuous function and contained in $L^\epsilon([0, \infty))$, it is bounded. Hence we have

$$|(H(s)\vec{w}, \vec{w})| \leq N \times \sup_{s \in [0, \infty), 1 \leq i, j \leq d} |H_{ij}(s)| \cdot |\vec{w}|^2 \equiv \alpha |\vec{w}|^2.$$

Thus we can see that for sufficiently large $\kappa > 0$

$$|(\mathbb{D}_\pm(s, \kappa)\vec{w}, \vec{w})| \geq \frac{m}{\kappa^2} \left(1 - \frac{1}{\kappa} \frac{\lambda}{m} \alpha \right) |\vec{w}|^2.$$

Hence we get (3). We shall prove (4). From (2) it follows that for any $\eta > 0$, there exists $\epsilon_0 > 0$ independent of $s \in [0, \infty)$ and $\kappa > 0$ such that for $0 <^\forall \epsilon \leq \epsilon_0$,

$$|(\mathbb{D}_\pm(s, \kappa)\vec{w}, \vec{w})| - \frac{\eta}{\kappa^3} |\vec{w}|^2 \leq |(\mathbb{D}(s \pm i\epsilon, \kappa)\vec{w}, \vec{w})|.$$

Hence we have

$$|(\mathbb{D}(s \pm i\epsilon, \kappa)\vec{w}, \vec{w})| \geq \left\{ \frac{m}{\kappa^2} \left(1 - \frac{1}{\kappa} \frac{\lambda}{m} \alpha \right) - \frac{\eta}{\kappa^3} \right\} |\vec{w}|^2. \quad (2.3)$$

On the other hand, put $\Pi_{\epsilon_0} = \mathbb{C} \setminus \{x + iy | x \geq 0, |y| \leq \epsilon_0\}$. Then we see that for $x + iy \in \Pi_{\epsilon_0}$

$$\begin{aligned} (\mathbb{D}(x + iy, \kappa)\vec{w}, \vec{w}) &= \frac{m}{\kappa^2} \left(|\vec{w}|^2 - \frac{1}{\kappa} \frac{\lambda}{m} \int_0^\infty \frac{(x-s) |\sum_{i=1}^N w_i \hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2 + y^2} ds \right) \\ &\quad + i \frac{\lambda}{\kappa^3} \int_0^\infty \frac{y |\sum_{i=1}^N w_i \hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2 + y^2} ds. \end{aligned}$$

Noting that $|ab|/a^2 + b^2 \leq 1/2$, we have

$$\begin{aligned} \left| \int_0^\infty \frac{(x-s) |\sum_{i=1}^N w_i \hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1}}{(x-s)^2 + y^2} ds \right| &\leq \frac{1}{2|y|} \int_0^\infty |\sum_{i=1}^N w_i \hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1} ds, \\ &\leq \frac{1}{2\epsilon_0} \int_0^\infty \sum_{i=1}^N |\hat{\rho}_i(\sqrt{s})|^2 s^{\frac{d}{2}-1} ds |\vec{w}|^2 \\ &\equiv \frac{\beta}{\epsilon_0} |\vec{w}|^2. \end{aligned}$$

Since ϵ_0 is independent of $\kappa > 0$, we see that for sufficiently large $\kappa > 0$,

$$|(\mathbb{D}(x + iy, \kappa)\vec{w}, \vec{w})| \geq \frac{m}{\kappa^2} \left(1 - \frac{1}{\kappa} \frac{\lambda}{m} \frac{\beta}{\epsilon_0} \right) |\vec{w}|^2, \quad x + iy \in \Pi_{\epsilon_0}. \quad (2.4)$$

Combining (2.3) and (2.4), we get (4). $\square$

From Lemma 2.4 (3) and (4) it follows that for sufficiently large $\kappa > 0$, there exist the inverse matrices to $\mathbb{D}(z, \kappa)$ and $\mathbb{D}_\pm(s, \kappa)$, which satisfy

$$\sup_{s \in [0, \infty)} \left| (\mathbb{D}_\pm^{-1}(s, \kappa)\vec{w}_1, \vec{w}_2) \right| < \frac{1}{d_1(\kappa)} |\vec{w}_1| |\vec{w}_2|, \quad (2.5)$$

$$\sup_{z \in \mathbb{C} \setminus [0, \infty)} \left| (\mathbb{D}^{-1}(z, \kappa)\vec{w}_1, \vec{w}_2) \right| < \frac{1}{d_2(\kappa)} |\vec{w}_1| |\vec{w}_2|. \quad (2.6)$$

We set for $\vec{\rho} \in \tilde{P}$ and sufficiently large $\kappa > 0$

$$Q(k, \kappa) \equiv \mathbb{D}_+^{-1}(k^2, \kappa) \begin{pmatrix} \hat{\rho}_1(k) \\ \vdots \\ \hat{\rho}_N(k) \end{pmatrix} \equiv (Q_1(k, \kappa), \dots, Q_N(k, \kappa)).$$

For later use in Appendix, we note that for all $s \in [0, \infty)$,

$$D_{+ij}(s, \kappa) - D_{-ij}(s, \kappa) = 2\pi i \frac{1}{\kappa^3} \left(\frac{e^2}{c^2} \mathbf{V}_d \frac{d-1}{d} \right) \hat{\rho}_i(\sqrt{s}) \hat{\rho}_j(\sqrt{s}) s^{\frac{d}{2}-1}. \quad (2.7)$$

Put $\mathbb{D}^{-1}(z) = (D_{ij}^{-1}(z))_{1 \leq ij \leq N}$, $\mathbb{D}_{\pm}^{-1}(s) = (D_{\pm ij}^{-1}(s))_{1 \leq ij \leq N}$. Then (2.7) implies that

$$\frac{2\pi i}{\kappa^3} \left(\frac{e^2}{c^2} \mathbf{V}_d \frac{d-1}{d} \right) \sum_{k,l=1}^N D_{-ik}^{-1}(s, \kappa) D_{+jl}^{-1}(s, \kappa) \hat{\rho}_k(\sqrt{s}) \hat{\rho}_l(\sqrt{s}) s^{\frac{d}{2}-1} = D_{-ij}(s, \kappa) - D_{+ji}(s, \kappa). \quad (2.8)$$

Remark 2.5

(1) In [3,4,6], the authors define functions $D_{\pm}(s)$ corresponding to $\mathbb{D}_{\pm}(s)$ defined in this paper. The function $1/D_{\pm}(s, \kappa)$ can be well defined for some ρ and any $\kappa > 0$. However, in our case, we do not know whether $\mathbb{D}_{\pm}(s, \kappa)$ has the inverse or not for all $\kappa > 0$. But since, in this paper, we focus on an asymptotic behavior as $\kappa \rightarrow \infty$, it is sufficient to consider the case where κ is sufficiently large.

(2) For the proof of Lemma 2.4, we do not need Definition 2.1 (4).

We define operators G_h ($h > 0$) by

$$(G_h f)(k) = \int_{\mathbb{R}^d} \frac{f(k')}{(k^2 - k'^2 + ih)(kk')^{\frac{d}{2}-1}} dk'.$$

It is well known and not so hard to see that G_h are bounded linear operators on $L^2(\mathbb{R}^d)$ and the strong limits $\lim_{h \rightarrow 0} G_h \equiv G$ exists ([4]). Furthermore G is skew symmetric ($G^* = -G$).

For sufficiently large $\kappa > 0$, we can define the following operators:

$$T_{\mu\nu}(\kappa)f \equiv \delta_{\mu\nu}f + \frac{1}{\kappa^3} \frac{e^2}{c^2} \sum_{j=1}^N Q_j(\kappa) \omega^{\frac{d}{2}-1} G \omega^{\frac{d}{2}-1} d_{\mu\nu} \hat{\rho}_j f, \quad 1 \leq \mu, \nu \leq d.$$

Lemma 2.6 Suppose that $\vec{\rho} \in \tilde{P}$ and $\kappa > 0$ is sufficiently large. Then the following holds.

(1) $T_{\mu\nu}(\kappa)$ and $T_{\mu\nu}^*(\kappa)$ are bounded operators on M_{α} , $\alpha = -1, 0, 1$ and $(T_{\mu\nu}(\kappa)f)^{\sim} = T_{\mu\nu}(\kappa)\tilde{f}$.

(2) Put $D_{ij}^{-1}(0, \kappa) \equiv D_{\pm ij}^{-1}(0, \kappa)$ and let $f \in M_{-1}$. Then

$$\left\langle d_{\nu\alpha} \frac{Q_i(\kappa)}{\sqrt{\omega^3}}, \frac{1}{\sqrt{\omega}} T_{\mu\nu}(\kappa) f \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle d_{\mu\alpha} \sum_{j=1}^N D_{ij}^{-1}(0, \kappa) \frac{\hat{\rho}_j}{\sqrt{\omega^3}}, \frac{f}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)}, i = 1, \dots, N.$$

(3) $[\omega^2, T_{\mu\nu}^*(\kappa)] = -\sum_{i=1}^N \frac{1}{\kappa^3} \frac{e^2}{c^2} \langle Q_i(\kappa), \cdot \rangle_{L^2(\mathbb{R}^d)} d_{\mu\nu} \hat{\rho}_i.$

(4) $T_{\mu\nu}(\kappa) \hat{\rho}_j = \delta_{\mu\nu} \frac{m}{\kappa^2} Q_j(\kappa).$

(5) $T_{\mu\nu}^*(\kappa) d_{\nu\alpha} T_{\alpha\beta}(\kappa) = d_{\mu\beta}.$

(6) $e_\mu^r T_{\mu\nu}(\kappa) d_{\nu\alpha} T_{\alpha\beta}(\kappa) e_\beta^s = \delta_{rs}.$

Proof: See Appendix. □

In the rest of this section, we fix sufficiently large $\kappa > 0$ and omit κ in notations for simplicity. Define $\hat{A}_\mu(f) = A_\mu(\hat{f})$ and $\hat{\Pi}_\mu(f) = \Pi_\mu(\hat{f})$. We put

$$\begin{aligned} B^{(r)}(f, p) &= \frac{1}{\sqrt{2}} \left\{ \hat{A}_\mu \left(\frac{1}{\sqrt{\hbar}} T_{\mu\nu}^* e_\nu^r \sqrt{c\omega} f \right) + i \hat{\Pi}_\mu \left(\frac{1}{\sqrt{\hbar}} T_{\mu\nu}^* e_\nu^r \frac{f}{\sqrt{c\omega}} \right) \right. \\ &\quad \left. + \sum_{j=1}^N \hbar p_\nu^j \left\langle \frac{e}{\sqrt{\hbar}} \frac{Q_j e_\nu^r}{(c\omega)^{\frac{3}{2}}}, f \right\rangle_{L^2(\mathbb{R}^d)} \right\}, \quad f \in M_0, \\ B^{\dagger(r)}(f, p) &= \frac{1}{\sqrt{2}} \left\{ \hat{A}_\mu \left(\frac{1}{\sqrt{\hbar}} \bar{T}_{\mu\nu}^* \tilde{e}_\nu^r \sqrt{c\omega} \tilde{f} \right) - i \hat{\Pi}_\mu \left(\frac{1}{\sqrt{\hbar}} \bar{T}_{\mu\nu}^* \tilde{e}_\nu^r \frac{\tilde{f}}{\sqrt{c\omega}} \right) \right. \\ &\quad \left. + \sum_{j=1}^N \hbar p_\nu^j \left\langle \frac{e}{\sqrt{\hbar}} \frac{\bar{Q}_j e_\nu^r}{(c\omega)^{\frac{3}{2}}}, f \right\rangle_{L^2(\mathbb{R}^d)} \right\}, \quad f \in M_0, p = (p^1, \dots, p^N) \in \mathbb{R}^{dN}. \end{aligned}$$

By the definition of $A_\mu(f)$ and $\Pi_\mu(f)$, for the vector of the form $\mathbf{f} = f_1 \oplus \dots \oplus f_{d-1} \in \mathcal{W}$, we see that for $p = (p^1, \dots, p^N) \in \mathbb{R}^{dN}$

$$\begin{aligned} B(\mathbf{f}, p) &\equiv \sum_{r=1}^{d-1} B^{(r)}(f_r, p) = a^\dagger(\mathbf{W}_- \mathbf{f}) + a(\mathbf{W}_+ \mathbf{f}) + \sum_{j=1}^N \langle \mathbf{L}_j p^j, \mathbf{f} \rangle_{L^2(\mathbb{R}^d)}, \\ B^\dagger(\mathbf{f}, p) &\equiv \sum_{r=1}^{d-1} B^{\dagger(r)}(f_r, p) = a^\dagger(\bar{\mathbf{W}}_+ \mathbf{f}) + a(\bar{\mathbf{W}}_- \mathbf{f}) + \sum_{j=1}^N \langle \bar{\mathbf{L}}_j p^j, \mathbf{f} \rangle_{L^2(\mathbb{R}^d)}, \quad (2.9) \end{aligned}$$

where

$$\begin{aligned}\mathbf{W}_\pm &= (W_\pm^{(r,s)})_{1 \leq r,s \leq d-1}, \\ \mathbf{L}_j &= (L_{\mu j}^r)_{1 \leq \mu \leq d, 1 \leq r \leq d-1} = \begin{pmatrix} L_{1j}^1 & \cdots & L_{dj}^1 \\ \vdots & & \vdots \\ L_{1j}^{d-1} & \cdots & L_{dj}^{d-1} \end{pmatrix}, j = 1, \dots, N,\end{aligned}$$

where

$$\begin{aligned}W_+^{(r,s)} f &= \frac{1}{2} \left(\frac{1}{\sqrt{\omega}} e_\mu^r T_{\mu\nu}^* e_\nu^s \sqrt{\omega} + \sqrt{\omega} e_\mu^r T_{\mu\nu}^* e_\nu^s \frac{1}{\sqrt{\omega}} \right) f, \\ W_-^{(r,s)} f &= \frac{1}{2} \left(\frac{1}{\sqrt{\omega}} e_\mu^r T_{\mu\nu}^* \tilde{e}_\nu^s \sqrt{\omega} - \sqrt{\omega} e_\mu^r T_{\mu\nu}^* \tilde{e}_\nu^s \frac{1}{\sqrt{\omega}} \right) \tilde{f}, \\ L_{\mu j}^r &= \frac{e \sqrt{\hbar} e_\mu^r Q_j}{\sqrt{2c^3 \omega^3}}.\end{aligned}$$

We see that, by Lemma 2.6 (1), $\mathbf{W}_\pm$ is a bounded operator on $\mathcal{W}$. By virtue of Lemma 2.6 (5) and (6), one can easily see that $\mathbf{W}_\pm$ satisfy the following algebraic relations:

$$\begin{aligned}\mathbf{W}_+^* \mathbf{W}_+ - \mathbf{W}_-^* \mathbf{W}_- &= I, \\ \overline{\mathbf{W}}_+^* \mathbf{W}_- - \overline{\mathbf{W}}_-^* \mathbf{W}_+ &= 0, \\ \mathbf{W}_+ \mathbf{W}_+^* - \overline{\mathbf{W}}_- \overline{\mathbf{W}}_-^* &= I, \\ \mathbf{W}_- \mathbf{W}_+^* - \overline{\mathbf{W}}_+ \overline{\mathbf{W}}_-^* &= 0.\end{aligned}\tag{2.10}$$

We put $\mathcal{W}_\alpha = \underbrace{M_\alpha \oplus \dots \oplus M_\alpha}_{d-1}$, $\alpha \in \mathbb{R}$. These relations (2.10) imply that on $\mathcal{F}^\infty(\mathcal{W})$ for $\mathbf{f}, \mathbf{g} \in \mathcal{W}_0$,

$$\begin{aligned}[B(\mathbf{f}, p), B^\dagger(\mathbf{g}, p)] &= \langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{W}}, \\ [B^\sharp(\mathbf{f}, p), B^\sharp(\mathbf{g}, p)] &= 0,\end{aligned}$$

and for $\Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W})$,

$$\langle B^\dagger(\mathbf{f}, p) \Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, B(\bar{\mathbf{f}}, p) \Psi \rangle_{\mathcal{F}(\mathcal{W})}.$$

Lemma 2.7 For $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$, $p \in \mathbb{R}^{dN}$ and $\vec{p} \in \tilde{P}$, we have

$$[H_{\vec{p}}(p), B^\sharp(\mathbf{f}, p)] = \pm B^\sharp(\hbar c \tilde{\omega} \mathbf{f}, p), \text{ on } \mathcal{F}^\infty(\mathcal{W}) \cap D(H_b^{\frac{3}{2}}), \quad (2.11)$$

where $+$ (resp. $-$) corresponds to $B^\dagger$ (resp. B).

Proof: Suppose that $\mathbf{f} \in \mathcal{W}_{-2} \cap \mathcal{W}_0 \cap \mathcal{W}_2$. Then by Lemma 2.6 (3) and (4), one can directly see that (2.11) holds. Next by a limiting argument, one can get (2.11) for $\mathbf{f} \in \mathcal{W}_0 \cap \mathcal{W}_2$. $\square$

Define

$$\begin{aligned} A(\mathbf{f}, p) &\equiv \frac{1}{\sqrt{2}} (B^\dagger(\mathbf{f}, p) + B(\bar{\mathbf{f}}, p)), \\ \Pi(\mathbf{f}, p) &\equiv \frac{i}{\sqrt{2}} (B^\dagger(\mathbf{f}, p) - B(\bar{\mathbf{f}}, p)), \quad \mathbf{f} \in \mathcal{W}_0. \end{aligned}$$

We can easily see that the operators $A(\mathbf{f}, p)|_{\mathcal{F}^\infty(\mathcal{W})}$ and $\Pi(\mathbf{f}, p)|_{\mathcal{F}^\infty(\mathcal{W})}$ are essentially self-adjoint by the Nelson analytic vector theorem [10, Theorem X.39]. We denote the self-adjoint extensions by the same symbols.

Theorem 2.8 Suppose $\vec{p} \in \tilde{P}$. Then for $\mathbf{f} \in \mathcal{W}_0$

$$\exp\left(i\frac{t}{\hbar}H_{\vec{p}}(p)\right) A(\mathbf{f}, p) \exp\left(-i\frac{t}{\hbar}H_{\vec{p}}(p)\right) = A(e^{i\tilde{\omega}t}\mathbf{f}, p), \quad (2.12)$$

$$\exp\left(i\frac{t}{\hbar}H_{\vec{p}}(p)\right) \Pi(\mathbf{f}, p) \exp\left(-i\frac{t}{\hbar}H_{\vec{p}}(p)\right) = \Pi(e^{i\tilde{\omega}t}\mathbf{f}, p). \quad (2.13)$$

Proof: We only show an outline of the proof. For simplicity, put $A(e^{i\tilde{\omega}t}\mathbf{f}, p) = A(\mathbf{f}, p, t)$. Let $C^\infty(H_b) = \cap_{n=1}^\infty D(H_b^n)$. We can easily see that, by Lemma 2.7, $\langle e^{iA(\mathbf{f}, p, t)}\Psi, \Phi \rangle$, $\Psi, \Phi \in C^\infty(H_b) \cap \mathcal{F}^\infty(\mathcal{W})$, $\mathbf{f} \in \mathcal{W}_{-2} \cap \mathcal{W}_0 \cap \mathcal{W}_2$, is differentiable in t with

$$\frac{d}{dt} \langle e^{iA(\mathbf{f}, p, t)}\Psi, \Phi \rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \frac{i}{\hbar} e^{iA(\mathbf{f}, p, t)}\Psi, H_{\vec{p}}(p)\Phi \right\rangle_{\mathcal{F}(\mathcal{W})} - \left\langle \frac{i}{\hbar} H_{\vec{p}}(p)\Psi, e^{-iA(\mathbf{f}, p, t)}\Phi \right\rangle_{\mathcal{F}(\mathcal{W})} \quad (2.14)$$

From (2.14) it follows that

$$\frac{d}{dt} \langle e^{-i\frac{t}{\hbar}H_{\vec{p}}(p)} e^{iA(\mathbf{f}, p, t)} e^{i\frac{t}{\hbar}H_{\vec{p}}(p)} \Psi, \Phi \rangle_{\mathcal{F}(\mathcal{W})} = 0, \quad \Psi, \Phi \in D(H_b), \mathbf{f} \in \mathcal{W}_0.$$

Hence

$$e^{isA(\mathbf{f}, p, 0)} = e^{i\frac{t}{\hbar}H_{\vec{p}}(p)} e^{isA(\mathbf{f}, p, t)} e^{-i\frac{t}{\hbar}H_{\vec{p}}(p)}, \quad \text{on } C^\infty(H_b) \cap \mathcal{F}^\infty(\mathcal{W}). \quad (2.15)$$

By a limiting argument, one can see that (2.15) holds for $\Phi, \Psi \in D(H_b), \mathbf{f} \in \mathcal{W}_0$. Since the both sides of (2.15) are one parameter unitary groups in $s \in \mathbb{R}$, Stone's theorem yields (2.12). (2.13) is quite similar to (2.12). Thus we get the desired result. $\square$

For $t \in \mathbb{R}$, we define operators in $\mathcal{F}(\mathcal{W})$ by

$$\begin{aligned} A_\mu(f, t|p) &= \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left\{ B^{\dagger(r)} \left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}} e^{ic\omega t} e_\nu^r \bar{T}_{\mu\nu} \hat{f}, p \right) + B^{(r)} \left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}} e^{-ic\omega t} e_\nu^r T_{\mu\nu} \tilde{\hat{f}}, p \right) \right\} \\ &\quad - e \sum_{i,j=1}^N \hbar p_\nu^i \left\langle d_{\mu\nu} D_{ij}^{-1}(0) \frac{\hat{\rho}_j}{\sqrt{(c\omega)^3}}, \frac{\hat{f}}{\sqrt{c\omega}} \right\rangle_{L^2(\mathbb{R}^d)}, \quad \hat{f} \in M_{-1}, \mu = 1, \dots, d. \end{aligned}$$

Form Lemma 2.5 (2) (5) and (6) it follows that

$$A_\mu(f, 0|p) = A_\mu(f). \quad (2.16)$$

Corollary 2.9 Suppose $\vec{p} \in \tilde{P}$. Then the operator $A_\mu(f, t|p)$ is the Heisenberg field with

$$\exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) A_\mu(f) \exp \left(-i \frac{t}{\hbar} H_{\vec{p}}(p) \right) = A_\mu(f, t|p). \quad (2.17)$$

Proof: It is enough to show (2.17) for a real-valued function f . For a real-valued function f such that $\hat{f} \in M_{-1}$, we can see that

$$\begin{aligned} A_\mu(f) &= \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \left\{ B^{\dagger(r)} \left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}} e_\nu^r \bar{T}_{\mu\nu} \hat{f}, p \right) + B^{(r)} \left(\frac{\sqrt{\hbar}}{\sqrt{c\omega}} e_\nu^r T_{\mu\nu} \hat{f}, p \right) \right\} \\ &\quad - e \sum_{i,j=1}^N \hbar p_\nu^i \left\langle d_{\mu\nu} D_{ij}^{-1}(0) \frac{\hat{\rho}_j}{\sqrt{(c\omega)^3}}, \frac{\hat{f}}{\sqrt{c\omega}} \right\rangle. \end{aligned}$$

Hence (2.17) follows from Theorem 2.8. $\square$

Corollary 2.10 Suppose $\vec{p} \in \tilde{P}$. Then for $\Phi \in D(H_b)$,

$$\exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) B^\sharp(\mathbf{f}, p) \exp \left(-i \frac{t}{\hbar} H_{\vec{p}}(p) \right) \Phi = B^\sharp(e^{ic\omega t} \mathbf{f}, p) \Phi$$

Proof: Note that $D(B^\sharp(\mathbf{f}, p)) \subset D(H_b) = D(H_{\bar{p}}(p))$. Thus from (2.12) and (2.13) it follows that on $D(H_b)$

$$\begin{aligned} e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}A(\mathbf{f}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} &= \frac{1}{\sqrt{2}} \left\{ e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}B^\dagger(\mathbf{f}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} + e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}B(\bar{\mathbf{f}}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} \right\}, \\ &= \frac{1}{\sqrt{2}} \left\{ B^\dagger(e^{i\tilde{\omega}t}\mathbf{f}) + B(e^{-i\tilde{\omega}t}\bar{\mathbf{f}}) \right\}, \\ e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}\Pi(\mathbf{f}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} &= \frac{i}{\sqrt{2}} \left\{ e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}B^\dagger(\mathbf{f}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} - e^{i\frac{t}{\hbar}H_{\bar{p}}(p)}B(\bar{\mathbf{f}}, p)e^{-i\frac{t}{\hbar}H_{\bar{p}}(p)} \right\} \\ &= \frac{i}{\sqrt{2}} \left\{ B^\dagger(e^{i\tilde{\omega}t}\mathbf{f}) - B(e^{-i\tilde{\omega}t}\bar{\mathbf{f}}) \right\}. \end{aligned}$$

Thus the corollary follows. $\square$

3 BOGOLIUBOV TRANSFORMATIONS AND SCALING LIMITS

In this section, we construct a unitary operator which implements a unitary equivalence of the Pauli-Fierz Hamiltonian and a decoupled Hamiltonian. Moreover we investigate a scaling limit of the Pauli-Fierz Hamiltonian. Unless otherwise stated in this section, we suppose that $\kappa > 0$ is sufficiently large. Since the bounded operators $W_-^{r,s}(\kappa)$ have integral kernels

$$W_-^{(r,s)}(k, k', \kappa) = \frac{1}{\kappa^3} \frac{e^2}{c^2} \frac{e_\mu^r(k) e_\mu^s(k') \sum_{j=1}^N \hat{\rho}_j(k) \bar{Q}_j(k', \kappa)}{2(|k| + |k'|)(|k||k'|)^{\frac{1}{2}}},$$

such that $W_-^{(r,s)}(\kappa) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, the operator $\mathbf{W}_-(\kappa)$ is a Hilbert Schmidt operator on $\mathcal{W}$. Then from (2.9) and (2.10) it follows that there exist two unitary operators $U(\kappa)$ (p independent) and $S(p, \kappa)$ such that ([6, Section III])

$$U^{-1}(\kappa)S(p, \kappa)^{-1}B^\sharp(\mathbf{f}, p, \kappa)S(p, \kappa)U(\kappa) = a^\sharp(\mathbf{f}), \quad \mathbf{f} \in \mathcal{W}. \quad (3.1)$$

Concretely $S(p, \kappa)$ is given by

$$S(p, \kappa) = \exp \left(\sum_{i,j=1}^N \frac{e\hbar}{\kappa^2} p_\mu^i \left\{ a \left(\bigoplus_{r=1}^{d-1} \frac{e_\mu^r D_{ij}^{-1}(0, \kappa) \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) - a^\dagger \left(\bigoplus_{r=1}^{d-1} \frac{e_\mu^r D_{ij}^{-1}(0, \kappa) \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) \right\} \right).$$

Theorem 3.1 Suppose $\vec{p} \in \tilde{P}$. Then putting $S(p, \kappa)U(\kappa) = \mathcal{U}(p, \kappa)$, we see that $\mathcal{U}(p, \kappa)$ maps $D(H_b)$ onto itself with

$$\mathcal{U}(p, \kappa)H_{\vec{p}}(p, \kappa)\mathcal{U}^{-1}(p, \kappa) = \kappa H_b + E(p, \kappa), \quad (3.2)$$

where

$$\begin{aligned} E(p, \kappa) &= \frac{\hbar^2}{2m} \sum_{i=1}^N \sum_{\mu=1}^d \left(\kappa p_{\mu}^i + \kappa \tilde{p}_{\mu}^i(\kappa) \right)^2 + \square(\kappa), \\ \tilde{p}_{\mu}^i(\kappa) &= \sum_{j=1}^N p_{\nu}^j \Delta_{\nu\mu}^{ji}(\kappa), \\ \Delta_{\nu\mu}^{ji}(\kappa) &= \frac{1}{\kappa^3} \frac{e^2}{2c^2} \sum_{k=1}^N \sum_{r,s=1}^{d-1} \left\langle \frac{e_{\nu}^r D_{jk}^{-1}(0, \kappa) \hat{\rho}_k}{\sqrt{\omega^3}}, \left(I + \mathbf{W}_{-}(\kappa) \mathbf{W}_{+}^{-1}(\kappa) \right)^{(r,s)} \frac{e_{\mu}^s \hat{\rho}_i}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)}, \\ \square(\kappa) &= \frac{e^2 \hbar}{4mc} \sum_{i=1}^N \sum_{r,s=1}^{d-1} \left\langle \frac{e_{\mu}^r \hat{\rho}_i}{\sqrt{\omega}}, \left(I - \mathbf{W}_{-}(\kappa) \mathbf{W}_{+}^{-1}(\kappa) \right)^{(r,s)} \frac{e_{\mu}^s \hat{\rho}_i}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Proof: For simplicity, we omit the symbol κ . Put $\mathcal{U}(p)\Omega \equiv \Omega(p)$. From [6, Proposition 2.4, Lemma 5.9] it follows that $\Omega(p) \in D(H_b)$. Then $\Omega(p) \in D(B(\mathbf{f}, p))$. By virtue of Corollary 2.10 and (3.1), we can see that for all $\mathbf{f} \in \mathcal{W}$

$$B(\mathbf{f}, p) \exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) \Omega(p) = 0. \quad (3.3)$$

The equation (3.3) implies that there exists a positive constant $E(p)$ such that

$$\exp \left(i \frac{t}{\hbar} H_{\vec{p}}(p) \right) \Omega(p) = \exp \left(i \frac{t}{\hbar} E(p) \right) \Omega(p). \quad (3.4)$$

Hence from Corollary 2.10, (3.1), (3.4) and the denseness of

$$\mathcal{L} \left\{ B^{\dagger}(\mathbf{f}_1) \dots B^{\dagger}(\mathbf{f}_n) \Omega(p), \Omega(p) \mid \mathbf{f}_j \in \mathcal{W}, j = 1, \dots, n, n \geq 1 \right\},$$

one can get (3.2) (we refer to [6, Lemma 5.12]). Noting that ([6, Lemma 2.2])

$$a \left(\bigoplus_{r=1}^{d-1} \frac{\sqrt{\hbar} e_{\mu}^r \hat{\rho}_i}{\sqrt{2c\omega}} \right) \Omega(p) = \left\{ -\tilde{p}_{\mu}^i - a^{\dagger} \left(\bigoplus_{r=1}^{d-1} \sum_{s=1}^{d-1} \left(\mathbf{W}_{-} \mathbf{W}_{+}^{-1} \right)^{(r,s)} \frac{\sqrt{\hbar} e_{\mu}^s \hat{\rho}_i}{\sqrt{2c\omega}} \right) \right\} \Omega(p),$$

one can easily get $E(p)$ by

$$E(p) = \frac{\langle H_{\bar{p}}(p)\Omega(p), \Omega \rangle_{\mathcal{F}(\mathcal{W})}}{\langle \Omega(p), \Omega \rangle_{\mathcal{F}(\mathcal{W})}}.$$

This completes the proof. $\square$

The positive constant $E(p, \kappa)$ can be rewritten as

$$E(p, \kappa) = \frac{\kappa^2 \hbar^2}{2m} p^2 + E^{REN}(p, \kappa) + \tilde{E}(p, \kappa),$$

where

$$\begin{aligned} \tilde{E}(p, \kappa) &= \frac{\kappa^2 \hbar^2}{2m} \sum_{i,j=1}^N \sum_{\mu,\nu=1}^d p_{\mu}^i b_{\mu\nu}^{ij}(\kappa) p_{\nu}^j, \\ b_{\mu\nu}^{ij}(\kappa) &= \sum_{k=1}^N \sum_{\alpha=1}^d \left(\frac{\Delta_{\nu\alpha}^{jk}(\kappa) + \overline{\Delta_{\nu\alpha}^{jk}(\kappa)}}{2} \right) \left(\frac{\Delta_{\mu\alpha}^{ik}(\kappa) + \overline{\Delta_{\mu\alpha}^{ik}(\kappa)}}{2} \right), \\ E^{REN}(p, \kappa) &= E(p, \kappa) - \frac{\kappa^2 \hbar^2}{2m} p^2 - \tilde{E}(p, \kappa). \end{aligned} \tag{3.5}$$

Let $M(K)$ be the set of $K \times K$ complex matrices. Note that since $(b_{\mu\nu}^{ij}(\kappa))_{1 \leq i,j \leq N, 1 \leq \mu,\nu \leq d} \in M(N) \otimes M(d) \cong M(dN)$ is nonnegative and symmetric, we have $\tilde{E}(p, \kappa) \geq 0$ for $p \in \mathbb{R}^{dN}$.

We define $H_{\bar{p}}^{REN}(p, \kappa)$ and $\widetilde{H}_{\bar{p}}(p, \kappa)$ by

$$\begin{aligned} H_{\bar{p}}^{REN}(p, \kappa) &= -E^{REN}(p, \kappa) + \kappa H_b + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-2\kappa e \hbar p_{\mu}^j A_{\mu}(\rho_j) + e^2 A_{\mu}(\rho_j)^2 \right), \\ \widetilde{H}_{\bar{p}}(p, \kappa) &= \tilde{E}(p, \kappa) + \kappa H_b. \end{aligned}$$

Then one can see that

$$\begin{aligned} H_{\bar{p}}^{REN}(\kappa) &\equiv \mathbf{F}^{-1} \left(\int_{\mathbb{R}^{dN}}^{\oplus} H_{\bar{p}}^{REN}(p, \kappa) dp \right) \mathbf{F} \\ &= -E^{REN}(D, \kappa) \otimes I + \kappa I \otimes H_b \\ &\quad + \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left(-2\kappa e \hbar D_{\mu}^j \otimes A_{\mu}(\rho_j) + e^2 I \otimes A_{\mu}(\rho_j)^2 \right), \\ \widetilde{H}_{\bar{p}}(\kappa) &\equiv \mathbf{F}^{-1} \left(\int_{\mathbb{R}^{dN}}^{\oplus} \widetilde{H}(p) dp \right) \mathbf{F} \\ &= \tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b, \end{aligned}$$

where $E^{REN}(D, \kappa)$ and $\tilde{E}(D, \kappa)$ are pseudo differential operators on $L^2(\mathbb{R}^{dN})$ with symbols $E^{REN}(p, \kappa)$ and $\tilde{E}(p, \kappa)$ respectively.

Theorem 3.2 Suppose $\tilde{\rho} \in \tilde{P}$. Then $H_{\tilde{\rho}}^{REN}(\kappa)$ and $\widetilde{H_{\tilde{\rho}}}(\kappa)$ are essentially self-adjoint on any core of H_0 and bounded from below.

Proof: By the definition of $E^{REN}(D, \kappa)$ and $\tilde{E}(D, \kappa)$, $H_{\tilde{\rho}}^{REN}(\kappa)$ and $\widetilde{H_{\tilde{\rho}}}(\kappa)$ are symmetric. For $f \in D(-\Delta)$, there exist $d_1(\kappa)$ and $d_2(\kappa)$ such that

$$\begin{aligned} \|\tilde{E}(D, \kappa)f\|_{L^2(\mathbb{R}^{dN})} &\leq d_1(\kappa)\|-\Delta f\|_{L^2(\mathbb{R}^{dN})}, \\ \|E^{REN}(D, \kappa)f\|_{L^2(\mathbb{R}^{dN})} &\leq d_2(\kappa)\|-\Delta f\|_{L^2(\mathbb{R}^{dN})}. \end{aligned}$$

Hence, similar to the proof of Theorem 2.2, the Nelson commutator theorem yields desired results. $\square$

Remark 3.3 Write

$$E(p, \kappa) = \frac{\hbar^2 \kappa^2}{2m} p^2 + \sum_{\mu=1}^d \sum_{i=1}^N \frac{\hbar^2 \kappa^2}{m} p_{\mu}^i \tilde{p}_{\mu}^i(\kappa) + \sum_{\mu=1}^d \sum_{i=1}^N \frac{\hbar^2 \kappa^2}{2m} \tilde{p}_{\mu}^i(\kappa)^2 + \square(\kappa). \quad (3.6)$$

Then the first and second terms on the right hand side of (3.6) diverge as $\kappa \rightarrow \infty$ for $p \neq 0$, but the rest terms not. Actually we see that

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \frac{\hbar^2 \kappa^2}{2m} \sum_{\mu=1}^d \sum_{i=1}^N \tilde{p}_{\mu}^i(\kappa)^2 &= \frac{1}{2m} \left(\frac{e^2}{2mc^2} \right) \left(\frac{d-1}{d} \right)^2 \sum_{\alpha=1}^d \sum_{k=1}^N \left(\sum_{j=1}^N \hbar p_{\alpha}^j \left\langle \frac{\hat{\rho}_j}{\sqrt{\omega^3}}, \frac{\hat{\rho}_k}{\sqrt{\omega}} \right\rangle_{L^2(\mathbb{R}^d)} \right)^2, \\ &\equiv E^{\infty}(p). \end{aligned}$$

Then, by (3.2), concerning an asymptotic behavior of $H_{\tilde{\rho}}(\kappa)$ as $\kappa \rightarrow \infty$, we should subtract the first and second terms in the right hand side of (3.6) from the original Hamiltonian $H_{\tilde{\rho}}(\kappa)$. However one can not say that $\tilde{p}_{\mu}^i(\kappa)^2$ is real and nonnegative for any $p \in \mathbb{R}^{dN}$. To guarantee the nonnegative self-adjointness of the Hamiltonian $H_{\tilde{\rho}}^{REN}(p, \kappa)$ with the divergence terms subtracted, we should define $\tilde{E}(p, \kappa)$ such as (3.5). In this sense, we may say that the operator $H_{\tilde{\rho}}^{REN}(\kappa)$ has an interpretation of the Hamiltonian $H_{\tilde{\rho}}(\kappa)$ with the infinite self-energy of the nonrelativistic particles subtracted.

We define

$$\mathcal{U}(\kappa) = \mathbf{F}^{-1} \left(\int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{U}(\kappa, p) dp \right) \mathbf{F}.$$

Then we have the following theorem.

Theorem 3.4 *Suppose that $\vec{\rho} \in \tilde{P}$. Then*

$$\begin{aligned} s - \lim_{\kappa \rightarrow \infty} \mathcal{U}(\kappa) &= \exp \left(\sum_{j=1}^N \frac{e\hbar}{m} D_{\mu}^j \otimes \left\{ a \left(\bigoplus_{r=1}^{d-1} \frac{e_{\mu}^r \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) - a^{\dagger} \left(\bigoplus_{r=1}^{d-1} \frac{e_{\mu}^r \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) \right\} \right), \\ &\equiv \mathcal{U}(\infty). \end{aligned} \quad (3.7)$$

Proof: From [6, Theorem 3.11] it follows (3.7). $\square$

We take scalar potentials V to be real-valued measurable functions on $\mathbb{R}^{dN}$ and put

$$C_{\kappa}(V) = \mathcal{U}^{-1}(\kappa)(V \otimes I)\mathcal{U}(\kappa), \quad C(V) = \mathcal{U}^{-1}(\infty)(V \otimes I)\mathcal{U}(\infty). \quad (3.8)$$

We introduce conditions **(V-1)** and **(V-2)** as follows.

(V-1) For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(V)$ and for $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda \rightarrow \infty} \|V(\tilde{E}(D, \kappa) + \lambda)^{-1}\| = 0, \quad (3.9)$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2) For $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in κ and

$$s - \lim_{\kappa \rightarrow \infty} V(\tilde{E}(D, \kappa) + \lambda)^{-1} = V(E^{\infty}(D) + \lambda)^{-1}.$$

The condition (3.9) yields that, by the Kato-Rellich theorem and commutativity of $\mathcal{U}(\kappa)$ and $(\tilde{E}(D, \kappa) + \lambda)^{-1}$, operators $\tilde{E}(D, \kappa) \otimes I + C_{\kappa}(V)$ are essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. Moreover since $I \otimes H_b$ is nonnegative and commute with $\tilde{E}(D, \kappa) \otimes I$, one can see that

$$\widetilde{H}_{\vec{\rho}}(V, \kappa) \equiv \tilde{E}(D, \kappa) \otimes I + C_{\kappa}(V) + \kappa I \otimes H_b$$

is essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. In particular, $D(H_0)$ is a core of $\widetilde{H_{\tilde{\rho}}}(V, \kappa)$. Put

$$H_{\tilde{\rho}}^{REN}(V, \kappa) \equiv H_{\tilde{\rho}}^{REN}(\kappa) + V \otimes I.$$

Theorem 3.5 *Let $\tilde{\rho} \in \tilde{P}$. Suppose that V satisfies $(\mathbf{V} - 1)$ and $(\mathbf{V} - 2)$. Then, for sufficiently large $\kappa > 0$, the operator $H_{\tilde{\rho}}^{REN}(V, \kappa)$ is essentially self-adjoint on $D(H_0)$ and bounded from below uniformly in sufficiently large $\kappa > 0$. Moreover the unitary operator $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself and for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$(H_{\tilde{\rho}}^{REN}(V, \kappa) - z)^{-1} = \mathcal{U}(\kappa) (\widetilde{H_{\tilde{\rho}}}(V, \kappa) - z)^{-1} \mathcal{U}^{-1}(\kappa). \quad (3.10)$$

Proof: Since $\mathcal{U}(\kappa)$ maps $D(I \otimes H_b)$ onto itself (see Theorem 3.1) and $-\Delta \otimes I$ commutes with $\mathcal{U}(\kappa)$ on $D(-\Delta \otimes I)$, $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself. Put

$$S_0^\infty(\mathbb{R}^{dN}) = \{f \in L^2(\mathbb{R}^{dN}) \mid \hat{f} \in C_0^\infty(\mathbb{R}^{dN})\}.$$

At first, by Theorem 3.1, we see that for $\Phi \in S_0^\infty(\mathbb{R}^{dN}) \hat{\otimes} D(H_b)$,

$$H_{\tilde{\rho}}^{REN}(V, \kappa)\Phi = \mathcal{U}(\kappa) \widetilde{H_{\tilde{\rho}}}(V, \kappa) \mathcal{U}^{-1}(\kappa) \Phi. \quad (3.11)$$

By a limiting argument we can extend (3.11) to $\Phi \in D(H_0)$. Since $D(H_0)$ is a core of $\widetilde{H_{\tilde{\rho}}}(V, \kappa)$ and $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself, the right hand side of (3.11) is essentially self-adjoint on $D(H_0)$. So is the left hand side of (3.11). (3.10) can be easily shown. $\square$

We want to consider a scaling limit of $H_{\tilde{\rho}}^{REN}(V, \kappa)$ as $\kappa \rightarrow \infty$. In [5], a general theory of the strong resolvent limit of self-adjoint operators including abstract versions like as the self-adjoint operator $\widetilde{H_{\tilde{\rho}}}(V, \kappa)$ has been established. We shall apply the theory in [5] with a little modification. Let V satisfy $(\mathbf{V} - 1)$. Then since $D(C(V)) \supset D(-\Delta) \hat{\otimes} D(H_b)$, one can define, for $\Phi \in \mathcal{F}(\mathcal{W})$ and $\Psi \in D(H_b)$, a symmetric operator $E_{\Phi, \Psi}(C(V))$ with $D(E_{\Phi, \Psi}(C(V))) = D(-\Delta)$ by

$$\langle f, E_{\Phi, \Psi}(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f \otimes \Phi, C(V)(g \otimes \Psi) \rangle_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^{dN}), g \in D(-\Delta).$$

In particular, we call $E_{\Omega,\Omega}(C(V)) \equiv E_{\Omega}(C(V))$ “the partial expectation of $C(V)$ with respect to Ω ” ([5, Section II]).

Theorem 3.6 *Let $\vec{\rho} \in \tilde{P}$. Suppose that V satisfies the conditions (V-1) and (V-2). Then for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$s - \lim_{\kappa \rightarrow \infty} (H_{\vec{\rho}}^{REN}(V, \kappa) - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^{\infty}(D) + E_{\Omega}(C(V)) - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty), \quad (3.12)$$

where P_0 is the projection from $\mathcal{F}(\mathcal{W})$ to the one dimensional subspace $\{\alpha\Omega | \alpha \in \mathbb{C}\}$.

Proof: By (V-1) and (V-2), we see that

(V-1)' For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(C_{\kappa}(V))$ and for $\lambda > 0$,

$C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda \rightarrow \infty} \|C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}\| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2)' For $\lambda > 0$, $C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in κ and

$$s - \lim_{\kappa \rightarrow \infty} C_{\kappa}(V)(\tilde{E}(D, \kappa) + \lambda)^{-1} = C(V)(E^{\infty}(D) + \lambda)^{-1}.$$

From [5, Section II], (V-1)' and (V-2)' imply that

$$s - \lim_{\kappa \rightarrow \infty} (\widetilde{H_{\vec{\rho}}}(V, \kappa) - z)^{-1} = (E^{\infty}(D) + E_{\Omega}(C(V)) - z)^{-1} \otimes P_0.$$

Thus by Theorems 3.4 and 3.5, we get (3.12). $\square$

We want to see $E_{\Omega}(C(V))$ more explicitly. For $\vec{\rho} \in \tilde{P}$, let $\tilde{\Delta}^{\infty} = (\tilde{\Delta}_{ij}^{\infty})_{1 \leq i, j \leq d}$, where $\tilde{\Delta}_{ij}^{\infty}$ is defined in (1.3). Let $\mathbf{I}_{d \times d}$ denote $d \times d$ -identity matrix. Since $\Delta^{\infty} \equiv \tilde{\Delta}^{\infty} \otimes \mathbf{I}_{d \times d} \in$

$M(N) \otimes N(d) \cong M(dN)$ is a nonnegative symmetric matrix, there exist unitary matrices $\mathbf{T} \in M(dN)$ so that

$$\mathbf{T} \Delta^\infty \mathbf{T}^{-1} = \begin{pmatrix} \lambda_1 \mathbf{I}_{d \times d} & & & \\ & \lambda_2 \mathbf{I}_{d \times d} & & \\ & & \ddots & \\ & & & \lambda_N \mathbf{I}_{d \times d} \end{pmatrix}, \quad (3.13)$$

where $\lambda_1 \geq \lambda_2 \dots \geq \lambda_N \geq 0$.

Theorem 3.7 Suppose $\lambda_1 \geq \lambda_2 \dots \geq \lambda_K > 0$, $\lambda_{K+1} = \dots = \lambda_N = 0$ and fix a unitary operator $\mathbf{T}$ in (3.13). Let $x = (x_1, \dots, x_N)$, $x_j \in \mathbb{R}^d$, $j = 1, \dots, N$ and V satisfy

$$\int_{\mathbb{R}^{dK}} dy_1 \dots dy_K |V| \circ \mathbf{T}^{-1}(y_1, \dots, y_K, (\mathbf{T}x)_{K+1}, \dots, (\mathbf{T}x)_N) \exp\left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 \dots \lambda_K}\right) < \infty. \quad (3.14)$$

Moreover we suppose that the left hand side of (3.14) is locally bounded. Then the partial expectation $E_\Omega(C(V))$ is given by a multiplication operator V_{eff} ;

$$\begin{aligned} V_{eff}(x) &= (2\pi \lambda_1 \dots \lambda_K)^{-\frac{d}{2}} \int_{\mathbb{R}^{dK}} dy_1 \dots dy_K V \circ \mathbf{T}^{-1}(y_1, \dots, y_K, (\mathbf{T}x)_{K+1}, \dots, (\mathbf{T}x)_N) \\ &\quad \times \exp\left(-\frac{\sum_{j=1}^K |(\mathbf{T}x)_j - y_j|^2}{2\lambda_1 \dots \lambda_K}\right). \end{aligned}$$

In particular, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, V_{eff} is given by

$$V_{eff}(x) = (2\pi \det \tilde{\Delta}^\infty)^{-\frac{d}{2}} \int_{\mathbb{R}^{dN}} V(y) \exp\left(-\frac{|x - y|^2}{2 \det \tilde{\Delta}^\infty}\right) dy.$$

Proof: Suppose $V \in \mathcal{S}(\mathbb{R}^{dN})$, which is the set of the rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^{dN}$. Then the direct calculation shows that for $f, g \in L^2(\mathbb{R}^{dN})$

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f, V_{eff}g \rangle_{L^2(\mathbb{R}^{dN})}. \quad (3.15)$$

We next consider the case where V is bounded. In this case we can approximate V by a sequence $\{V_n\}_{n=1}^\infty$, $V_n \in \mathcal{S}(\mathbb{R}^{dN})$, such that

$$\|V - V_n\|_\infty \rightarrow 0 \quad (n \rightarrow \infty),$$

where $\|\cdot\|_\infty$ denotes the sup norm. Then we have

$$E_\Omega(C(V_n)) \rightarrow E_\Omega(C(V)) \quad (n \rightarrow \infty),$$

strongly. Moreover $(V_n)_{eff}(x) \rightarrow V_{eff}(x)$ for all $x \in \mathbb{R}^{dN}$. Thus for $f, g \in L^2(\mathbb{R}^{dN})$, (3.15) follows for such V . Finally, let V satisfy (3.14). Define

$$V_n = \begin{cases} V(x) & |V(x)| \leq n, \\ n & |V(x)| > n. \end{cases}$$

Hence for $f \in L^2(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$, we have

$$\langle f, E_\Omega(C(V_n))g \rangle_{L^2(\mathbb{R}^{dN})} \rightarrow \langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} \quad (n \rightarrow \infty).$$

On the other hand, since the left hand side of (3.14) is locally bounded, we can see that for $f \in C_0^\infty(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$,

$$\langle f, (V_n)_{eff}g \rangle_{L^2(\mathbb{R}^{dN})} \rightarrow \langle f, V_{eff}g \rangle_{L^2(\mathbb{R}^{dN})} \quad (n \rightarrow \infty),$$

which completes the proof. □

Remark 3.8 In Theorem 3.7, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, since the left hand side of (3.14) is continuous in $x \in \mathbb{R}^{dN}$, it is necessarily locally bounded.

We call V_{eff} “the effective potential with respect to V ”. We give some examples of scalar potentials V and ultraviolet cut off functions ρ .

Example 3.9 ([non-degenerate case]) Let

$$\tilde{\Delta}_{ij}^\infty = \delta_{ij} \frac{1}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}_i(k)^2}{\omega(k)^3}.$$

Then there exist positive constants δ_1 and δ_2 such that for sufficiently large $\kappa > 0$

$$\delta_1 |p|^2 \leq \tilde{E}(p, \kappa) \leq \delta_2 |p|^2. \quad (3.16)$$

Let $d = 3$ and V be the Coulomb potential;

$$V(x_1, \dots, x_N) = - \sum_{j=1}^N \frac{\alpha_j}{|x_j|} + \sum_{i \neq j} \frac{\beta_{ij}}{|x_i - x_j|}, \quad \alpha_j \geq 0, \beta_{ij} \geq 0.$$

Then V is the Kato class potential ([10], Theorem X.16). Namely for any $\epsilon > 0$, there exists $b \geq 0$ such that $D(V) \supset D(-\Delta)$ and

$$\|V\Phi\|_{L^2(\mathbb{R}^{3N})} \leq \epsilon \|-\Delta\Phi\|_{L^2(\mathbb{R}^{3N})} + b \|\Phi\|_{L^2(\mathbb{R}^{3N})}. \quad (3.17)$$

Together with (3.16) and (3.17), one can see that V satisfies $(\mathbf{V} - 1)$, $(\mathbf{V} - 2)$ and for any $t > 0$

$$\int_{\mathbb{R}^{3d}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Then the scaling limit of the Pauli-Fierz Hamiltonian with the Coulomb potential exists and has the effective potential given by

$$V_{eff}(x) = (2\pi\gamma)^{-\frac{3}{2}} \int_{\mathbb{R}^{3N}} V(y) e^{-\frac{|x-y|^2}{2\gamma}} dy,$$

$$\gamma = \left\{ \frac{1}{3} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \right\}^N \prod_{j=1}^N \left(\int_{\mathbb{R}^3} dk \frac{\hat{\rho}_j^2(k)}{\omega(k)^3} \right).$$

Moreover

$$E^\infty(D) = -\frac{1}{2m} \left(\frac{e^2}{2mc^2} \right) \left(\frac{d-1}{d} \right)^2 \otimes_{j=1}^N \left\| \frac{\hat{\rho}_j}{\omega} \right\|^4 \hbar^2 \Delta_j,,$$

where Δ_j , $j = 1, \dots, N$, is the Laplacian in $L^2(\mathbb{R}^d)$.

Example 3.10 ([non-degenerate case]) Let $\tilde{\Delta}^\infty$ be non-degenerate and V be the Phillips perturbation with respect to $-\Delta$ ([12]). Then (3.16) holds with some δ_1 and δ_2 . Hence V satisfies $(\mathbf{V} - 1)$, $(\mathbf{V} - 2)$ and for any $t > 0$

$$\int_{\mathbb{R}^{dN}} |V|(y) e^{-t|x-y|^2} dy < \infty.$$

Hence the scaling limit of the Pauli-Fierz Hamiltonian with Phillips perturbation exists and has the effective potential in Theorem 3.7.

Example 3.11 ([degenerate case]) *Let V be a real-valued bounded function. Then V satisfies the conditions $(V - 1)$ and $(V - 2)$. Hence the scaling limit of the Pauli-Fierz Hamiltonian with the scalar potential V exists for all $\vec{\rho} \in \tilde{P}$.*

Example 3.12 ([degenerate case]) *Let $\rho_i = \rho$, $i = 1, \dots, N$ and V satisfy $(V - 1)$, $(V - 2)$ and the assumption stated in Theorem 3.7. Then $\text{rank } \tilde{\Delta}^\infty = 1$ and the non-zero eigenvalue C is given by*

$$C = \frac{N}{2} \frac{d-1}{d} \left(\frac{\hbar}{mc} \right)^2 \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \frac{\hat{\rho}(k)^2}{\omega(k)^3}.$$

Thus the scaling limit of the Pauli-Fierz Hamiltonian with the ultraviolet cut-off function ρ exists and has the following effective potential:

$$V_{eff}(x) = (2\pi C)^{-\frac{d}{2}} \int_{\mathbb{R}^d} dy_1 V \circ \mathbf{T}^{-1}(y_1, (\mathbf{T}x)_2, \dots, (\mathbf{T}x)_N) \exp \left(-\frac{|(\mathbf{T}x)_1 - y_1|^2}{2C} \right).$$

Moreover

$$E^\infty(D) = -\frac{1}{2m} \left(\frac{N^2 e^2}{2mc^2} \right) \left(\frac{d-1}{d} \right)^2 \left\| \frac{\hat{\rho}}{\omega} \right\|^4 \hbar^2 \Delta,$$

where Δ is the Laplacian in $L^2(\mathbb{R}^{dN})$.

4 CONCLUDING REMARK

As is seen in Theorem 3.7, the effective potential V_{eff} is characterized by the matrix-valued functional $\tilde{\Delta}^\infty = \tilde{\Delta}^\infty(\vec{\rho})$, which has the following mathematical meaning; putting

$$\mathcal{U}(\infty)(x_i \otimes I) \mathcal{U}^{-1}(\infty) - x_i \otimes I \equiv \Delta x_i, \quad i = 1, \dots, N,$$

we see that the partial expectation of $\Delta x_i \Delta x_j$ with respect to Ω is as follows;

$$E_\Omega[(\Delta x_i \Delta x_j)] = \tilde{\Delta}_{ij}^\infty(\vec{\rho}) I.$$

In one-nonrelativistic particle case, A.Arai [5] shows that the partial expectation $E_{\Omega}[(\Delta x)^2]$ with respect to Ω may be interpreted as the mean square fluctuation in position of one-nonrelativistic particle ([2]). In this sense, $\tilde{\Delta}_{ij}^{\infty}(\vec{\rho})$ may also be interpreted as correlation of fluctuations in position of the i -th and the j -th nonrelativistic particles under the action of quantized radiation fields.

Figure 1: Cut Plane $\mathbb{C}_{R,\delta,\epsilon}$

5 APPENDIX

In this Appendix we prove Lemma 2.6. For simplicity, in this proof, we omit κ in notations and put

$$\lambda = \frac{e^2}{c^2}, \quad \widehat{G} = \omega^{1-\frac{d}{2}} G \omega^{1-\frac{d}{2}}, \quad \widehat{G}_t = \omega^{1-\frac{d}{2}} G_t \omega^{1-\frac{d}{2}}, t > 0.$$

(1) : This follows from the definition of $T_{\mu\nu}$ and $T_{\mu\nu}^*$, and Definition 2.1 (4).

(2) : For $f \in M_{-1}$,

$$\begin{aligned} \left\langle \omega^{-\frac{3}{2}} Q_i, \omega^{-\frac{1}{2}} d_{\nu\alpha} T_{\mu\nu} f \right\rangle_{L^2(\mathbb{R}^d)} &= \left\langle d_{\mu\alpha} \omega^{-\frac{3}{2}} Q_i, \omega^{-\frac{1}{2}} f \right\rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \sum_{j=1}^N \left\langle d_{\nu\alpha} \omega^{-\frac{3}{2}} Q_i, \lambda \omega^{-\frac{1}{2}} Q_j \widehat{G} d_{\mu\nu} \hat{\rho}_j f \right\rangle_{L^2(\mathbb{R}^d)} \\ &= I + II \end{aligned}$$

Using (2.8), one can see that

$$II = \lim_{t \rightarrow 0} \lambda \sum_{j=1}^N \int \frac{\overline{Q_i}(k) Q_j(k) d_{\mu\nu}(k') d_{\nu\alpha}(k) \hat{\rho}_j(k') f(k')}{(k^2 - k'^2 + it) k^2} dk dk'$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0} \lambda \sum_{j=1}^N \frac{d-1}{d} \mathbf{V}_d \int \frac{\sum_{k,l=1}^N D_{-ik}^{-1}(s) D_{+jl}^{-1}(s) \hat{\rho}_k(\sqrt{s}) \hat{\rho}_l(\sqrt{s}) s^{\frac{d}{2}-1} F_j(k')}{(s - k'^2 + it)s} ds dk' \\
&= \lim_{t \rightarrow 0} \frac{1}{2\pi i} \sum_{j=1}^N \int \frac{1}{(s - k'^2 + it)s} (D_{-ij}^{-1}(s) - D_{+ji}^{-1}(s)) F_j(k') ds dk',
\end{aligned}$$

where $F_j(k') = d_{\mu\alpha}(k') \hat{\rho}_j(k') f(k')$. Using the contour integral on the cut plane $\mathbb{C}_{R,\delta,\epsilon}$ (Figure 1), by (2.5) and (2.6), we have

$$\begin{aligned}
&\frac{1}{2\pi i} \int_0^\infty \frac{1}{(s - k'^2 + it)s} (D_{-ij}^{-1}(s) - D_{+ji}^{-1}(s)) ds \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_\epsilon^\infty \frac{1}{(s - k'^2 + it)s} (D_{-ij}^{-1}(s) - D_{+ji}^{-1}(s)) ds \\
&= \lim_{\epsilon \rightarrow 0} \lim_{R \rightarrow \infty, \delta \rightarrow 0} -\frac{1}{2\pi i} \int_{\mathbb{C}_{R,\delta,\epsilon}} \frac{D_{ij}^{-1}(z)}{(z - k'^2 + it)z} dz - \frac{D_{ij}^{-1}(0)}{-k'^2 + it} \\
&= -\frac{D_{ij}^{-1}(k'^2 - it)}{k'^2 - it} - \frac{D_{ij}^{-1}(0)}{-k'^2 + it}.
\end{aligned}$$

Then

$$\begin{aligned}
II &= \lim_{t \rightarrow 0} \sum_{j=1}^N \int \frac{F_j(k') D_{ij}^{-1}(0)}{k'^2 - it} - \frac{D_{ij}^{-1}(k'^2 - it) F_j(k')}{k'^2 - it} dk' \\
&= -\left\langle d_{\mu\alpha} \omega^{-\frac{3}{2}} Q_i, \omega^{-\frac{1}{2}} f \right\rangle_{L^2(\mathbb{R}^d)} + \left\langle d_{\mu\alpha} \omega^{-\frac{3}{2}} \sum_{j=1}^N D_{ij}^{-1}(0) \hat{\rho}_j, \omega^{-\frac{1}{2}} f \right\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence we get (2).

(3),(4) : They are direct calculations.

(5) : For $f, g \in M_0$,

$$\begin{aligned}
\langle T_{\mu\nu}^* d_{\nu\alpha} T_{\alpha\beta} f, g \rangle_{L^2(\mathbb{R}^d)} &= \langle d_{\mu\beta} f, g \rangle_{L^2(\mathbb{R}^d)} + \lambda \sum_{j=1}^N \langle d_{\nu\beta} f, Q_i \hat{G} d_{\mu\nu} \hat{\rho}_i g \rangle_{L^2(\mathbb{R}^d)} \\
&\quad + \lambda \sum_{j=1}^N \langle d_{\mu\alpha} Q_j \hat{G} d_{\alpha\beta} \hat{\rho}_j f, g \rangle_{L^2(\mathbb{R}^d)} \\
&\quad + \lambda^2 \frac{d-1}{d} \sum_{i,j=1}^N \langle Q_i \hat{G} d_{\alpha\beta} \hat{\rho}_i f, Q_j \hat{G} d_{\mu\alpha} \hat{\rho}_j g \rangle_{L^2(\mathbb{R}^d)} \\
&= I + II + III + IV.
\end{aligned}$$

Then,

$$\begin{aligned} IV &= \lim_{t \rightarrow 0} \lambda^2 \sum_{i,j,k,l=1}^N \mathbf{V}_d \frac{d-1}{d} \int \frac{D_{-ik}^{-1}(s) D_{+jl}^{-1}(s) \hat{\rho}_k(\sqrt{s}) \hat{\rho}_l(\sqrt{s}) s^{\frac{d}{2}-1} F_{ij}(k', k'')}{(s - k'^2 - it)(s - k''^2 + it)} ds dk' dk'', \\ &\equiv \lim_{t \rightarrow 0} IV_t, \end{aligned}$$

where $F_{ij}(k', k'') = d_{\mu\alpha}(k'') \hat{\rho}_j(k'') d_{\alpha\beta}(k') \hat{\rho}_i(k') \bar{f}(k') g(k'')$. By using the cut plane integral method as in (2), we have

$$\begin{aligned} IV_t &= \sum_{i,j=1}^N \int \frac{-\lambda D_{ij}^{-1}(k' + it) F_{ij}(k', k'')}{k'^2 - k''^2 + 2it} + \frac{-\lambda D_{ij}^{-1}(k'' - it) F_{ij}(k', k'')}{k''^2 - k'^2 - 2it} dk' dk'' \\ &= -\lambda \sum_{i,j=1}^N \left\langle d_{\alpha\beta} f, D_{ji}^{-1}(\cdot + it) \hat{\rho}_i \hat{G}_{2t} d_{\mu\alpha} \hat{\rho}_j g \right\rangle_{L^2(\mathbb{R}^d)} \\ &\quad - \lambda \sum_{i,j=1}^N \left\langle d_{\mu\alpha} D_{ij}^{-1}(\cdot + it) \hat{\rho}_j \hat{G}_{2t} d_{\alpha\beta} \hat{\rho}_i f, g \right\rangle_{L^2(\mathbb{R}^d)}. \end{aligned}$$

By a limiting argument as $t \rightarrow 0$, we get

$$\lim_{t \rightarrow 0} IV_t = -II - III.$$

(6) : For $f, g \in M_0$,

$$\begin{aligned} \left\langle e_\mu^r T_{\mu\nu} d_{\nu\alpha} T_{\alpha\beta} e_\beta^s f, g \right\rangle_{L^2(\mathbb{R}^d)} &= \langle \delta_{rs} f, g \rangle_{L^2(\mathbb{R}^d)} - \lambda \langle e_\beta^r \rho_j \hat{G} \bar{Q}_j e_\beta^s f, g \rangle_{L^2(\mathbb{R}^d)} \\ &\quad - \lambda \langle f, e_\mu^s \rho_j \hat{G} \bar{Q}_j e_\mu^r g \rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \lambda^2 \langle d_{\mu\beta} \rho_j \hat{G} \bar{Q}_j e_\beta^s f, \rho_j \hat{G} \bar{Q}_j e_\mu^r g \rangle_{L^2(\mathbb{R}^d)} \\ &= I - II - III + IV. \end{aligned}$$

We see that

$$\begin{aligned} IV &= \lambda^2 \lim_{\epsilon \rightarrow 0} \sum_{i,j=1}^N \mathbf{V}_d \frac{d-1}{d} \int \frac{\hat{\rho}_j(\sqrt{s}) \hat{\rho}_i(\sqrt{s}) s^{\frac{d}{2}-1} H_{ij}(k', k'')}{(s - k'^2 - i\epsilon)(s - k''^2 + i\epsilon)} ds dk' dk'' \\ &= \lambda \lim_{\epsilon \rightarrow 0} \sum_{i,j=1}^N \int \left(\frac{D_{ij}(k'^2 + i\epsilon)}{k'^2 - k''^2 + 2i\epsilon} + \frac{D_{ij}(k''^2 - i\epsilon)}{k''^2 - k'^2 - 2i\epsilon} \right) H_{ij}(k', k'') dk' dk'' \\ &= \lambda \lim_{\epsilon \rightarrow 0} \sum_{i,j=1}^N \langle f, e_\mu^s Q_j D_{ij}(\cdot + i\epsilon) \hat{G}_{2\epsilon} \bar{Q}_i e_\mu^r g \rangle_{L^2(\mathbb{R}^d)} \\ &\quad + \lambda \sum_{i,j=1}^N \langle e_\mu^r Q_i \bar{D}_{ij}(\cdot - i\epsilon) \hat{G}_{2\epsilon} \bar{Q}_j e_\mu^s f, g \rangle_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where $H_{ij}(k', k'') = Q_j(k') \overline{Q}_i(k'') e_\mu^s(k') e_\mu^r(k'') \bar{f}(k') g(k'')$. Note that $\sum_{j=1}^N Q_j D_{+ij} = \hat{\rho}_i$.

Then

$$\begin{aligned} &= \lambda \sum_{i=1}^N \langle f, e_\mu^s \hat{\rho}_i \widehat{G} \overline{Q}_i e_\mu^r g \rangle_{L^2(\mathbb{R}^d)} + \lambda \sum_{j=1}^N \langle e_\mu^r \hat{\rho}_j \widehat{G} \overline{Q}_j e_\mu^s f, g \rangle_{L^2(\mathbb{R}^d)} \\ &= II + III. \end{aligned}$$

Hence we get the desired results. □

6 REFERENCES

- [1] H.A.Bethe, The electromagnetic shift of energy levels, Phys.Rev.72,(1947)339-342.
- [2] T.A.Welton, Some observable effects of the quantum mechanical fluctuations of the electromagnetic field, Phys.Rev.74(1948)1157-1167.
- [3] A.Arai, A note on scattering theory in non-relativistic quantum electrodynamics, J.Phys.A.Math.Gen.16,(1983)49-70.
- [4] A.Arai, Rigorous theory of spectra and radiation for a model in a quantum electrodynamics, J.Math.Phys.24(1983)1896-1910.
- [5] A.Arai, An asymptotic analysis and its applications to the nonrelativistic limit of the Pauli-Fierz and a spin-boson model, J.Math.Phys.31(1990)2653-2663.
- [6] F.Hiroshima, Scaling limit of a model in quantum electrodynamics, J.Math.Phys.34(1993)4478-4578.
- [7] F.Hiroshima, Diamagnetic inequalities for a systems of nonrelativistic particles with a quantized radiation field. to appear Rev.Math.Phys.Vol.8 No.2.
- [8] F.Hiroshima, Functional integral representation of a model in quantum electrodynamics, submitted to J.Funct.Anal..
- [9] E.C.Titchmarsh, Theory of Fourier Integrals, 2nd.ed.(Oxford Univ.Press, Oxford,1948).
- [10] M.Reed, B.Simon, Method of Modern Mathematical Physics II Fourier Analysis and

Self-Adjoint operator, Academic Press(1975).

[11] E.A. Berezin, "Second quantization", Academic Press(1966).

[12] E.B.Davies, Properties of the Green's functions of some Schrödinger operators, J.London Math.Soc.7,483-493(1976).