ABEL-TAUBER THEOREMS FOR FOURIER-STIELTJES COEFFICIENTS

Akihiko Inoue

Series #340. July 1996

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#316 Y. Shibukawa, Vertex-face correspondence in elliptic solutions of the Yang-Baxter equation, 8 pages. 1995.

#317 M.-H. Giga and Y. Giga, Consistency in evolutions by crystalline curvature, 16 pages. 1995.

#318 Wei-Zhi Sun, Shadows of moving surfaces, 19 pages. 1995.

#319 S. Izumiya and G.T. Kossioris, Bifurcations of shock waves for viscosity solutions of Hamilton-Jacobi equations of one space variable, 39 pages. 1995.

#320 T. Teruya, Normal intermediate subfactors, 44 pages. 1995.

#321 M. Ohnuma, Axisymmetric solutions and singular parabolic equations in the theory of viscosity solutions, 26 pages. 1995.

#322 T. Nakazi, An outer function and several important functions in two variables, 12 pages. 1995.

#323 N. Kawazumi, An infinitesimal approach to the stable cohomology of the moduli of Riemann surfaces, 22 pages. 1995.

#324 A. Arai, Factorization of self-adjoint operators by abstract Dirac operators and its application to second quantizations on Boson Fermion Fock spaces, 15 pages. 1995.

#325 K. Sugano, On strongly separable Frobenius extensions, 11 pages. 1995.

#326 D. Lehmann and T. Suwa, Residues of holomorphic vector fields on singular varieties, 21 pages. 1995.

#327 K. Tsutaya, Local regularity of non-resonant nonlinear wave equations, 23 pages. 1996.

#328 T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, 25 pages. 1996.

#329 O. Ogurisu, Anticommutativity and spin 1/2 Schrödinger operators with magnetic fields, 12 pages. 1996.

#330 Y. Kurokawa, Singularities for projections of contour lines of surfaces onto planes, 24 pages. 1996.

#331 M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, 94 pages. 1996.

#332 M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the p-laplace diffusion equation, 20 pages. 1996.

#333 T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.

#334 B. Khanedani and T. Suwa, First variation of horomorphic forms and some applications, 11 pages. 1996.

#335 J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations[1], 28 pages. 1996.

#336 Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.

#337 I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.

#338 F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.

#339 N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.

ABEL-TAUBER THEOREMS FOR FOURIER-STIELTJES COEFFICIENTS

AKIHIKO INOUE

Abstract. We prove Abel-Tauber theorems which link the asymptotics of a function and its Fourier-Stieltjes coefficients. Both cosine and sine coefficients are studied. The results in the cosine case can be applied to stationary time series with long-time memory. The analogues for Fourier-Stieltjes transforms are also given.

1. Introduction and results. We are concerned with relations between the asymptotics of a function and its Fourier-Stieltjes coefficients, and our results are Abel-Tauber theorems of this type. The results we pass from the Fourier-Stieltjes coefficients to the original function are Abelian, while the results in the converse direction are Tauberian. As usual, the latter need an extra condition, known as a *Tauberian condition*, but in the present paper it is weak enough to be non-restrictive.

The class $BV[0,\pi]$ is that of all right-continuous $f : [0,\pi] \to \mathbf{R}$ that have bounded variation on $[0,\pi]$. For $F \in BV[0,\pi]$ we define its *Fourier-Stieltjes cosine coefficients (FS cosine coefficients)*

$$(1.1) \qquad a_n := \frac{2}{\pi}\int_{[0,\pi]} \cos n\theta dF(\theta) \quad (n = 1, 2, \cdots), \quad := \frac{F(\pi)}{\pi} \quad (n = 0),$$

where $dF\{0\} = F(0)$.

We write R_0 for the class of slowly varying functions at infinity: the class of positive, measurable l, defined on some neighbourhood $[X, \infty)$ of infinity, such that

$$\forall\lambda > 0, \quad \lim_{x\to\infty} l(\lambda x)/l(x) = 1.$$

For $l \in R_0$ the class Π_l is the class of measurable g, defined on some neighbourhood $[X, \infty)$ of infinity, satisfying

$$\forall\lambda > 0, \quad \lim_{x\to\infty} \{g(\lambda x) - g(x)\}/l(x) \to c \log\lambda$$

with $c \in \mathbf{R}$ called the *l-index* of f (cf. Bingham et al. [BGT], Ch.3).

A real sequence (c_n) is called *slowly decreasing* if

$$\lim_{\lambda\downarrow 1} \liminf_{n\to\infty} \inf_{n\leq m\leq\lambda n} (c_m - c_n) \geq 0 \qquad (\text{hence} = 0),$$

1991 *Mathematics Subject Classification.* Primary 42A32; Secondary 40E05, 60G10.

slowly increasing if $(-c_n)$ is slowly decreasing. A real sequence (a_n) is said to satisfy the Tauberian condition (T) if

(a_n) is eventually positive, and $(\log a_n)$ is either slowly decreasing or slowly increasing.

For example, (a_n) satisfies (T) if $a_n = n^\rho c_n$, where $\rho \in \mathbf{R}$ and (c_n) is eventually positive and monotone.

THEOREM 1.1. *Let $l \in R_0$ and $0 < \alpha < 1$. Let $F \in BV[0,\pi]$ with FS cosine coefficients (a_n). Then*

$$a_n \sim n^{-\alpha} l(n) \quad (n \to \infty) \tag{1.2}$$

implies

$$F(\theta) \sim \theta^\alpha l(1/\theta) \cdot \frac{\pi}{2\Gamma(\alpha+1)\cos(\pi\alpha/2)} \quad (\theta \to 0+). \tag{1.3}$$

Conversely , (1.3) implies (1.2) if (a_n) satisfies (T).

THEOREM 1.2. *Let $l \in R_0$ and $F \in BV[0,\pi]$ with FS cosine coefficients (a_n). We write $\bar{F}(x) := xF(1/x)$ for $x \geq 1/\pi$. Then*

$$a_n \sim n^{-1} l(n) \qquad (n \to \infty) \tag{1.4}$$

implies

$$\bar{F} \in \Pi_l \text{ with } l\text{-index } 1. \tag{1.5}$$

Conversely, (1.5) implies (1.4) if (a_n) satisfies (T).

The theorems above can be applied to stationary time series. Let $X = (X(n) : n \in \mathbf{Z})$ be a real, weakly stationary time series with expectation zero, and let R be its correlation function: $R(n) = E[X(n)X(0)]$ for $n \in \mathbf{Z}$. By the spectral representation theorem for correlation functions,

$$R(n) = \int_{[0,\pi]} \cos n\theta dF(\theta) \quad (n \in \mathbf{Z})$$

with non-decreasing $F \in BV[0,\pi]$ called the spectral distribution function of X. Now X is called *long-time memory* or *long-range dependent* if it exhibits the property

$$\sum_{n=-\infty}^{\infty} |R(n)| = \infty$$

(see e.g. Beran [Be]). The prototype of such correlations functions is R with

$$R(n) \sim n^{-\alpha} l(n) \quad (n \to \infty),$$

where $0 < \alpha < 1$ and $l \in R_0$. The boundary case $\alpha = 1$ is delicate; the value of $\sum_{n=-\infty}^{\infty} |R(n)|$ is infinite if and only if $\int^{\infty} l(t)dt/t = \infty$. The theorems above characterize such R in terms of F rather than the spectral density of X, which does not always exist, under the weak condition (T).

To consider the analogues of the theorems above for sine coefficients, it will be convenient to restrict the class of functions. The class $NBV[0,\pi]$ is the subclass of $BV[0,\pi]$ consisting of all G that are normalized by $G(0) = 0$. For $G \in NBV[0,\pi]$ we define its *Fourier-Stieltjes sine coefficients (FS sine coefficients)*

$$b_n = \frac{2}{\pi} \int_{[0,\pi]} \sin n\theta dG(\theta) \quad (n = 1, 2, \cdots). \tag{1.6}$$

THEOREM 1.3. *Let $l \in R_0$ and $0 < \alpha < 2$. Let $G \in NBV[0,\pi]$ with FS sine coefficients (b_n). Then*

$$b_n \sim n^{-\alpha} l(n) \quad (n \to \infty) \tag{1.7}$$

implies

$$G(\theta) \sim \theta^{\alpha} l(1/\theta) \cdot \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)} \quad (\theta \to 0+). \tag{1.8}$$

Conversely , (1.8) implies (1.7) if (b_n) satisfies (T).

THEOREM 1.4. *Let $l \in R_0$, and $G \in NBV[0,\pi]$ with FS sine coefficients (b_n). We write $\tilde{G}(x) := x^2 G(1/x)$ for $x \geq 1/\pi$. Then*

$$b_n \sim n^{-2} l(n) \qquad (n \to \infty) \tag{1.9}$$

implies

$$\tilde{G} \in \Pi_l \text{ with } l\text{-index } 1/2. \tag{1.10}$$

Conversely, (1.10) implies (1.9) if (b_n) satisfies (T).

If c_n decreases to zero as $n \to \infty$, then the Fourier cosine series $f(\theta) := \sum_{n=0}^{\infty} c_n \cos n\theta$ converges for any $\theta \in (0, 2\pi)$. For this, we have Abel-Tauber theorems which link the asymptotics of (c_n) and $f(1/\cdot)$, and similarly for Fourier sine series; see Aljančić et al. [ABT], Yong [Y], and Inoue [I1], [I2]. Here monotonicity of (c_n) fill the two roles of a sufficient condition for convergence and a Tauberian condition. However, though monotonicity is simple, it is far from best-possible as each of these conditions. In contrast, the Tauberian condition (T) is a consequence of each of the final assertions (1.2), (1.4), (1.7) and (1.9), hence it does not restrict the class of FS coefficients that the theorems cover; see [BGT], p.43. We refer to Bingham [B, Theorem 3b] and [BGT, Theorem 4.10.1] for Tauberian theorems with such weak Tauberian conditions for Fourier and Jacobi series. For the classical results on Abelian and Tauberian theorems for Fourier series and integrals, we refer to [BGT, Ch.4], where proofs and references to earlier work — by Hardy and Rogosinski, Bojanic and Karamata, Zygmund, Vuilleumier, Pitman, K. Soni and R. P. Soni, and others — are given. For recent work we refer to Bingham and Inoue [BI], where a result of different – *Mercerian* – type is given.

After preliminary results in §2, we prove Theorems 1.1 and 1.2 in §3, and Theorems 1.3 and 1.4 in §4. We close in §5 with the analogues of the results above for Fourier-Stieltjes transforms.

ACKNOWLEDGEMENT. Part of this work was carried out during the author's visit to Royal Holloway College, under the Anglo-Japanese Exchange Programme of the Japan Society for the Promotion of Science (JSPS) and the Royal Society, during the academic year 1994-1995. Thanks are due to Professor N. H. Bingham for his hospitality and helpful discussions.

2. Preliminaries. First we consider Π-variation for sequences. In what follows, $(a_n)_{n=0}^{\infty}$ is a real sequence.

DEFINITION. For $l \in R_0$ and $c \in \mathbf{R}$, (a_n) is in Π_l with l-index c if for any $\lambda > 0$,

$$(a_{[\lambda n]} - a_n)/l(n) \to c \log \lambda \quad (n \to \infty).$$

LEMMA 2.1. *Let $l \in R_0$ and $c \in \mathbf{R}$. If (a_n) is in Π_l with l-index c, then $(a_{n-1} - a_n)/l(n) \to 0$ as $n \to \infty$.*

PROOF. Choose an irrational number $\lambda > 1$, say, $\lambda = \sqrt{2}$. Then $[[\lambda n]/\lambda] = n-1$ for $n = 1, 2, \cdots$. Since $l([\lambda n])/l(n) \to 1$ as $n \to \infty$ by the uniform convergence theorem (cf. [BGT, Theorem 1.5.2]),

$$\begin{aligned}\frac{a_{n-1} - a_n}{l(n)} &= \frac{(a_{[[\lambda n]/\lambda]} - a_{[\lambda n]})}{l([\lambda n])} \cdot \frac{l([\lambda n])}{l(n)} + \frac{a_{[\lambda n]} - a_n}{l(n)} \\ &\to c \log(1/\lambda) + c \log \lambda = 0 \qquad (n \to \infty),\end{aligned}$$

whence the lemma. □

THEOREM 2.2. *Let $l \in R_0$ and $c \in \mathbf{R}$. Then (a_n) is in Π_l with l-index c if and only if the function $f(x) := a_{[x]}$ is in Π_l with l-index c.*

PROOF. Suppose (a_n) is in Π_l with l-index c. For $\lambda > 0$, we write

$$\frac{f(\lambda x) - f(x)}{l(x)} = \frac{a_{[\lambda x]} - a_{[\lambda [x]]}}{l(x)} + \frac{a_{[\lambda [x]]} - a_{[x]}}{l(x)}.$$

Since $l([x])/l(x) \to 1$ as $x \to \infty$, the second term on the right tends to $c \log \lambda$ as $x \to \infty$. Now $0 \le [\lambda x] - [\lambda [x]] < \lambda + 1$, so repeated application of Lemma 2.1 gives $(a_{[\lambda x]} - a_{[\lambda [x]]})/l(x) \to 0$ as $x \to \infty$, hence f is in Π_l with l-index c. The converse is trivial. □

THEOREM 2.3. *Let $l \in R_0$ and $c \in \mathbf{R}$. We write $s_n := \sum_{k=0}^{n} a_k$ for $n = 0, 1, 2, \cdots$. Then*

$$a_n \sim cn^{-1}l(n) \qquad (n \to \infty) \tag{2.1}$$

implies

$$(s_n) \in \Pi_l \text{ with } l\text{-index } c. \tag{2.2}$$

Conversely, (2.2) implies (2.1) if (a_n) satisfies (T).

We omit the proof, since it is almost the same as that of the function case (see Theorems 3.6.8 (due to de Haan) and 3.6.10 in [BGT]).

THEOREM 2.4. *Let $l \in R_0$, $\rho > -1$ and $c \in \mathbf{R}$, and let (s_n) be as above. Assume the series $B(x) := \sum_{n=0}^{\infty} a_n e^{-n/x}$ absolutely converges for $x > 0$. Then (2.2) implies*

$$B \in \Pi_l \text{ with } l\text{-index } c. \tag{2.3}$$

Conversely, (2.3) implies (2.2) if $a_n \geq 0$ for all sufficiently large n.

PROOF. We write $V(x) = s_{[x]}$ for $x \geq 0$. By Theorem 2.2, (2.2) holds if and only if V is in Π_l with l-index c. Let $\hat{V}$ be the Laplace-Stieltjes transform of V:

$$\hat{V}(x) := \int_{[0,\infty)} e^{-tx} dV(t) = x \int_0^{\infty} e^{-xt} V(t) dt \qquad (x > 0).$$

Then $B(x) = \hat{V}(1/x)$ for $x > 0$, and so the implication (2.2) $\Rightarrow$ (2.3) follows from the argument in [BGT, p.246]. Conversely, since $e^{-n/(\lambda x)} - e^{-n/x} = o(l(x))$ as $x \to \infty$ for any $\lambda > 0$, we may assume $a_n \geq 0$ for all n. Therefore, (2.3) gives (2.2) by de Haan's theorem (see e.g. [BGT, Theorem 3.9.1]). □

Next we consider stability of Π-variation under change of variables.

LEMMA 2.5. *Let $l \in R_0$ and $c \in \mathbf{R}$. Assume $\phi : (X, \infty) \to (Y, \infty)$ is measurable and satisfies $\phi(x) \sim \alpha x$, $\alpha > 0$, as $x \to \infty$. If the measurable function $f : (Y, \infty) \to \mathbf{R}$ is in Π_l with l-index c, then $f \circ \phi$ is also in Π_l with l-index c.*

PROOF. Since $\phi(\lambda x)/\phi(x) \to \lambda$ as $x \to \infty$, the uniform convergence theorem for Π_l due to Balkema (see [BGT, Theorem 3.1.16]) gives

$$\frac{f(\phi(\lambda x)) - f(\phi(x))}{l(\phi(x))} \to c \log \lambda \quad (x \to \infty),$$

while by the uniform convergence theorem for R_0, $l(\phi(x))/l(x)$ tends to 1 as $x \to \infty$. Combining,

$$\frac{f(\phi(\lambda x)) - f(\phi(x))}{l(x)} \to c \log \lambda \qquad (x \to \infty),$$

hence the lemma. □

PROPOSITION 2.6. *Let $l \in R_0$, $c \in \mathbf{R}$ and $A : (0,1) \to \mathbf{R}$ be measurable. Write $B(x) := A(e^{-1/x})$ for $x > 0$ and $C(x) := A((x-1)/(x+1))$ for $x > 1$. Then B is in Π_l with l-index c if C is in Π_l with l-index c.*

PROOF. For

$$\phi_1(x) := \frac{1+e^{-1/x}}{1-e^{-1/x}} \quad (x > 0),$$

we have $B = C \circ \phi_1$. Since $\phi_1(x) \sim 2x$ as $x \to \infty$, we obtain the assertion by Lemma 2.5. □

PROPOSITION 2.7. *Let $l \in R_0$, $c \in \mathbf{R}$ and $\phi(x) := 1/\{2\arctan(1/x)\}$ for $x > 0$. If the function $f : (1/\pi, \infty) \to \mathbf{R}$ is in Π_l with l-index c, then $f \circ \phi$ is also in Π_l with l-index c.*

PROOF. Since $\phi(x) \sim x/2$ as $x \to \infty$, the assertion follows from Lemma 2.5. □

3. Proofs of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1. Since

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1-r^2}{1-2r\cos\theta + r^2} \qquad (|r| < 1), \tag{3.1}$$

Fubini's theorem yields

$$\sum_{n=0}^{\infty} a_n r^n = \frac{1}{\pi}\int_{[0,\pi]} \frac{1-r^2}{1-2r\cos\theta + r^2} dF(\theta) \quad (|r| < 1). \tag{3.2}$$

First we prove (1.2) implies (1.3). Since (1.2) implies $a_n \to 0$ as $n \to \infty$,

$$(1-r)\sum_{n=0}^{\infty} a_n r^n \to 0 \quad (r \uparrow 1).$$

But

$$\begin{aligned}&\int_{[0,\pi]} \frac{(1-r)^2}{1-2r\cos\theta + r^2} dF(\theta)\\ &= F(0) + \int_{(0,\pi]} \frac{1}{1+\{2r(1-\cos\theta)/(1-r)^2\}} dF(\theta) \to F(0) \quad (r \uparrow 1),\end{aligned}$$

whence (3.2) gives $F(0) = 0$.

Since $\sum_{n=1}^{\infty} |a_n/n| < \infty$, we may write

$$C(\theta) := a_0\theta + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin n\theta \quad (\theta \in [0,\pi]).$$

Clearly $F(\pi) = C(\pi)$. By the inversion formula (see e.g. Kawata [K, Theorem 4.2.1]), if x and y are continuity points of F such that $0 < x < y < \pi$, then $F(y) - F(x) = C(y) - C(x)$. Take two sequences of continuity points (x_n), (y_n) such that $x_n \downarrow 0$, $y_n \downarrow \theta \in [0, \pi)$ as $n \to \infty$. Letting $n \to \infty$ in $F(y_n) - F(x_n) = C(y_n) - C(x_n)$, we obtain

$$F(\theta) = a_0\theta + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin n\theta \quad (0 \le \theta \le \pi), \tag{3.3}$$

where we used $F(0) = 0$. Therefore, by an Abelian result due to Vuilleumier and others (see e.g. [BGT, Proposition 4.3.1a]), (1.3) follows.

Next we prove (1.3) with (T) implies (1.2). By (1.3), we have $dF\{0\} = 0$. We write

$$\begin{aligned}
R(x) &:= \frac{x-1}{x+1} \quad (x > 1),\\
\Theta(\xi) &:= 2\arctan \xi \quad (0 \le \xi < \infty),\\
\mu(d\theta) &:= \mathrm{I}_{(0,\pi/2]}(\theta) dF(\theta),\\
F_1(\xi) &:= \int_{(0,\xi]} (t^2+1)\mu \circ \Theta(dt) \quad (0 < \xi < \infty),\\
\bar{F}_1(x) &:= xF_1(1/x) \quad (0 < x < \infty),\\
k_1(x) &:= \frac{2}{\pi} \cdot \frac{x^3}{(1+x^2)^2} \quad (0 < x < \infty).
\end{aligned}$$

Since $F_1(\xi) = F_1(1)$ for all $\xi > 1$, $\bar{F}_1$ is locally-bounded on each interval $(0, a]$.

For $x > 1$ and $\theta \in (\pi/2, \pi]$,

$$1 - 2R(x)\cos\theta + R(x)^2 \ge 1 + R(x)^2,$$

hence

$$\begin{aligned}
&\left| \int_{(\pi/2,\pi]} \frac{1 - R(x)^2}{1 - 2R(x)\cos\theta + R(x)^2} dF(\theta) \right|\\
&\le \frac{1 - R(x)^2}{1 + R(x)^2} |dF|((\pi/2, \pi]) = O(x^{-1}) \quad (x \to \infty),
\end{aligned} \tag{3.4}$$

where $|dF|$ is the total variation measure of F.

Since $\cos\Theta(\xi) = (1 - \xi^2)/(1 + \xi^2)$,

$$\frac{1 - R(x)^2}{1 - 2R(x)\cos\Theta(\xi) + R(x)^2} = \frac{x(\xi^2+1)}{\xi^2 x^2 + 1} \quad (x > 1,\ \xi > 0),$$

and so, for $x > 1$,

$$\begin{aligned}
\frac{1}{\pi}\int_{(0,\pi/2]} \frac{1 - R(x)^2}{1 - 2R(x)\cos\theta + R(x)^2} dF(\theta) &= \frac{1}{\pi}\int_{(0,\infty)} \frac{x(\xi^2+1)}{\xi^2x^2+1} \mu \circ \Theta(d\xi)\\
&= \frac{1}{\pi}\int_{(0,\infty)} \frac{x}{x^2\xi^2+1} dF_1(\xi).
\end{aligned}$$

By integration by parts (see e.g. [BGT, Theorem A6.1]), the right-hand side is

$$\frac{1}{\pi}\int_0^\infty \frac{2\xi x^3}{(\xi^2x^2+1)^2}F_1(\xi)d\xi = k_1 * \bar{F}_1(x) \quad (0 < x < \infty),$$

where $k_1 * \bar{F}_1$ denotes the Mellin convolution of k_1 and $\bar{F}_1$:

$$k_1 * \bar{F}_1(x) := \int_0^\infty k_1(x/t)\bar{F}_1(t)dt/t \quad (0 < x < \infty).$$

This with (3.2) and (3.4) gives

$$\text{(3.5)} \qquad \sum_{n=0}^\infty a_n R(x)^n = k_1 * \bar{F}_1(x) + O(x^{-1}) \quad (x \to \infty).$$

The Mellin transform

$$\check{k}_1(z) := \int_0^\infty t^{-z}k_1(t)dt/t = \frac{2}{\pi}\int_0^\infty \frac{t^{2-z}}{(1+t^2)^2}dt$$

converges absolutely for $-1 < \Re z < 3$, and is equal to

$$\frac{1}{\pi}\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{1+z}{2}\right).$$

Now

$$\text{(3.6)} \qquad F_1(\xi) = F(\Theta(\xi)) + \int_{(0,\Theta(\xi)]} \tan^2(\theta/2)dF(\theta) \quad (0 < \xi \le 1),$$

and the integral on the right is

$$\text{(3.7)} \qquad O\left(\xi^2 \int_{(0,\Theta(\xi)]} |dF(\theta)|\right) = o(\xi^2) \quad (\xi \to 0+).$$

Hence (1.3) gives

$$F_1(\xi) \sim F(\Theta(\xi)) \sim \xi^\alpha l(1/\xi)\frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\cos(\pi\alpha/2)} \quad (\xi \to 0+)$$

or

$$\bar{F}_1(x) \sim x^{1-\alpha}l(x)\frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\cos(\pi\alpha/2)} \quad (x \to \infty).$$

So by Arandelović's theorem (see e.g. [BGT, Theorem 4.1.6]), we obtain

$$k_1 * \bar{F}_1(x) \sim \check{k}_1(1-\alpha)\bar{F}_1(x) \sim x^{1-\alpha}l(x)2^{\alpha-1}\Gamma(1-\alpha) \quad (x \to \infty).$$

Referring back to (3.5), this gives

$$\sum_{n=0}^{\infty} a_n R(x)^n \sim x^{1-\alpha} l(x) 2^{\alpha-1}\Gamma(1-\alpha) \quad (x \to \infty)$$

or

$$\sum_{n=0}^{\infty} a_n r^n \sim \left(\frac{1+r}{1-r}\right)^{1-\alpha} l\left(\frac{1+r}{1-r}\right) 2^{\alpha-1}\Gamma(1-\alpha)$$
$$\sim (1-r)^{\alpha-1} l\left(\frac{1}{1-r}\right)\Gamma(1-\alpha) \quad (r \uparrow 1)$$

Since individual terms $a_n r^n$ are $o((1-r)^{\alpha-1})$, we may assume $a_n > 0$ for all n, which gives by Karamata's tauberian theorem for power series (see e.g. [BGT, Corollary 1.7.3]),

$$\sum_{k=0}^{n} a_k \sim \frac{n^{1-\alpha} l(n)}{1-\alpha} \quad (n \to \infty). \tag{3.8}$$

Finally, the Tauberian condition (T) corresponds to (1.7.10") in [BGT, Theorem 1.7.5], whence it gives (1.2). □

PROOF OF THEOREM 1.2. First we prove (1.4) implies (1.5). In the same way as above, (1.4) gives (3.3). Write $A(x) := \sum_{j=0}^{[x]} a_j$ for $x > 0$. Then for $x > 0$,

$$\bar{F}(x) - A(x) = \int_0^{\infty} f_1(x,t) l_1(t) dt + \int_0^{\infty} f_2(x,t) l_1(t) dt,$$

where for $x > 0$ and $t > 0$,

$$f_1(x,t) := \frac{1}{[t]}\left(\frac{\sin([t]/x)}{([t]/x)} - 1\right) \quad (1 \le t < [x]+1), \quad := 0 \quad (\text{otherwise}),$$
$$f_2(x,t) := \frac{1}{[t]} \cdot \frac{\sin([t]/x)}{([t]/x)} \quad ([x]+1 \le t < \infty), \quad := 0 \quad (0 < t < [x]+1),$$
$$l_1(t) := a_{[t]} \cdot [t] \quad (1 \le t < \infty), \quad := 1 \quad (0 < t < 1).$$

If $0 < \delta < 2$, then as $x \to \infty$,

$$\int_0^x t^{-\delta} |f_1(x,t)| dt \le \sum_{j \le x} \frac{1}{j^{1+\delta}} \left(1 - \frac{\sin(j/x)}{(j/x)}\right) = O\left(\sum_{j \le x} \frac{(j/x)^2}{j^{\delta+1}}\right) = O(x^{-\delta}).$$

Also,

$$\int_0^{\infty} f_1(x,t) dt = \frac{1}{x} \sum_{0 < j/x \le 1} \frac{1}{(j/x)} \left(\frac{\sin(j/x)}{(j/x)} - 1\right)$$
$$\to c_1 := \int_0^1 \frac{1}{u}\left(\frac{\sin u}{u} - 1\right) du \quad (x \to \infty).$$

So by Vuilleumier's theorem (see e.g. [BGT, Theorem 4.1.4]), (1.4) gives

$$\int_0^\infty f_1(x,t)l_1(t)dt \sim c_1 l_1(x) \sim c_1 l(x) \quad (x \to \infty).$$

Similarly, if $0 < \delta < 1$, then there exists $C > 0$ such that for $x \geq 1$ and $M \geq 1$,

$$\int_{Mx}^\infty t^\delta |f_2(x,t)|dt \leq 2^\delta x \sum_{j \geq Mx} \frac{1}{j^{2-\delta}} = CM^{\delta-1}x^\delta. \tag{3.9}$$

Choose $\epsilon > 0$ small enough; then for large enough M and all $x \geq 1$, the right-hand side with $\delta = 0$ is less than ϵ, while

$$\begin{aligned}\int_0^{[Mx]+1} f_2(x,t)dt &= \frac{1}{x} \sum_{0<j/x\leq M} \frac{\sin(j/x)}{(j/x)^2} \\ &\to \int_1^M \frac{\sin u}{u^2} du \quad (x \to \infty),\end{aligned}$$

hence

$$\int_0^\infty f_2(x,t)dt \to c_2 := \int_1^\infty \frac{\sin u}{u^2} du \quad (x \to \infty).$$

This and (3.9) with $M = 1$ imply that the conditions of Vuilleumier's theorem are satisfied, hence

$$\int_0^\infty f_2(x,t)l_1(t)dt \sim c_2 l_1(x) \sim c_2 l(x) \quad (x \to \infty).$$

Combining,

$$\{\bar{F}(x) - A(x)\}/l(x) \to c_1 + c_2 \quad (x \to \infty).$$

Since $l \in R_0$, this gives for any $\lambda > 0$,

$$\{\bar{F}(\lambda x) - A(\lambda x)\}/l(x) \to c_1 + c_2 \quad (x \to \infty).$$

Subtract and use Theorems 2.3 and 2.4:

$$\{\bar{F}(\lambda x) - \bar{F}(x)\}/l(x) \to \log \lambda \quad (x \to \infty),$$

which gives (1.5).

Next we prove (1.5) with (T) implies (1.4). By [BGT, Theorem 3.7.4], we find $|\bar{F}| \in R_0$, and so $dF\{0\} = F(0) = 0$. Hence as above, we obtain (3.5). Write

$$\begin{aligned} D(x) &:= \frac{F(\Theta(1/x))}{\Theta(1/x)} \quad (x > 0), \\ a(x) &:= x\Theta(1/x) \quad (x > 0). \end{aligned}$$

Then by (3.6) and (3.7),

$$\bar{F}_1(x) = a(x)D(x) + o(x^{-1}) \quad (x \to \infty). \tag{3.10}$$

Proposition 2.7 shows that $D \in \Pi_l$ with l-index 1, so that in particular $|D| \in R_0$ by [BGT, Theorem 3.7.4]. Hence, since $a(x) \to 2$ as $x \to \infty$ and there exists $C > 0$ such that for all $\lambda > 1$ and $x \geq 2$

$$|a(\lambda x) - a(x)| \leq C\frac{(1-\lambda^{-1})}{x},$$

we have

$$\begin{aligned}&\{a(\lambda x)D(\lambda x) - a(x)D(x)\}/l(x)\\&= a(\lambda x)\frac{D(\lambda x) - D(x)}{l(x)} + \{a(\lambda x) - a(x)\}\frac{D(x)}{l(x)} \to 2\log\lambda \quad (x \to \infty).\end{aligned}$$

So, by (3.10), $\bar{F}_1$ is in Π_l with l-index 2.

Since $\check{k}_1(0) = 1/2$, $k_1 * \bar{F}_1$ is in Π_l with l-index 1 (see [BGT], p.242). This and (3.5) imply that $\sum_{n=0}^{\infty} a_n R(\cdot)^n$ is in Π_l with l-index 1, hence by Proposition 2.6 the function $\sum_{n=0}^{\infty} a_n e^{-n/x}$ in x is also in Π_l with l-index 1. Applying Theorem 2.4 to this, $a_n > 0$ for all sufficiently large n then shows

$$\left(\sum_{k=0}^{n} a_k\right) \in \Pi_l \text{ with } l\text{-index } 1. \tag{3.11}$$

Finally, under the Tauberian condition (T), Theorem 2.3 gives (1.4). □

REMARK 1. In the proof of Theorem 1.1, the Tauberian condition (T) except for the positivity of (a_n) was used only to deduce (1.2) from (3.8). So (T) can be replaced by each Tauberian condition of [BGT, Theorem 4.10.1] if we further assume that $a_n \geq 0$ for all sufficiently large n; similarly for Theorems 1.2–1.4, and Theorems 5.1–5.4 below. □

REMARK 2. In the cases of Theorems 1.1 and 1.2, F is continuous on $[0, \pi]$ by (3.3).

4. Proofs of Theorems 1.3 and 1.4.

PROOF OF THEOREM 1.3. First we prove (1.7) implies (1.8). As above, the inversion formula gives

$$G(\theta) = \sum_{n=1}^{\infty} b_n \cdot \frac{1 - \cos n\theta}{n} \quad (0 \leq \theta < \pi), \tag{4.1}$$

hence for $x > 0$,

$$x^{\alpha} G(1/x) = \int_0^{\infty} g_0(x,t) l_2(t) dt,$$

where for $x > 0$ and $t > 0$,

$$g_0(x,t) := \frac{1}{[t]} \cdot \frac{1-\cos([t]/x)}{([t]/x)^\alpha} \quad (1 \le t < [x]+1), \quad := 0 \quad \text{(otherwise)},$$
$$l_2(t) := [t]^\alpha \cdot b_{[t]} \quad (1 \le t < \infty), \quad := 1 \quad (0 < t < 1).$$

By an argument similar to that in §3, Vuilleumier's theorem gives

$$\int_0^\infty g_0(x,t) l_2(t) dt \sim c_3 l_2(x) \sim c_3 l(x) \quad (x \to \infty)$$

with

$$c_3 := \int_0^\infty \frac{1-\cos u}{u^{\alpha+1}} du.$$

Since

$$c_3 = \frac{1}{\alpha} \int_0^{\infty-} \frac{\sin u}{u^\alpha} du = \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)},$$

we obtain (1.8)

Next we prove (1.8) with (T) implies (1.7). Differentiating the both sides of (3.1) in θ,

$$\sum_{n=1}^\infty r^n n \sin n\theta = \frac{r(1-r^2)\sin\theta}{(1-2r\cos n\theta + r^2)^2} \quad (|r| < 1),$$

hence by Fubini's theorem,

$$\sum_{n=1}^\infty n b_n r^n = \frac{2}{\pi} \int_{(0,\pi]} \frac{r(1-r^2)\sin\theta}{(1-2r\cos n\theta + r^2)^2} dG(\theta) \quad (|r| < 1). \tag{4.2}$$

Let $R(x)$ and $\Theta(\xi)$ be as in §3. Then

$$\left| \int_{(\pi/2,\pi]} \frac{R(x)\{1-R(x)^2\}\sin\theta}{\{1-2R(x)\cos n\theta + R(x)^2\}^2} dG(\theta) \right| \tag{4.3}$$
$$\le \frac{R(x)\{1-R(x)^2)\}}{\{1+R(x)^2\}^2} |dG|((\pi/2,\pi]) = O(x^{-1}) \quad (x \to \infty).$$

We write

$$\nu(d\theta) := \mathrm{I}_{(0,\pi/2]}(\theta) dG(\theta),$$
$$G_1(\xi) := \int_{(0,\xi]} (t^2+1)\nu \circ \Theta(dt) \quad (0 < \xi < \infty),$$
$$\tilde{G}_1(x) := x^2 G_1(1/x) \quad (0 < x < \infty),$$
$$k_2(x) := \frac{1}{\pi} \cdot \frac{3x^5 - x^3}{(1+x^2)^3} \quad (0 < x < \infty).$$

Since $G_1(\xi) = G_1(1)$ for all $\xi > 1$, $\tilde{G}_1$ is bounded on each interval $(0, a]$.

Since $\cos\Theta(\xi) = (1-\xi^2)/(1+\xi^2)$, $\sin\Theta(\xi) = 2\xi/(1+\xi^2)$, we find, for $x > 1$,

$$\frac{R(x)\{1-R(x)^2\}\sin\Theta(\xi)}{\{1-2R(x)\cos\Theta(\xi)+R(x)^2\}^2} = \frac{(x^3-x)}{2}\cdot\frac{\xi(\xi^2+1)}{(\xi^2x^2+1)^2} \quad (x>1,\ \xi>0),$$

so that

$$\begin{aligned}&\frac{2}{\pi}\int_{(0,\pi/2]}\frac{R(x)\{1-R(x)^2\}\sin\theta}{\{1-2R(x)\cos\theta+R(x)^2\}^2}dG(\theta)\\ &=\frac{(x^3-x)}{\pi}\int_{(0,\infty)}\frac{\xi(\xi^2+1)}{(\xi^2x^2+1)^2}\nu\circ\Theta(d\xi) = \frac{(x^3-x)}{\pi}\int_{(0,\infty)}\frac{\xi}{(\xi^2x^2+1)^2}dG_1(\xi).\end{aligned}$$

By integration by parts, the right-hand side is

$$\frac{(x^3-x)}{\pi}\int_0^\infty\frac{(3x^2\xi^2-1)}{(\xi^2x^2+1)^3}G_1(\xi)d\xi = (1-x^{-2})k_2 * \tilde{G}_1(x) \quad (0<x<\infty),$$

where $k_2 * \tilde{G}_1$ is the Mellin convolution of k_2 and $\tilde{G}_1$. Hence

$$\text{(4.4)} \qquad \sum_{n=1}^{\infty} nb_n R(x)^n = (1-x^{-2})k_2 * \tilde{G}_1(x) + O(x^{-1}) \quad (x\to\infty).$$

The Mellin transform $\check{k}_2(z)$ converges absolutely for $-1 < \Re z < 3$, and is equal to

$$\begin{aligned}&\frac{1}{\pi}\int_0^\infty t^{2-z}\frac{3t^2-1}{(t^2+1)^3}dt = \frac{1}{\pi}\int_0^\infty t^{2-z}\frac{d}{dt}\left\{-\frac{t}{(t^2+1)^2}\right\}dt\\ &=\frac{(2-z)}{\pi}\int_0^\infty\frac{t^{2-z}}{(t^2+1)^2}dt = \frac{(2-z)}{2\pi}\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{1+z}{2}\right).\end{aligned}$$

Now as $\xi \to 0+$,

$$\text{(4.5)} \qquad G_1(\xi) = G(\Theta(\xi)) + \int_{(0,\Theta(\xi)]}\tan^2(\theta/2)dG(\theta) = G(\Theta(\xi)) + o(\xi^2),$$

hence by (1.8),

$$\tilde{G}_1(x) \sim x^2 G(\Theta(1/x)) \sim x^{2-\alpha}l(x)\frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\sin(\pi\alpha/2)} \quad (x\to\infty).$$

By Arandelović's theorem,

$$k_2 * \tilde{G}_1(x) \sim \check{k}_2(2-\alpha)\tilde{G}_1(x) \sim x^{2-\alpha}l(x)2^{\alpha-2}\Gamma(2-\alpha) \quad (x\to\infty).$$

Referring back to (4.4), this gives

$$\sum_{n=1}^{\infty} nb_n R(x)^n \sim x^{2-\alpha}l(x)2^{\alpha-2}\Gamma(2-\alpha) \quad (x\to\infty)$$

or

$$\sum_{n=1}^{\infty} nb_n r^n \sim (1-r)^{\alpha-2} l\left(\frac{1}{1-r}\right) \Gamma(2-\alpha) \quad (r \uparrow 1).$$

Therefore by Karamata's Tauberian theorem for power series,

$$\sum_{k=1}^{n} kb_k \sim \frac{n^{2-\alpha} l(n)}{2-\alpha} \quad (n \to \infty).$$

Since the series (nb_n) also satisfies the Tauberian condition (T), [BGT, Theorem 1.7.5] gives (1.7). □

PROOF OF THEOREM 1.4. First we prove (1.9) implies (1.10). In the same way as above, (1.9) gives (4.1). Write $B(x) := \frac{1}{2}\sum_{j=1}^{[x]} jb_j$ for $x > 0$. Then for $x > 0$,

$$\tilde{G}(x) - B(x) = \int_0^{\infty} g_1(x,t) l_2(t) dt + \int_0^{\infty} g_2(x,t) l_2(t) dt,$$

where for $x > 0$ and $t > 0$,

$$\begin{aligned}
g_1(x,t) &:= \mathrm{I}_{[1,[x]+1)}(t) \cdot \frac{1}{[t]} \cdot \frac{1 - \frac{1}{2}([t]/x)^2 - \cos([t]/x)}{([t]/x)^2}, \\
g_2(x,t) &:= \mathrm{I}_{[[x]+1,\infty)}(t) \cdot \frac{1}{[t]} \cdot \frac{1 - \cos([t]/x)}{([t]/x)^2}, \\
l_2(t) &:= [t]^2 \cdot b_{[t]} \quad (1 \le t < \infty), \quad := 1 \quad (0 < t < 1).
\end{aligned}$$

By Vuilleumier's theorem,

$$\int_0^{\infty} g_1(x,t) l_2(t) dt \sim c_3 l_2(x) \sim c_3 l(x) \quad (x \to \infty),$$

where

$$c_3 := \int_0^1 \frac{1 - \frac{1}{2}u^2 - \cos u}{u^3} du \quad (x \to \infty).$$

Similarly,

$$\int_0^{\infty} g_2(x,t) l_2(t) dt \sim c_4 l_2(x) \sim c_4 l(x) \quad (x \to \infty),$$

where

$$c_4 := \int_0^{\infty} \frac{1 - \cos u}{u^2} du.$$

Combining,

$$\{\tilde{G}(x) - B(x)\}/l(x) \to c_3 + c_4 \quad (x \to \infty),$$

which implies (1.10).

Next we prove (1.10) with (T) implies (1.9). We set $\tilde{l}(x) := |\tilde{G}(x)|$ for $x > 1/\pi$. Then by [BGT, Theorem 3.7.4], (1.10) shows $\tilde{l} \in R_0$. By integration by parts, for some $C > 0$ and all $\xi \in (0,1)$,

$$\left|\int_{(0,\Theta(\xi)]} \tan^2(\theta/2)dG(\theta)\right| = \left|\xi^2 G(\Theta(\xi)) - \int_{(0,\Theta(\xi)]} \frac{\sin(\theta/2)}{\cos^3(\theta/2)} G(\theta)d\theta\right|$$
$$\leq \xi^2\Theta(\xi)^2\tilde{l}(1/\Theta(\xi)) + C\int_{(0,\Theta(\xi)]} \theta^3\tilde{l}(1/\theta)d\theta,$$

which is $O(\xi^3)$ as $\xi \to 0+$ (see [BGT, Theorem 1.3.6(v)]). Write

$$E := \frac{G(\Theta(1/x))}{\Theta(1/x)^2} \quad (x > 0),$$
$$b(x) := x^2\Theta(1/x)^2 \quad (x > 0).$$

Then by the estimate above

$$\tilde{G}_1(x) = b(x)E(x) + O(x^{-1}) \quad (x \to \infty).$$

By Proposition 2.7, E is in Π_l with l-index $1/2$, hence, arguing as in §3, $\tilde{G}_1$ is in Π_l with l-index 2. Since $\check{k}_2(0) = 1/2$, [BGT, p.242] shows that $k_2 * \tilde{G}_1$ is in Π_l with l-index 1. By (4.4), this implies that $\sum_{n=1}^{\infty} nb_nR(\cdot)^n$ is in Π_l with l-index 1. So under the Tauberian condition (T), Proposition 2.6 and Theorems 2.4 and 2.3 give (1.9). □

5. Fourier-Stieltjes transforms. In this section, we show the analogues of Theorems 1.1–1.4 for Fourier-Stieltjes transforms. The classes $BV[0,\infty)$ and $NBV[0,\infty)$ are defined similarly. In particular, each function in $BV[0,\infty)$ is bounded on $[0,\infty)$. For $F \in BV[0,\infty)$, we define its *Fourier-Stieltjes cosine transform (FS cosine transform)*

$$f(t) := \frac{2}{\pi}\int_{[0,\infty)} \cos t\xi dF(\xi) \quad (0 \leq t < \infty),$$

where as above $dF\{0\} = F(0)$. Similarly, for $G \in NBV[0,\infty)$, we define its *Fourier-Stieltjes sine transform (FS sine transform)*

$$g(t) := \frac{2}{\pi}\int_{[0,\infty)} \sin t\xi dG(\xi) \quad (0 \leq t < \infty).$$

The function $h : [0,\infty) \to \mathbf{R}$ is called *slowly decreasing* if

$$\lim_{\lambda \downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} (h(tx) - h(x)) \geq 0 \qquad (\text{hence} = 0),$$

slowly increasing if $-h$ is slowly decreasing. The function $f : [0,\infty) \to \mathbf{R}$ is said to satisfy the Tauberian condition (T) if

f is eventually positive, and $\log f$ is either slowly decreasing or slowly increasing.

THEOREM 5.1. *Let $l \in R_0$ and $0 < \alpha < 1$. Let $F \in BV[0,\infty)$ with FS cosine transform f. Then*

$$(5.1) \qquad f(t) \sim t^{-\alpha} l(t) \quad (t \to \infty)$$

implies

$$(5.2) \qquad F(\xi) \sim \xi^{\alpha} l(1/\xi) \cdot \frac{\pi}{2\Gamma(\alpha+1)\cos(\pi\alpha/2)} \quad (\xi \to 0+).$$

Conversely , (5.2) implies (5.1) if f satisfies (T).

THEOREM 5.2. *Let $l \in R_0$. Let $F \in BV[0,\infty)$ with FS cosine transform f. We write $\bar{F}(x) := xF(1/x)$ for $x > 0$. Then*

$$(5.3) \qquad f(t) \sim t^{-1} l(t) \qquad (t \to \infty)$$

implies

$$(5.4) \qquad \bar{F} \in \Pi_l \text{ with } l\text{-index } 1.$$

Conversely, (5.4) implies (5.3) if f satisfies (T).

As Theorems 1.1 and 1.2, the theorems above can be applied to stationary processes. Let $X = (X(t) : t \in \mathbf{R})$ be a real, centered, weakly stationary process with correlation function $R(t) := E[X(t)X(0)]$ and spectral distribution function F:

$$R(t) = \int_{[0,\infty)} \cos t\xi dF(\xi) \quad (t \in \mathbf{R}).$$

Then the theorems above link the asymptotics of R and $F(1/\cdot)$.

THEOREM 5.3. *Let $l \in R_0$ and $0 < \alpha < 2.$, Let $G \in NBV[0,\infty)$ with FS sine transform g. Then*

$$(5.5) \qquad g(t) \sim t^{-\alpha} l(t) \quad (t \to \infty)$$

implies

$$(5.6) \qquad G(\xi) \sim \xi^{\alpha} l(1/\xi) \cdot \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)} \quad (\xi \to 0+).$$

Conversely , (5.6) implies (5.5) if g satisfies (T).

THEOREM 5.4. *Let $l \in R_0$ and $G \in NBV[0,\infty)$ with FS sine transform g. We write $\tilde{G}(x) := x^2 G(1/x)$ for $x > 0$. Then*

$$(5.7) \qquad g(t) \sim t^{-2} l(t) \qquad (t \to \infty)$$

implies

$$(5.8) \qquad \tilde{G} \in \Pi_l \text{ with } l\text{-index } 1/2.$$

Conversely, (5.8) implies (5.7) if g satisfies (T).

The proofs of the theorems above are similar to and easier than those of Theorems 1.1–1.4, hence we omit the details; we only note that, as (3.2) and (4.4), the following equalities are keys to the proofs:

$$\int_0^{\infty} e^{-xt} f(t) dt = \frac{2}{\pi} \int_{[0,\infty)} \frac{x}{x^2+\xi^2} dF(\xi),$$

$$\int_0^{\infty} e^{-xt} t g(t) dt = \frac{4}{\pi} \int_{(0,\infty)} \frac{x\xi}{(x^2+\xi^2)^2} dG(\xi).$$

REFERENCES

[ABT] S. ALJANČIĆ, R. BOJANIĆ AND M. TOMIĆ, Sur le comportement asymptotique au voisinage de zéro des séries trigonométriques de sinus à coefficients monotones, Publ. Inst. Math. Acad. Sci. 10 (1956), 101–120.

[Be] J. BERAN, Statistical methods for data with long-range dependence (with discussion), Statistical Science 7 (1992), 404–427.

[B] N. H. BINGHAM, Tauberian theorems for Jacobi series, Proc. London Math. Soc. 36 (1978), 285–309.

[BGT] N. H. BINGHAM, C. M. GOLDIE AND J. L. TEUGELS, Regular Variation, Cambridge Univ. Press, Cambridge, 1987.

[BI] N. H. BINGHAM AND A. INOUE, The Drasin-Shea-Jordan theorem for Fourier and Hankel transforms, preprint.

[I1] A. INOUE, On Abel-Tauber theorems for Fourier cosine transforms, J. Math. Anal. Appl. 196 (1995), 764–776.

[I2] A. INOUE, An Abel-Tauber theorem for Fourier sine transforms, J. Math. Sci. Univ. Tokyo 2 (1995), 303–309.

[K] T. KAWATA, Fourier Analysis in Probability Theory, Academic Press, New York, 1972.

[Y] C. H. YONG, On the asymptotic behaviour of cosine series with monotone coefficients, Publ. Inst. Math. 9 (1969), 29–35.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO 060
JAPAN
E-mail address: inoue@math.hokudai.ac.jp