A REMARK ON ESTIMATES OF BILINEAR FORMS OF GRADIENTS IN HARDY SPACE

Yasuyuki Shimizu

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YASUYUKI SHIMIZU

Department of Mathematics
Hokkaido University
Sapporo 060,Japan

§0. Introduction.

In a recent interesting paper[1] L.C.Evans and S.Müller established the estimate of local Hardy space norm of gradients ψ_{x_1}, ψ_{x_2}:

$$\|\phi\psi_{x_1}\psi_{x_2}\|_{h^1} + \|\phi(\psi_{x_1}^2 - \psi_{x_2}^2)\|_{h^1} \leqq C(\|\psi_{x_1}\|^2_{L^2(B(0,R))} + \|\psi_{x_2}\|^2_{L^2(B(0,R))}) \tag{0.1}$$

provided that

$$-\Delta\psi = \omega \geqq 0 \text{ in } \mathbb{R}^2. \tag{0.2}$$

Here ϕ is in $C_0^\infty(\mathbb{R}^2)$ and the constants C and R depends only on ϕ ; h^1 is a local Hardy space defined in §1 and $B(x,R)$ denotes the closed ball of radius R centered at $x \in \mathbb{R}^2$. (Another proof based on harmonic analysis is given by Semmes[2].)

This estimate is useful in proving the existence of weak solutions for the initial value problem of the two-dimensional Euler equation when the vorticity of the initial value is nonnegative measure ([1] and Delort[3]). The assumption $\omega \geqq 0$ in (0.2) is essential for the estimate (0.1); in fact, Evans and Müller[1] gave a counterexample for (0.1) when the condition $\omega \geqq 0$ is violated. However, in their example the set where ω is nonnegative may be complicated.

In this paper we give another counterexample for (0.1) even when ω is odd in the second variable x_2 i.e. $\omega(x_1,x_2) = -\omega(x_1,-x_2)$ and $\omega(x_1,x_2) \geqq 0$ for $x_2 \geqq 0$. This suggests that it is difficult to extend weak solutions for the initial-boundary value problem of the Euler equation when the domain is a half space $\mathbb{R}^2_+$ even if initial value is nonnegative in $\mathbb{R}^2_+$.

To get our counterexample we construct a sequence ψ^ϵ of form $\psi^\epsilon(x) = \psi(x/\epsilon)$. A key observation is the existence of function ψ that satisfies

$$\int_{\mathbb{R}^2} \psi_{x_1}^2\,dx \neq \int_{\mathbb{R}^2} \psi_{x_2}^2\,dx$$

with $-\Delta\psi = \omega$, where $\omega \in C_0^\infty(\mathbb{R}^2)$ is odd in the second variable x_2 and $\omega \geqq 0$ in $\mathbb{R}^2_+$, and $\psi \in H^1(\mathbb{R}^2)$; $H^1(\mathbb{R}^2)$ denotes the Sobolev space, i.e. the space of $f \in L^2(\mathbb{R}^2)$ with $f_{x_1}, f_{x_2} \in L^2(\mathbb{R}^2)$.

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§1. Definition and Main Theorem.

We begin with definition of local Hardy space as in [1].

Definition 1.1. Let η be in $C_0^\infty(\mathbb{R}^n)$ with supp$\eta \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta dx = 1$. For a function$f$ in $L^1_{loc}(\mathbb{R}^n)$, f^{**} is defined by

$$f^{**}(x) = \sup_{0<r<1} \left| \int_{\mathbb{R}^n} \eta \left(\frac{x-y}{r} \right) f(y) dy \right| . \tag{1.1}$$

The local Hardy space $\mathcal{H}^1_{loc}$ is defined by

$$\mathcal{H}^1_{loc}(\mathbb{R}^n) = \{ f \in L^1_{loc}(\mathbb{R}^n) |\ f^{**} \in L^1_{loc}(\mathbb{R}^n) \}. \tag{1.2}$$

We recall the normed local Hardy space h^1 defined by

$$h^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) |\ f^{**} \in L^1(\mathbb{R}^n) \} \tag{1.3}$$

with the norm

$$\| f \|_{h^1(\mathbb{R}^n)} = \| f^{**} \|_{L^1(\mathbb{R}^n)}.$$

Definition 1.2. For a function f in $C_0^\infty(\mathbb{R}^2)$, we define the operator $(-\Delta)^{-1}$ by

$$(-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y) \log |x-y| dy. \tag{1.4}$$

Theorem 1.3. *Let T and S be the spaces of form*

$$T = \left\{ \omega \in C_0^\infty(\mathbb{R}^2) |\ \omega(x_1,x_2) \geqq 0 \text{ for } x_2 \geqq 0,\ \omega(x_1,x_2) = -\omega(x_1,-x_2) \right\},$$
$$S = \left\{ (-\Delta)^{-1} \omega |\ \omega \in T \right\}.$$

Then there exists a sequence $\{\psi^\epsilon\}_{0<\epsilon<1}$ such that

$$\sup_{0<\epsilon<1} \| \psi^\epsilon \|_{H^1(\mathbb{R}^2)} < \infty$$

and

$$\lim_{\epsilon \downarrow 0} \left\| \phi \left\{ (\psi^\epsilon_{x_1})^2 - (\psi^\epsilon_{x_2})^2 \right\} \right\|_{h^1(\mathbb{R}^2)} = \infty \tag{1.5}$$

where $\phi \in C_0^\infty(\mathbb{R}^2), 0 \leqq \phi \leqq 1, \phi|_{B(0,1/8)} \equiv 1$ and supp$\phi \subset B(0,1/2)$.

§2. Proof of Theorem

At first, we show a fundamental estimate in normed local Hardy space; this is an extension of a result of Evans and Müller[1].

Lemma 2.1. *Assume that f is in $L^1(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} f(x)dx \neq 0$. Let $f^\epsilon = \frac{1}{\epsilon^n} f\left(\frac{x}{\epsilon}\right)$. Then $\|f^\epsilon\|_{L^1} = \|f\|_{L^1} < \infty$, and*

$$\lim_{\epsilon \downarrow 0} \|\phi f^\epsilon\|_{h^1(\mathbb{R}^n)} = \infty \tag{2.1}$$

for a function ϕ in $C_0^\infty(\mathbb{R}^n)$ with $0 \leqq \phi \leqq 1$, $\phi|_{B(0,1/8)} \equiv 1$, and $\operatorname{supp}\phi \subset B(0, 1/4)$.

Proof. L^1-estimate is easily obtained by scaling variables. To show the estimate (2.1), assume that the function η in (1.1) satisfies $0 \leqq \eta \leqq 1$ and $\eta|_{B(0,1/2)} \equiv 1$. Now we estimate the function $(\phi f^\epsilon)^{**}$:

$$\begin{aligned}(\phi f^\epsilon)^{**}(x) &= \sup_{0<r<1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-y}{r}\right) \phi(y) f^\epsilon(y) dy \right| \\ &= \sup_{0<r<1} \left| \frac{1}{r^n} \int_{\mathbb{R}^n} \eta\left(\frac{x-\epsilon z}{r}\right) \phi(\epsilon z) f(z) dz \right|.\end{aligned} \tag{2.2}$$

Now take a parameter $R > 0$, and let $r = 4|x|$. Then

$$\begin{aligned}(\phi f^\epsilon)^{**}(x) \geqq {} & \frac{1}{4^n |x|^n} \left| \int_{B(0,R)} \eta\left(\frac{x-\epsilon z}{4|x|}\right) \phi(\epsilon z) f(z) dz \right| \\ & - \frac{1}{4^n |x|^n} \left| \int_{\mathbb{R}^n \setminus B(0,R)} \eta\left(\frac{x-\epsilon z}{4|x|}\right) \phi(\epsilon z) f(z) dz \right| \\ = {} & I_1^\epsilon - I_2^\epsilon \text{ for } \epsilon R < |x| < 1/4.\end{aligned} \tag{2.3}$$

We show that $\lim_{\epsilon\downarrow 0} \|I_2^\epsilon\|_{L^1(B(0,1/4))} = 0$ and $\lim_{\epsilon\downarrow 0} \|I_1^\epsilon\|_{L^1(B(0,1/4))} = \infty$ to complete the proof. Firstly, we estimate the term I_2^ϵ :

$$\begin{aligned} I_2^\epsilon(x) &\leqq \frac{1}{(4\epsilon R)^n} \int_{\mathbb{R}^n \setminus B(0,R)} |f(z)|\, dz = \frac{F(R)}{(4\epsilon R)^n} \\ &\text{with } F(R) = \int_{\mathbb{R}^n \setminus B(0,R)} |f(z)|\, dz. \end{aligned}$$

Since $F(R)$ is continuous, nonincreasing, and $\lim_{R\to\infty} F(R) = 0$, for sufficiently small ϵ, there exists $R = R(\epsilon)$ such that

$$\frac{\{F(R)\}^{1/2}}{R^n} = \epsilon^n. \tag{2.4}$$

By (2.4), we get

$$I_2^\epsilon(x) \leqq \frac{\{F(R)\}^{1/2}}{4^n} \longrightarrow 0 \text{ as } \epsilon \downarrow 0 \tag{2.5}$$

and get $\lim_{\epsilon\downarrow 0} \|I_2^\epsilon\|_{L^1(B(0,1/4))} = 0$.

Secondly, we estimate the term I_1^ϵ. As $\left|\frac{x-\epsilon z}{4|x|}\right| \leqq \frac{|x|}{4|x|} + \frac{\epsilon R}{4|x|} \leqq \frac{1}{2}$,

$$I_1^\epsilon = \frac{1}{4^n|x|^n}\left|\int_{B(0,R)} \phi(\epsilon z) f(z)dz\right| \tag{2.6}$$
$$= \frac{1}{4^n|x|^n}\left|\int_{B(0,R)} f(z)dz\right| \text{ for } \epsilon R \leqq |x| \leqq 1/4.$$

for $\epsilon R < 1/8$.

Now let $\epsilon \downarrow 0$. For $\epsilon R = \{F(R)\}^{1/2n} \longrightarrow 0$ by (2.4),

$$\lim_{\epsilon\downarrow 0} I_1^\epsilon = \frac{1}{4^n|x|^n}\left|\int_{\mathbb{R}^n} f(z)dz\right| = \frac{\|f\|_{L^1(\mathbb{R}^n)}}{4^n|x|^n} \text{ for } 0 \leqq |x| \leqq 1/4. \tag{2.7}$$

Since $\frac{1}{|x|^n}$ is not in $L^1(\mathbb{R}^n)$, we get a conclusion. □

Next lemma is important to show Lemma 2.3 which is the key to show Theorem 1.3:

Lemma 2.2. *For any function ψ in S, there exists a constant C depending only on ψ such that*

$$|\psi(x)| \leqq \frac{C}{1+|x|} \tag{2.8}$$

$$|\psi_{x_j}(x)| \leqq \frac{C}{1+|x|^2},\ j = 1,2 \tag{2.9}$$

for $x \in \mathbb{R}^2$.

Proof. If ψ is in S, then there exists a function ω in T such that

$$\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \omega(y)\log|x-y|dy. \tag{2.10}$$

Notice that ω is in $C_0^\infty(\mathbb{R}^2)$, so there is a constant $R = R(\omega)$ such that supp$\omega \subset B(0,R)$. Since ω is odd in x_2, we get

$$\psi(x) = \frac{1}{2\pi}\int_{B(0,R)} \omega(y)\log|x-y|dy \tag{2.11}$$
$$= \frac{1}{2\pi}\int_{B_+(0,R)} \omega(y)\log\frac{|x-y|}{|x-\bar{y}|}dy,$$

$$\psi_{x_1}(x) = \frac{1}{2\pi}\int_{B(0,R)} \omega(y)\frac{x_1-y_1}{|x-y|^2}dy \tag{2.12}$$
$$= \frac{2x_2}{\pi}\int_{B_+(0,R)} \omega(y)\frac{y_2(x_1-y_1)}{|x-y|^2|x-\bar{y}|^2}dy,$$

$$\psi_{x_2}(x) = \frac{1}{2\pi}\int_{B(0,R)} \omega(y)\frac{x_2-y_2}{|x-y|^2}dy \tag{2.13}$$
$$= \frac{1}{\pi}\int_{B_+(0,R)} \omega(y)\frac{y_2\left\{(x_2-y_2)(x_2+y_2)-(x_1-y_1)^2\right\}}{|x-y|^2|x-\bar{y}|^2}dy,$$

where $B_+(0,R)$ denotes $B(0,R)\cap\mathbb{R}^2_+$ and $\bar{y}=(y_1,-y_2)$. Now we show that ψ and ψ_{x_j} is bounded in $B(0,R)$ and that there exists a constant C such that

$$\begin{aligned}|\psi(x)| &\leqq C|x|^{-1},\\ |\psi_{x_j}(x)| &\leqq C|x|^{-2} \text{ for } |x|\geqq 2R.\end{aligned}\tag{2.14}$$

The boundedness of ψ and ψ_j on $B(0,2R)$ is obtained by the fact that $\log|x|$ and $|x|^{-1}$ are in $L^1_{loc}(\mathbb{R}^2)$:

$$\begin{aligned}|\psi(x)| &\leqq \frac{\sup|\omega|}{2\pi}\int_{B(0,R)}|\log|x-y||\,dy\\ &= C_\omega\int_{B(x,R)}|\log|y||\,dy\\ &\leqq C_\omega\int_{B(0,3R)}|\log|y||\,dy\\ &= C_{\omega,R}<\infty,\end{aligned}\tag{2.15}$$

$$\begin{aligned}|\psi_{x_j}(x)| &\leqq \frac{\sup|\omega|}{2\pi}\int_{B(0,R)}\frac{1}{|x-y|}dy\\ &\leqq C_\omega\int_{B(0,3R)}|y|^{-1}dy\\ &= C_{\omega,R}<\infty\end{aligned}\tag{2.16}$$

for $|x|\leqq 2R$. (Notice that $|y|\leqq|x-y|+|x|\leqq 3R$.)

Now we show (2.14) to complete the proof. We may assume $x_2\geqq 0$ to estimate ψ, because ψ is odd in x_2. By this assumption, we get the following inequality:

$$1\leqq\frac{|x-\bar{y}|}{|x-y|}\leqq\frac{|x-y|+|y-\bar{y}|}{|x-y|}\leqq 1+\frac{2R}{|x|-R}\leqq 1+\frac{4R}{|x|}\tag{2.17}$$

for $|x|\geqq 2R$ and $|y|\leqq R$. The inequality (2.17) leads the estimate of ψ :

$$\begin{aligned}|\psi(x)| &\leqq \frac{1}{2\pi}\log\left(1+\frac{4R}{|x|}\right)\int_{B_+(0,R)}\omega(y)dy\\ &\leqq \frac{1}{2\pi}\frac{4R}{|x|}\log\left(1+\frac{4R}{|x|}\right)^{\frac{|x|}{4R}}\|\omega\|_{L^1(B_+(0,R))}\\ &\leqq \frac{2R}{\pi}\|\omega\|_{L^1(\mathbb{R}^2)}|x|^{-1}.\end{aligned}\tag{2.18}$$

Notice that the inequality

$$\frac{|x|}{2}\leqq|x-y|,\frac{|x|}{2}\leqq|x-\bar{y}|$$

holds for $|x| \geqq 2R, |y| \leqq R$. This inequality leads the estimate of ψ_{x_j} :

$$
\begin{aligned}
(2.19) \qquad |\psi_{x_1}(x)| &\leqq \frac{2|x|}{\pi} \int_{B_+(0,R)} |\omega(y)| \frac{|y|}{|x-y|^2|x-\bar{y}|} dy \\
&\leqq \frac{16R\|\omega\|_{L^1}}{\pi|x|^2}, \\
(2.20) \qquad |\psi_{x_2}(x)| &\leqq \frac{1}{\pi} \int_{B_+(0,R)} |\omega(y)| \frac{2|y|}{|x-y||x-\bar{y}|} dy \\
&\leqq \frac{8R\|\omega\|_{L^1}}{\pi|x|^2}.
\end{aligned}
$$

Combining the estimate (2.15), (2.16), (2.18), (2.19), and (2.20) leads the conclusion (2.8) and (2.9). □

Now we are ready to show the key lemma.

Lemma 2.3. *There exists a function ψ in S such that*

$$(2.21) \qquad \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 \, dx \neq \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 \, dx.$$

Proof. Assume that the conclusion is not true, i.e.

$$(2.22) \qquad \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 \, dx = \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 \, dx$$

for any ψ in S. Let ψ is in S, and let $\psi^h(x) = \psi(x_1 - h, x_2)$. Then the function ψ^h and $\psi + \psi^h$ are in S. In fact,there exists a function ω in T such that $\psi = (-\Delta)^{-1}\omega$, and we can write $\psi^h = (-\Delta)^{-1}\omega^h$ and $\psi + \psi^h = (-\Delta)^{-1}(\omega + \omega^h)$, where $\omega^h(x) = \omega(x_1 - h, x_2)$. It is obvious that ω^h and $\omega + \omega^h$ are in T.

By the assumption (2.22), we get

$$
\begin{aligned}
(2.23) \qquad \int_{\mathbb{R}^2} \{\psi_{x_1}(x)\}^2 \, dx &= \int_{\mathbb{R}^2} \{\psi_{x_2}(x)\}^2 \, dx, \\
\int_{\mathbb{R}^2} \{(\psi^h)_{x_1}(x)\}^2 \, dx &= \int_{\mathbb{R}^2} \{(\psi^h)_{x_2}(x)\}^2 \, dx, \\
\int_{\mathbb{R}^2} \{(\psi + \psi^h)_{x_1}(x)\}^2 \, dx &= \int_{\mathbb{R}^2} \{(\psi + \psi^h)_{x_2}(x)\}^2 \, dx.
\end{aligned}
$$

Combining the equalities (2.23), we get

$$(2.24) \qquad \int_{\mathbb{R}^2} \psi_{x_1}(x)(\psi^h)_{x_1}(x) dx = \int_{\mathbb{R}^2} \psi_{x_2}(x)(\psi^h)_{x_2}(x) dx.$$

Now we integrate the equality (2.24) by h:

$$(2.25) \qquad \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_1}(x)(\psi^h)_{x_1}(x) dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} \psi_{x_2}(x)(\psi^h)_{x_2}(x) dx.$$

Notice that we can change the order of integration by the estimate (2.9). Firstly, we compute the left term of (2.25):

(2.26)
$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\mathbb{R}^2}\psi_{x_1}(x)(\psi^h)_{x_1}(x)dx\,dh &= \int_{\mathbb{R}^2}\psi_{x_1}(x)\int_{-\infty}^{\infty}\psi_{x_1}(x_1-h,x_2)dh\,dx\\ &= \int_{\mathbb{R}^2}\psi_{x_1}(x)\Big[\psi(h,x_2)\Big]_{-\infty}^{\infty}dx\\ &= 0.\end{aligned}$$

The equality is obvious by the estimate (2.8). Secondly, we compute the right term of (2.25):

(2.27)
$$\begin{aligned}\int_{-\infty}^{\infty}\int_{\mathbb{R}^2}\psi_{x_2}(x)(\psi^h)_{x_2}(x)dx\,dh &= \int_{\mathbb{R}^2}\psi_{x_2}(x)\int_{-\infty}^{\infty}\psi_{x_2}(x_1-h,x_2)dh\,dx\\ &= \int_{-\infty}^{\infty}\left\{\int_{\infty}^{\infty}\psi_{x_2}(x_1,x_2)dx_1\right\}^2dx_2\\ &> 0.\end{aligned}$$

The positivity of integrals is shown by computing the form of ψ_{x_2}. Notice that ψ_{x_2} is continuous, so it is sufficient to check that $\psi_{x_2}(x_1,0)$ is not zero. By (2.13), we can write ψ_{x_2} as

$$\psi_{x_2}(x)=\frac{1}{\pi}\int_{B_+(0,R)}\omega(y)\frac{y_2\left\{(x_2-y_2)(x_2+y_2)-(x_1-y_1)^2\right\}}{|x-y|^2|x-\bar{y}|^2}dy \tag{2.28}$$

where ω is a function in T. Putting $x_2=0$ in (2.28) leads

$$\psi_{x_2}(x_1,0)=-\frac{1}{\pi}\int_{B_+(0,R)}\omega(y)\frac{y_2}{(x_1-y_1)^2+y_2^2}dy. \tag{2.29}$$

Since ω and the integral kernel are positive in $B_+(0,R)$, $\psi_{x_2}(x_1,0)$ is always nonzero. The results of compution (2.26) and (2.27) leads a contradiction, and we get a conclusion of the lemma. □

Proof of Theorem 1.3. Let ψ the function in S that satisfies (2.20). Let $\psi^\epsilon(x)=\psi(x/\epsilon)$. Then

$$\left\{(\psi^\epsilon)_{x_1}(x)\right\}^2-\left\{(\psi^\epsilon)_{x_2}(x)\right\}^2=\frac{1}{\epsilon^2}\left\{(\psi_{x_1})^2-(\psi_{x_2})^2\right\}\left(\frac{x}{\epsilon}\right)$$

and Lemma 2.1 is appliable. (Notice that $\psi_{x_1}^2-\psi_{x_2}^2$ is in $L^1(\mathbb{R}^2)$: the estimate (2.9) leads this fact.) □

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