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1 INTRODUCTION

We are concerned with a parabolic equation of form

$$u_t - a(u_x)\{W'(u_x)_x - C(x)\} = 0, \qquad x \in \mathbf{R}, \quad t > 0. \tag{1}$$

Here W' is the derivative of a given convex function $W : \mathbf{R} \to \mathbf{R}$ and $a : \mathbf{R} \to \mathbf{R}$ is a given positive function; $C : \mathbf{R} \to \mathbf{R}$ is a given function. This equation stems from equations of curves moved by its curvature. In fact, if $a(p) = W(p) = (1 + p^2)^{1/2}$ and $C \equiv 0$, the equation (1) becomes the curve shortening equation for a curve $y = u(t, x)$. Such equations are important to describe motion of phase-boundaries. For general references the reader is referred to [Gu] and [TCH].

Recently, several authors discussed (1) when W is not necessarily C^1. Important examples include crystalline energy ([AG], [T1], [T3]) where W is piecewise linear. For background of this problem the reader is referred to recent articles [GG1], [GG2], [GG3], review papers [T2], [GirK] and papers cited there. In this case the diffusion is very strong so that the equation (1) is no longer a partial differential equation. One should introduce right notion of solutions to solve (1). This idea is implemented when C is independent of the spatial variable [GG3]. However, there is essentially no mathematical analysis when C depends on the spatial variable except the work of Roosen [R], though there are extensive numerical calculation when W is crystalline e. g. [RT].

In this note we study subdifferential formulation of the quantity

$$-\Lambda^C = -W'(u_x)_x + C(x)$$

proposed by [FG] and observe that Λ^C is consistent with the calculation given by Roosen [R]. Contrary to the case when C is constant, Λ^C may not be a constant on a facet (linear part of the graph of u). So facets should be shattered (splitted) into small pieces to solve (1). Several interesting methods are proposed in [RT] by approximating C by piecewise *constant* functions. Unfortunately, we do not know which is the best way to track or at least approximate reasonable solutions. When C is piecewise *linear*, we conjecture that reasonable solutions of (1) should be piecewise linear if a is a constant and initial data is piecewise linear. In this note we give several exact solutions of (1) with constant a and $W(p) = \text{const.}|p|$ when C is a special piecewise linear function. Initial data we give are rather special but typical. Our solutions are exact at least in the sense of nonlinear semigroups [FG]. Our exact solutions indicate a reasonable way of shattering facets. To our knowledge these are first explicit examples of solutions of (1) with $W(p) = \text{const.}|p|$ and nonconstant C. We also give general criteria so that the speed of facets are constant. This note also reviews results of [GG1] – [GG4] from semigroup point of view.

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2 SUBDIFFERENTIAL FORMULATION

As in [FG] to simplify the situation we assume that u is spatially periodic with period ω. The quantity $-\Lambda^C = -W'(u_x)_x + C$ is the Euler-Lagrange operator of the integral

$$\Phi^C(u) = \int_0^\omega \{W(u_x) + Cu\}dx.$$

In other words $-\Lambda^C$ is a 'derivative' of Φ^C but one should clarify the norm of the space of functions to state it rigorously. For technical reasons we shall always *assume* the *coerciveness condition*

$$\lim_{p\to\infty} W(p)/|p| = \infty$$

for the convex function W when we consider Φ^C and that $C \in L^\infty(\mathbf{T})$ with $\mathbf{T} = \mathbf{R}/\omega\mathbf{Z}$ i.e. $C \in L^\infty(\mathbf{R})$ and C is ω-periodic. We now rigorously define Φ^C as a functional on the Hilbert space $H = L^2(\mathbf{T})$:

$$\Phi^C(u) = \begin{cases} \int_0^\omega \{W(u_x) + Cu\}dx & \text{if } u_x \in L^1(\mathbf{T}) \text{ and } W(u_x) \in L^1(\mathbf{T}), \\ \infty & \text{otherwise,} \end{cases}$$

where $u \in H$. The subdifferential $\partial \Phi^C(u)$ is the set of all $f \in H$ that satisfies

$$\Phi^C(u+h) - \Phi^C(u) \geq < f, h >$$

for all $h \in H$, where $<,>$ denotes the inner product of H:

$$< f, h >= \int_0^\omega fh dx.$$

The subdifferential $\partial \Phi^C$ is a substitute of derivative of Φ^C when $\Phi^C(u)$ is not differentiable in u. Under our assumptions we have a characterization of the subdifferential.

LEMMA 1. *(i) The fucntional $\Phi^C (\not\equiv \infty)$ is lower semicontinuous and convex in H. (ii) For $u \in H$, $f \in H$ belongs to $\partial \Phi^C(u)$ if and only if there is an absolutely continuous function η on $\mathbf{T}$ that satisfies*

$$\eta(x) \in \partial W(u_x(x)) \quad \text{a.e. } x \qquad \text{and } f = -\eta_x + C.$$

These results are standard and obtained by adapting the argument in [Br, example 3]. If W is smooth, (ii) says $f = -\Lambda^C$. In other words, $\partial \Phi^C(u)$ is a singleton (for smooth u) and its element is $-\Lambda^C$. This explains the reason why subdifferentials play a role of derivatives (for convex functionals). If W is not C^1, $\partial \Phi^C(u)$ may not be a singleton. We should examine what element of $\partial \Phi^C(u)$ is chosen in evolutions.

Subdifferential Equations

If we use subdifferentials the equation (1) can be written as

$$u_t \in -a(u_x) \partial \Phi^C(u) \tag{2}$$

at least formally. If a is a constant function we may assume that a equals one by a normalization. In this case (2) is a subdifferential equation (i.e. an equation of divergence type). Since $\Phi^C (\not\equiv \infty)$ is a lower semicontinuous and convex function (densely defined) on H by Lemma 1 (i), the theory of nonlinear semigroup (initiated by Kōmura [Ko]) applies (2) with constant a. It yields a unique global-in-time solution for initial value problem to (2) for any initial data in H; see e.g. [Ba, IV Theorem 2.1].

For general nonconstant a, if C vanishes identically, then (1) becomes a subdifferential equation

$$u_t \in -\partial \overline{\Phi}(u) \tag{3}$$

with

$$\overline{\Phi}(u) = \int_0^\omega \overline{W}(u_x) dx,$$

where $\overline{W}$ is a convex function that satisfies $\overline{W}'' = aW''$. This is exactly the case studied in [FG]. Again we obtain a unique global solution.

For both (2) and (3), the solution u we get is continuous in $[0,\infty)$ with values in H, i.e., $u \in C([0,\infty), H)$ and locally absolutely continuous in $(0,\infty)$ with values in H so that du/dt exists for a.e. $t \geq 0$. The equations (2) and (3) are fulfilled for a.e. $t \geq 0$, respectively.

It is also known [Ba] that the solution u is right differentiable for all $t > 0$ as a function on $[0,\infty)$ with valued in H. Its right derivative d^+u/dt equals $-A^0u$ for all $t > 0$, where A^0 is the *canonical restriction* of $A = \partial\Phi^C$ and $\partial\overline{\Phi}$ for (2) and (3), respectively. We shall recall its definition.

Let A be a multivalued operator in H with domain $D(A)$. For $v \in D(A)$ if there is a unique minimizer of

$$\{\|w\|_H; w \in Av\}, \tag{4}$$

we denote it by A^0v. If the operator A is maximal monotone, then the set Av is a closed convex set so there is a unique minimizer for (4), which is the element Av closest to the origin. The single valued operator A^0 is called the *canonical restriction* or *minimal section* of A. Fortunately, if $\Phi(\not\equiv \infty)$ is lower semicontinuous and convex, then its subdifferential $\partial\Phi$ is always maximal monotone so that its canonical restriction is well-defined. What is important in this section is that the canonical restriction is a nice selection of multivalued subdifferential operator. Although we won't use any details of the theory of subdifferential equations, the reader is referred to a book [Ba] for details if they are interested in the theory.

REMARK 1. In applications the coerciveness condition for W may not be fulfilled. However, if we restrict ourselves in Lipschitz-in-space solutions, we may assume that W is coercive by modifying $W(p)$ for large p as is done in [FG]. There is one caution when C is involved for (2) with constant a. For (3) if initial data u_0 is Lipschitz i.e. $|u_{0x}| \leq K$, then the solution satisfies $|u_x| \leq K$ for all $t \geq 0$ [FG]. However, this is no longer true when C exists. To observe this we formally differentiate (1) in x, assuming that W and C are smooth enough. We then have a parabolic equation for u_x:

$$u_{xt} = a\{W''(u_x)u_{xx}\}_x + C_x,$$

where a is a constant. The parabolic maximum principle yields

$$|u_x(t,x)| \leq K + t\sup_x |C_x|. \tag{5}$$

This is the best estimate we expect when C exists. Actually by an approximation argument, one can prove (5) for the solution of (2) as in [FG], when C is Lipschitz. From the estimate (5) we observe that we may assume that W is coercive when we consider (2) for each finite time interval $(0,T)$ (but not for $(0,\infty)$.)

3 GENERAL EQUATIONS WITH CONSTANT C

When a is nonconstant and C is nonzero, it is hard to analyse (2) by using the nonlinear semigroup theory in H (even if C is a constant) since (2) ceases to be a subdifferential equations. In [GG1]-[GG4] the authors analyzed (2) (or (1)) by extending the theory of viscosity solution [CIL] where C is independent of the spatial variable. Let us review some of their results from the semigroup point of view.

THEOREM 2. *Assume that $a \in C(\mathbf{R})$ and $a \geq 0$ and that C is a constant. Assume that a convex function W is C^2 except a discrete set P in $\mathbf{R}$ and that W'' is bounded on every bounded set of $\mathbf{R} \setminus P$. (We do not need to assume that W is coercive).*
(i) For each $u_o \in C(\mathbf{T})$ there is a unique solution $u \in C([0,\infty) \times \mathbf{T})$ of (1) with $u(0,x) = u_o(x)$ in the sense of [GG3].
(ii) Let T_t be the mapping $u_o \longmapsto u(t,\cdot)$ from $C(\mathbf{T})$ into itself, where u is the solution of (1) with initial data u_o . Then $\{T_t\}_{t\geq 0}$ is an order-preserving semigroup on $C(\mathbf{T})$. In other words:
(a) $T_0 =$ identity, $T_{t+s} = T_t \cdot T_s \quad (t, s \geq 0)$;
(b) $T_t u_o$ is continuous in $t \in [0,\infty)$ with values in $C(\mathbf{T})$ for each $u_o \in C(\mathbf{T})$;
(c) $\|(T_t u_o - T_t v_o)_+\|_\infty \leq \|(u_o - v_o)_+\|_\infty$ for all $u_o, v_o \in C(\mathbf{T})$, where $f_+ = \max(f, 0)$ and $\|f\|_\infty$ denotes the norm of f in $C(\mathbf{T})$, i.e. the maximum norm of f. (This property in particular implies that $\|T_t u_o - T_t v_o\|_\infty \leq \|u_o - v_o\|_\infty$, i.e. $\{T_t\}_{t\geq 0}$ is a contraction semigourp in $C(\mathbf{T})$.)

The part (a) is contained in [GG3] where the unique existence of solutions is established for fully nonlinear version of (1). The part (ii)(a), (b) is a trivial consequence of (i). The part (ii)(c) follows from the comparison theorem in [GG3]. Indeed, the comparison theorem implies that $T_t u_o \leq T_t u_1$ if $u_o \leq u_1$. Note that if u solves (1), so does $u + m$ with a constant m. Since $u_o \leq v_o + m$ with $m = \|(u_o - v_o)_+\|_\infty$, we observe that

$$T_t u_o \leq T_t(v_o + m) = T_t v_o + m$$

which yields (c).

The reader may be curious whether the solution in Theorem 2 agrees with that in [FG] when C is zero. Recently the authors [GG4] obtained a stability theorem which guarantees that both solutions agree with each others since they are obtained by the same approximation. Let us present a simplest version of their general results in [GG4].

THEOREM 3. *Let u^ϵ be a classical solution of*

$$u_t - a^\epsilon(u_x)((W^{\epsilon\prime}(u_x))_x - C) = 0$$

with initial data $u_o^\epsilon \in C^\infty(\mathbf{T})$, where $a^\epsilon, W^\epsilon \in C^\infty$ with $a^\epsilon > 0$ and $W^{\epsilon\prime\prime} > 0$. Suppose that $u_o^\epsilon \to u_o$ in $C(\mathbf{T})$ and $a^\epsilon \to a$, $W^\epsilon \to W$ (locally uniform) in $\mathbf{R}$. Then u^ϵ converges to a function u locally uniform on $[0,\infty) \times \mathbf{T}$ as $\epsilon \to 0$ and u solves (1) in the sense of [GG3].

The existence of u^ϵ is classical [LSU]. Theorem 3 says that our solution is constructed as the limit of u^ϵ. If $C = 0$, with a special choice of a^ϵ and W^ϵ the solution by the nonlinear semigroup theory is also approximated by the same u^ϵ[FG]. Thus, the two solution should agree.

4 CALCULATION OF CANONICAL RESTRICTION

To simplify the situation here and hereafter we consider the case that W is piecewise linear and that the set P of jumps of W' is finite, i.e. $P = \{p_1 < p_2 < \dots < p_m\}$.

We shall calculate the canonical restriction of $\partial\Phi^C(v)$ for a piecewise linear function $v \in C(\mathbf{T})$ with finitely many nondifferentiable points Σ. We say slopes of v are *consistent* with P if at each $s \in \Sigma$ the open interval between $v_x(s-0)$ and $v_x(s+0)$ does not intersect P. This is of course compatible with the definition in [GG1]-[GG3]. If all slopes v_x belongs to P, then clearly, slopes of v are consistent with P if and only if $v_x = p_{i-1}$ or p_{i+1} on all faceted regions adjacent to a faceted region where $v_x = p_i$; v is often called an admissible crystal [GG1]-[GG3]. Here $[a,b]$ is called a *faceted region* if it is the closure of a maximal interval in $\mathbf{T}\backslash\Sigma$. If $v_x(x_0) = p_i$, $v_x(x_1) = q \in P$ with $x_1 < x_2$ and $v_x \notin P$ on all facetd regions contained in (x_0, x_1), then $q \in \{p_{i+1}, p_i, p_{i-1}\}$ and $v_x(x)$ lies in either (p_{i-1}, p_i) or (p_i, p_{i+1}) faceted regions in (x_0, x_1), provided that all slopes of v are consistent with P. (A piecewise linear nonadmissible function whose slopes are consistent with P is also considered in [EGS].) The next result is an extension of [FG, Lemma 4.1].

THEOREM 4. *Let $v \in C(\mathbf{T})$ be a piecewise linear function with finitely many nondifferentiable points Σ. (For sufficiently large $|p| > \sup|v_x|$ the function W is modified so that it is coercive to give the meaning of $\partial\Phi^C$.)*
(i) The set $\partial\Phi^C(v)$ is non-empty if and only if all slopes of v are consistent with P.
(ii) Suppose that C is Lipschitz. Suppose that $Av = \partial\Phi^C(v)$ is non-empty. Let (a,b) be the interior of a faceted region of v with slope p.
(a) $(A^0v)(x) = C(x), x \in (a,b)$ if $p \notin P$;
(b) $(A^0v)(x) = -\zeta_x(x), x \in (a,b)$ if $p \in P$; here, $\bar{\zeta}$ is a unique $C^{1,1}$ solution of an obstacle problem

$$\begin{aligned} min\{ \int_a^b |\zeta_x(x)|^2 dx; \quad & \zeta(a) = \lim_{\epsilon \downarrow 0} W'(p + \epsilon(v_x(a-0) - p)), \\ & \zeta(b) = \lim_{\epsilon \downarrow 0} W'(p + \epsilon(v_x(b+0) - p)) - \int_a^b C(z)dz, \\ & Z(x) \le \zeta(x) \le \bar{Z}(x) \text{ for } x \in (a,b)\} \end{aligned}$$

with

$$\begin{aligned} & Z(x) = W'(p-0) - \int_a^x C(z)dz, \qquad \bar{Z}(x) = Z(x) + \Delta, \\ & \Delta = W'(p+0) - W'(p-0). \end{aligned}$$

The proof is essentially along the line of that of [FG, Lemma 4.1] by setting $\zeta = \eta - \int C$. The main ingredient is Lemma 1(ii). We omit the detail of the proof. The obstacle problem is well studied even for the multi-dimensional case [Fr]. The existence of $C^{1,1}$ solution is known for Lipschitz C. (Here a function is called $C^{1,1}$ if its derivative is Lipschitz). This theorem indicates that it is natural to set $-\Lambda^C = A^0v$ for the quantity Λ^C in the introduction.

We shall give a sufficient condition so that $A^0 v$ is a constant on a faceted region. We recall the *transition number* χ of v on the faceted region (a, b). The value of χ is assigned to be equal to one if $v_x(a-0) < p < v_x(b+0)$; minus one if $v_x(a-0) > p > v_x(b+0)$; zero otherwise.

LEMMA 5. *Under the assumption of Theorem 4 let (a, b) be a faceted region of v with slope $p \in P$. Assume that its transition number χ equals either one or minus one. If*

$$- \inf_{a \le x \le b} C + L^{-1} \int_a^b C(z)dz \le \Delta/L, \tag{6}$$

then

$$\Lambda^C = -(A^0 v)(x) = \chi\Delta/L - L^{-1} \int_a^b C(z)dz, \quad x \in (a, b), \tag{7}$$

where L is the length of the faceted region (a, b), i.e., $L = b - a$. In particular, if L is sufficiently small so that

$$-\inf C + \sup C \le \Delta/L, \tag{8}$$

then by (7), Λ^C is a constant on the faceted region (a, b) since (6) is fulfilled.

Proof. Since the proof for $\chi = -1$ parallels the case $\chi = 1$, we may assume that $\chi = 1$. We may also assume that $W'(p-0) = 0$. It suffices to prove that an affine function

$$\zeta_0(x) = (\Delta - \int_a^b C(z)dz)(x-a)/L, \quad a < x < b$$

is a minimizer of the obstacle problem in Theorem 4(b). For this purpose it suffices to prove that ζ_0 satisfies $Z \le \zeta_0 \le \bar{Z}$, i.e.,

$$-\int_a^x C \le \zeta_0(x) \le \Delta - \int_a^x C, \quad a < x < b \tag{9}$$

since $\zeta_{0x} = \Delta/L - L^{-1} \int_a^b C$ and $\zeta_0(a) = W'(p-0) = 0$,

$$\zeta_0(b) = \Delta - \int_a^b C = W'(p+0) - \int_a^b C.$$

The inequality (6) yields

$$-(x-a)^{-1} \int_a^x C + L^{-1} \int_a^b C \le \Delta/L, \tag{10_1}$$

$$-(b-x)^{-1} \int_x^b C + L^{-1} \int_a^b C \le \Delta/L, \tag{10_2}$$

for all $x \in (a, b)$. These two inequalities are equivalent to (9) so the proof is now complete.

REMARK 2. (i) As in [FG] if C is a constant, then the solution of the obstacle problem in Theorem 4 is an affine function and

$$-(A^0 v)(x) = \Lambda^C = \bar{\zeta}_x = \chi\Delta/L - C, \quad x \in (a,b). \tag{11}$$

Indeed, if χ is either one or minus one, then (8) is always fulfilled to get (7) or (11). If $\chi = 0$, then it is clear that Z or $\bar{Z}$ minimizes the obstacle problem since they are affine. Again we obtain (11).

In this case we have

$$(\partial\Phi^C)^0 = (\partial\Phi^0)^0 + C.$$

However, this formula needs not to be true when C depends on x. Note also that Λ^C needs not to be a constant on a faceted region if C depends on x.
(ii) In [Ry] a crystalline version of the Mullins-Sekerka type problem was studied, There the quantity corresponding to Λ^C was defined like (7) by using the avarage of C on each facet. Lemma 5 indicates that this choice is good if χ is not zero and the facet is sufficiently small.
(iii) In the case of $\chi = 0$, we do not expect that $A^0 v$ is a constant in general even if L is sufficiently small. However, sometimes $A^0 v$ is easy to calculate as follows.

LEMMA 6. *Assume the same hypotheses of Lemma 5 except for χ. Assume that χ equals zero.*
(i) If C is nondecreasing on (a,b) and $v_x(a-0) < p$ (or C is nonincreasing and $v_x(a-0) > p$), then

$$(A^0 v)(x) = C(x), \qquad \text{for all } x \in (a,b). \tag{12}$$

(ii) If C is nondecreasing on (a,b) and $v_x(a-0) > p$ (or C is nonincreasing and $v_x(a-0) < p$), then $A^0 v$ is a constant on (a,b) and it equals

$$(A^0 v)(x) = L^{-1}\int_a^b C(z)dz. \tag{13}$$

Proof. We may assume tht C is nondecreasing since the other case can be treated in the same way.
(i) By the concavity of Z and $v_x(a-0) < p$, the minimizer of the obstacle problem in Theorem 4 should be Z itself so we obtain (12).
(ii) In this case the minimizer is affine with slope $-L^{-1}\int_a^b C$ so (13) follows.

REMARK 3. It turns out that our canonical restriction satisfies all properties [R, Theorem 3] of a minimal velocity profile for his implicit time discretization scheme [AT], [ATW] with a trivial modification. Although it is not explicitly written in [R], a minimal velocity profile is characterized by properties in [R, Theorem 3] at least for a large class of C. Thus our canonical restriction essentially agrees with his minimal velocity profile. We do not give a detailed statement and proof. Instead we give a typical property of the canonical restriction which has been proved for the minimal velocity profile [R, Theorem 3(ii),(i),(iii)].

LEMMA 7. *Assume the hypotheses of Theorem 4(b). Assume that C is Lipschitz and that $[a_1, b_1]$ is a maximal interval in the interior of the faceted region $[a, b]$ such that $A^0 v$ is constant on (a_1, b_1). Then $C(a_1) = C(b_1)$. Assume furthermore that C is differentiable at a_1 and b_1.*
(i) Assume that $C'(a_1) > 0$ and $C'(b_1) < 0$. Then

$$\int_{a_1}^{b_1} C dx - \Delta = (b_1 - a_1)C(a_1).$$

(ii) Assume that $C'(a_1) < 0$ and $C'(b_1) > 0$. Then

$$\int_{a_1}^{b_1} C dx + \Delta = (b_1 - a_1)C(a_1).$$

(iii) Assume that $C'(a_1)C'(b_1) > 0$. Then

$$\int_{a_1}^{b_1} C dx = (b_1 - a_1)C(a_1).$$

Proof. There is a unique minimizer $\bar{\zeta}$ of the obstacle problem in Theorem 4(b) such that $Z \le \bar{\zeta} \le \bar{Z}$ and $\bar{\zeta}_x$ is a constant on $[a_1, b_1]$. Since $\bar{\zeta}$ is $C^{1,1}$ [Fr] and since $[a_1, b_1]$ is maximal so that $\bar{\zeta}$ agrees with either Z or $\bar{Z}$ at a_1 and b_1, we have

$$Z_x(a_1) = \bar{\zeta}_x(a_1) = \bar{\zeta}_x(b_1) = Z_x(b_1)$$

to get $C(a_1) = C(b_1)$.

If $C'(a_1) > 0$ and $C'(b_1) < 0$, then $Z \le \bar{\zeta} \le \bar{Z}$ implies that

$$\bar{\zeta}(a_1) = Z(a_1) \text{ and } \bar{\zeta}(b_1) = \bar{Z}(b_1).$$

Since the slope of $\bar{\zeta}$ on $[a_1, b_1]$ equals $-C(a_1)$, we have

$$(\bar{Z}(b_1) - Z(a_1))/(b_1 - a_1) = -C(a_1).$$

Rearranging this identity completes the proof of (i). The proof of (ii),(iii) is similar so is omitted.

REMARK 4. (i) The theory of viscosity solutions has not yet been extended when C is not spatially constant. However, Theorem 4 suggests that the 'time derivative' of test function ϕ in the definition of viscosity solutions for (1) should be given by $-a(\phi_x)(A^0\phi)(x)$.
(ii) We give a remark on the semigroup $\{T_t\}_{t\ge 0}$ on $C(\mathbf{T})$ in Theorem 2. Different from the semigroup on the Hilbert space $L^2(\mathbf{T})$, $u = T_t u_o$ may not be right differentiable in $(0, \infty)$ with values in $C(\mathbf{T})$. For example, when W is piecewise linear and $0 \in P$, one can prove that

$$\lim_{t \downarrow 0}(T_t u_o - u_o)/t \in C(\mathbf{T})$$

if and only if u_o equals a constant. This says that the domain of generator of $\{T_t\}_{t\geq 0}$ equals the space of constant functions. So $T_t u_o$ is not right differentiable except at the time t_0 that $T_{t_0}u_o$ is constant.

5. EXAMPLES OF SOLUTIONS WITH NONCONSTANT C

We consider an example of (2) (or (1)) with constant $a \equiv 1$ and piecewise linear C when W is piecewise linear. As our examples show later, we expect that solution stays spatially piecewise linear when it is initially piecewise linear for such a situation. We give definition of solutions for such a class of functions.

Definition. Let u be a Lipschitz continuous function of $[0,T)\times\mathbf{R}$. Assume that $u(t,\cdot)$ is piecewise linear whose slopes are consistent with P for $t>0$. We say that u *solves* (1) if for each $t\in(0,T)$ and the set of interior $I(t)$ of each faceted region $\overline{I}(t)$ of $u(t,\cdot)$, the time derivative u_t exists and

$$u_t(t,x) = -C(x), \qquad x\in I(t)$$

when $u_x(t,x)\notin P$, and

$$u_t(t,x) = \overline{\zeta_x}(x), \qquad x\in I(t)$$

when $u_t(t,x)\in P$, where $\overline{\zeta}$ is the unique minimizer of the obstacle problem in Theorem 4 (b) with $(a,b)=I(t)$.

It is easy to check that if u is spatially periodic, a solution in the above sense is a solution of (2) in the sense of subdifferential equation in $L^2(\mathbf{T})$ by modifying $W(p)$ for large p so that W is coercive. Such a solution is unique if initial data $u(0,x)=u_0(x)$ is given and piecewise linear.

Unfortunately, we shall consider non-periodic initial data u_0. However if u_{0x} is constant on $(-\infty,-R]$ and $[R,\infty)$ for large R and $u_{0x}\notin P$ on these intervals, it is easy to construct a periodic piecewise linear function v_0 that satisfies $u_0=v_0$ on $[-R,R]$ and that $u_{0x}=v_{0x}$ near $\pm R$. (If u_0 is consistent with P, so is v_0.) If C is compactly supported, say $C\equiv 0$ outside $(-R/2,R/2)$ our solution u with initial data u_0 should agree with the solution v of (2) with initial data v_0 near $(-R,R)$ for sufficiently short time. This is because both u and v do not move near $x=\pm R$ at least for a short time and the length of faceted regions containing $x=\pm R$ does not affect the evolution since $u_{0x}(\pm R)\notin P$. Since solution of (2) is unique, this observation shows that our solution of (1) is *unique* for piecewise linear initial data u_0 if u_0 is periodic or $u_{0x}(x)$ is constant for large x with $u_{0x}\notin P$.

As an special example of (1) we take

$$W(p)=\Delta|p|/2, \qquad C(x)=h(\beta_0-|x|)_+/\beta_0 \tag{14}$$

where Δ, h, β_0 are positive. Since P is a singleton, slopes of a piecewise linear function are always consistent with P. As an initial data we consider

$$u_0(x;\alpha_0,q_1,q_2)=u_0(x)=\begin{cases}0 & |x|\le\alpha_0,\\ q_1(x-\alpha_0) & x\ge\alpha_0,\\ q_2(x+\alpha_0) & x\le-\alpha_0,\end{cases}$$

where $\alpha_0>\beta_0$ and $|q_1|=|q_2|(\neq 0)$. The interval $[-\alpha_0,\alpha_0]$ is a faceted region with slope zero. The situation is classified in three cases.

Case 1. $q_1>0>q_2$.

Clearly, the transition number χ of u_0 on $[-\alpha_0,\alpha_0]$ equals 1.

Case 2. $0<q_1=q_2$ or $0>q_1=q_2$. In this case $\chi=0$.

Case 3. $q_1<0<q_2$. In this case $\chi=-1$.

We may assume that $\beta_0=1$ and $h=1$ by rescaling variables as follows:

$$x=y\beta_0,\quad t=s/h,\quad \tilde{u}(y,s)=u(x,t).$$

The rescaled function solves

$$\tilde{u}_s-\left(\tilde{W}(\tilde{u}_y)_y-\frac{C(y\beta_0)}{h}\right)=0$$

with $\tilde{W}(p)=\tilde{\Delta}|p|/2, \tilde{\Delta}=\Delta/(\beta_0 h)$. Of course, for the initial data for rescaled variable is

$$\tilde{u}_0(y)=u_0(y;\alpha_0/\beta_0,q_1\beta_0,q_2\beta_0).$$

By this scaling transformation remaining parameters are Δ,α_0 and $q=|q_1|=|q_2|$. We shall give explicit solutions for each case.

Case 1. There are two situations depending on the size of Δ. We first assume $\beta_0=h=1$.

(i) the case $\Delta\ge 1$. Since $\int_{-1}^{1}C=1$ is less than Δ and $\alpha_0>\beta_0=1$, by (10_1) and (10_2), the solution of the obstacle problem is affine. Thus Λ^C is a constant and

$$\Lambda^C=(\Delta-1)/L,$$

where L is the length of the faceted region. This observation shows that

$$u(t,x)=\begin{cases}u_0(\alpha(t)) & |x|<\alpha(t),\\ u_0(x) & \text{otherwise}\end{cases}$$

is the unique solution of (1) and (14) with initial data u_0 provided that $\alpha(t)$ solves

$$\frac{d}{dt}u_0(\alpha(t))=(\Delta-1)/(2\alpha(t))\quad(<0),\quad \alpha(0)=\alpha_0.$$

The left hand side equals $q\alpha'(t)$ so this equation is integrable to get

$$\alpha(t)=\left(\alpha_0^2+(\Delta-1)t/q\right)^{1/2}.$$

(ii) the case $\Delta < 1$. The solution $\overline{\zeta}$ of the obstacle problem is no longer affine so we should split the faceted region $[-\alpha_0, \alpha_0]$. There is a unique δ_0 such that $\overline{\zeta} = Z$ on $[-\alpha_0, -\delta_0]$, $\overline{\zeta} = \overline{Z}$ on $[\delta_0, \alpha_0]$ and $\overline{\zeta}$ is affine on $[-\delta_0, \delta_0]$. The value of δ_0 is computable directly of by Lemma 7 (i). It yields $\delta_0 = \Delta^{1/2}$. The derivative $\overline{\zeta_x}$ on $[-\delta_0, \delta_0]$ equals $-C(\delta_0) = -(1 - \Delta^{1/2})$. We set

$$u(t,x) = \begin{cases} -C(\delta_0)t & |x| \le \delta_0, \\ -C(x)t & \delta_0 < |x| < 1, \\ u_0(x) & \text{otherwise.} \end{cases}$$

This is the unique solution of (1) and (14) with initial data u_0. Indeed, for $t > 0$, on the faceted region $|x| \le \delta_0$ we have $\Lambda^C = -C(\delta_0)$. On $[\delta_0, 1]$ or $[-1, -\delta_0]$ the transition number equals zero and C is monotone. Applying Lemma 6, we see that the speed there equals $-C(x)$, so this u is a solution.

Note that in both cases u solves (1) with (14) globally in time.

It is easy to rewrite a formula of solutions for arbitrary β_0 and h by rescaling. We list them below.

(i) the case $\Delta/(h\beta_0) \ge 1$.

$$u(t,x) = \begin{cases} u_0(\alpha(t)) & |x| \le \alpha(t), \\ u_0(x) & \text{otherwise} \end{cases}$$

with $\alpha(t) = (\alpha^2 + (\Delta/(h\beta_0) - 1)th\beta_0/q)^{1/2}$.

(ii) the case $\Delta/(h\beta_0) < 1$.

$$u(t,x) = \begin{cases} -C(\delta_0)t & |x| \le \delta_0, \\ -C(x)t & \delta_0 < |x| < \beta_0, \\ u_0(x) & \text{otherwise} \end{cases}$$

with $\delta_0 = (\beta_0 \Delta/h)^{1/2}$.

Case 2. We discuss only the case $0 < q_1 = q_2 = q$ since the case $0 > q_1 = q_2$ can be treated parallelly. The situation is again divided into two cases. Here we also assume that β_0 and h equals one.

If Δ is sufficiently large compared with the length of the faceted region $(-\alpha_0, \alpha_0)$, the minimizer $\overline{\zeta}$ is affine on $[-\alpha_0, \gamma_0]$ and $\overline{\zeta} = \overline{Z}$ on $[\gamma_0, \alpha_0]$ for some γ_0, $0 < \gamma_0 < 1$. Note that $\overline{\zeta}(\pm\alpha_0) = \overline{Z}(\pm\alpha_0)$ since $q_1 = q_2 > 0$. Since the slope of $\overline{\zeta}$ on $[-\alpha_0, \gamma_0]$ agrees with the slope of $\overline{Z}$ at γ_0 (cf. Lemma 7), it must satisfy

$$C(\gamma_0) = D(\gamma_0)/(\gamma_0 + \alpha_0), \qquad D(x) = \int_{-1}^{x} C(z)dz. \tag{15}$$

(If fact, γ_0 is the unique positive solution of the quadratic equation (15).)

(i) the case $C(\gamma_0) < (\Delta + D(-\gamma_0))/(\alpha_0 - \gamma_0)$. This is a necessary and sufficient condition the segment between $(-\alpha_0, \overline{Z}(-\alpha_0))$ and $(\gamma_0, \overline{Z}(\gamma_0))$ does not intersect the graph of Z. We set

$$u(t,x)=\begin{cases} u_0(-\alpha(t)) & -\alpha(t)<x\le\gamma(t),\\ -C(x)t & \gamma(t)<x<1=\beta_0,\\ u_0(x) & \text{otherwise.}\end{cases} \tag{16}$$

(Between $-\alpha(t)$ and α_0 there are three faceted regions for $t>0$ close to zero.) Here $\alpha(t)$ and $\gamma(t)$ are determined by

$$\begin{cases} u_0(-\alpha(t))=-C(\gamma(t))t,\\ \frac{d}{dt}u_0(-\alpha(t))=\frac{-1}{\alpha(t)+\gamma(t)}D(\gamma(t)),\\ \text{with } \alpha(0)=\alpha_0, \quad \gamma(0)=\gamma_0.\end{cases} \tag{17}$$

If (17) has a solution α,γ in $C[0,T)\cap C^1(0,T)$ for small $T>0$, it is not difficult to observe that u given by (16) solves (1) with (14) (at least locally in time) with initial data u_0 by checking Λ^C on each faceted region. The first equation in (17) implies the continuity of u at $x=\gamma(t)$. The second one describes the speed of u in the faceted region $[-\alpha(t),\gamma(t)]$. It says that the speed is negative, so that $\alpha'(t)>0$.

The existence of solution is not trivial because the ODE (ordinary differential equation) of γ obtained from (17) has a singularity at $t=0$; it is a Briot-Bouquet equation. At the last part of this paper we give a proof for existence of unique local-in-time solution α,γ in $C[0,T)\cap C^1(0,T)$ of (17) for small $T>0$. It turns out that a local solution of (17) extends to a global solution by solving the ODE for γ. Indeed, checking the sign of derivatives of γ at $\gamma=0$ and $\gamma=1$ in (17), we get a priori bound $0<\gamma(t)<1$ for solution.

If $\Delta\ge 1$ so that the graph of Z does not affect the speed of u, then u given by (16) solves (1) with (14) globally- in-time because each faceted region never disappears. However, if not, the length of $(-\alpha(t),\gamma(t))$ *may* become large so that the segment between $(-\alpha(t),\bar{Z}(-\alpha(t)))$ and $(\gamma(t),\bar{Z}(\gamma(t)))$ intersects the graph of Z at some time t_0. The faceted region $[-\alpha(t_0),\gamma(t_0)]$ is expected to split into three pieces. Their evolution is described (globally-in-time) in a similar way to the next case(ii) (but not exactly the same).

(ii) the case $C(\gamma_0)\ge(\Delta+D(-\gamma_0))/(\alpha_0-\gamma_0)$. As in (i), the unique local solution is given by

$$u(t,x)=\begin{cases} u_0(-\alpha(t)) & -\alpha(t)<x\le-\rho(t),\\ -C(x)t & -\rho(t)<x\le-\delta_0,\\ -C(\delta_0)t & |x|<\delta_0,\\ -C(x)t & \delta_0\le x<1=\beta_0,\\ u_0(x) & \text{otherwise.}\end{cases} \tag{18}$$

(Between $-\alpha(t)$ and α_0 there are five faceted regions for $t>0$ close to zero.) Here $\alpha(t)$ and $\rho(t)$ are determined by

$$\begin{cases} u_0(-\alpha(t))=-C(\rho(t))t,\\ \frac{d}{dt}u_0(-\alpha(t))=\frac{-1}{\alpha(t)-\rho(t)}\{\Delta+D(-\rho(t))\},\\ \text{with } \alpha(0)=\alpha_0, \quad \rho(0)=\rho_0\end{cases} \tag{19}$$

and δ_0 is as in Case 1. Here ρ_0 ($\delta_0 < \rho_0 < 1$) is the unique solution of quadratic equation

$$C(-\rho_0) = \frac{\Delta + D(-\rho_0)}{\alpha_0 - \rho_0}.$$

In other words ρ_0 is determined, so that $\overline{\zeta}$ is affine on $(-\alpha_0, -\rho_0)$ and that the graph of $\overline{\zeta}$ is tangent to the graph of Z at $-\rho_0$. Here $\overline{\zeta}$ is the minimizer of the obstacle problem on the faceted region $[-\alpha_0, \alpha_0]$ (of u_0). Note that $\overline{\zeta}(-\alpha_0) = \overline{Z}(-\alpha_0)$, $\overline{\zeta}$ is affine on $[-\delta_0, \delta_0]$ and $\overline{\zeta} = Z$ on $[-\rho_0, -\delta_0]$ and $\overline{\zeta} = \overline{Z}$ on $[\delta_0, \alpha_0]$. The equation (19) has the same structure as (17) and it turns out that it has a unique global solution with a priori bound $\delta_0 < \rho(t) < 1$ and $\alpha'(t) > 0$. Thus u given by (18) solves (1) with (14) globally in time.

Case 3. As in Cases 1, 2, we have solutions of following forms, where we set $\beta_0 = h = 1$.
(i) the case $C(\gamma_0) < (\Delta + D(-\gamma_0))/(\alpha_0 - \gamma_0)$. The unique local solution is given by

$$u(t,x) = \begin{cases} u_0(-\alpha(t)) & -\alpha(t) < x < \alpha(t), \\ u_0(x) & \text{otherwise} \end{cases}$$

with

$$\frac{du_0(-\alpha(t))}{dt} = -\frac{\Delta + 1}{2\alpha(t)}, \qquad \alpha(0) = \alpha_0.$$

This is easy to integrate:

$$\alpha(t) = \{\alpha_0^2 + (\Delta + 1)t/q\}^{1/2}, \qquad q = |q_1| = |q_2|.$$

The function u solves (1) until the time t_0 when $\alpha(t_0)$ satisfies

$$C(\delta_0) = \frac{\Delta + D(-\delta_0)}{\alpha(t_0) - \delta_0}.$$

If $\Delta \geq 1$, such situation does not occur so the above u is a global solution. If $\Delta < 1$, then we should break the facet and the next case(ii) applies to solve (1) from $t = t_0$.
(ii) the case $C(\gamma_0) \geq (\Delta + D(-\gamma_0))/(\alpha_0 - \gamma_0)$. This assumption implicitly implies $\delta_0 = \Delta^{1/2} < 1$. As in Case 2 (ii) the unique local solution is given by

$$u(t,x) = \begin{cases} u_0(-\alpha(t)) & \rho(t) < |x| \leq \alpha(t), \\ -C(x)t & \delta_0 < |x| \leq \rho, \\ -C(\delta_0)t & |x| \leq \delta_0, \\ u_0(x) & \text{otherwise.} \end{cases}$$

(There are five faceted regions between $-\alpha(t)$ and $\alpha(t)$.) Here $\alpha(t)$ and $\rho(t)$ are given by (19). As in Case 2, u given in Case 3(ii) solves (1) with (14) globally-in-time.

We conclude this paper by studying (17) and (19). By a substitution we get an integral equation from (17) for γ of form

$$\gamma(t) = \frac{1}{t}\int_0^t G(s, \gamma(s))ds \tag{20}$$

with

$$G(s,\gamma) = 1 - \frac{D(\gamma)}{\gamma + \alpha_0 + sC(\gamma)/q}.$$

By definition (15) of γ_0 we see

$$G(0,\gamma_0) = 1 - C(\gamma_0) = \gamma_0. \tag{21}$$

Moreover

$$\frac{\partial G}{\partial \gamma}(0,\gamma_0) = \frac{C(\gamma_0)(\gamma_0 + \alpha_0) - D(\gamma_0)}{(\gamma_0 + \alpha_0)^2} = 0. \tag{22}$$

For such a type of integral equation there exists a unique local solution by the standard contraction mapping principle.

LEMMA 8. *Let $F = F(s,x)$ be C^1 in (a neighborhood of) $E = [0,T] \times B_r(x_0)$, where $B_r(x_0)$ is the closed ball of radius r centered at a point x_0 in $\mathbf{R}^N$. Assume that*

$$F(0,x_0) = x_0 \quad \text{and} \quad \frac{\partial F}{\partial x}(0,x_0) = 0, \tag{23}$$

where $\partial F/\partial x$ denotes the gradient of F in x variable. Then there exists $T_0 > 0$ and $x \in C([0,T_0);\mathbf{R}^N)$ that satisfies

$$x(t) = \frac{1}{t}\int_0^t F(s,x(s))ds \qquad \text{with } x(0) = x_0. \tag{24}$$

Solution x of (24) is unique and x is C^1 on $(0,T_0)$.

Proof. Let $\mathcal{F}$ be a mapping in $C([0,t_0];\mathbf{R}^N)$ of form

$$(\mathcal{F}x)(t) = \frac{1}{t}\int_0^t F(s,x(s))ds.$$

We take σ and η small $(0 < \sigma < r, 0 < \eta < T)$ so that

$$L = \sup\left\{\left|\frac{\partial F}{\partial x}(s,x)\right| ; x \in B_\sigma(x_0) \quad 0 \le s \le \eta\right\} \le \frac{1}{2}.$$

This is possible because of our assumption on $\partial F/\partial x$. We take $t_0(\le \eta)$ small so that

$$t_0 M < \sigma \quad \text{with} \quad M = \sup\left\{\left|\frac{\partial F}{\partial s}(s,x_0)\right|; \quad 0 \le s \le \eta\right\}.$$

By this choice of t_0 and σ, $\mathcal{F}$ is a (strict) contraction mapping from $X = C([0,t_0], B_\sigma(x_0))$ into itself.

Indeed for $x_1, x_2 \in X$ estimating

$$(\mathcal{F}x_0 - \mathcal{F}x_2)(t) = \frac{1}{t}\int_0^t (F(s,x_1(s)) - F(s,x_2(s)))ds$$

yields

$$|(\mathcal{F}x_1 - \mathcal{F}x_2)(t)| \leq L \sup_{0\leq s\leq t} |x_1(s) - x_2(s)|. \tag{25}$$

Since $F(0, x_0) = x_0$ we have

$$\mathcal{F}x_0(t) - x_0 = \frac{1}{t}\int_0^t (F(s, x_0) - F(0, x_0))ds$$

to get

$$|\mathcal{F}x_0(t) - x_0| \leq \frac{1}{t}\int_0^t sMds = \frac{M}{2}t. \tag{26}$$

Since $\mathcal{F}x_1 - x_0 = \mathcal{F}x_1 - \mathcal{F}x_0 + \mathcal{F}x_0 - x_0$, (25) and (26) yield

$$\sup_{0\leq t\leq t_0} |\mathcal{F}x_1 - x_0| \leq L\sigma + \frac{Mt_0}{2} < \sigma$$

so $\mathcal{F}$ maps X into X. The inequality (25) implies that $\mathcal{F}$ is a contraction in X.

By the Banach fixed point theorem, there is a unique fixed point x in X and this x is a solution of (24). This fixed point satisfies $x(0) = x_0$ since $F(0, x_0) = x_0$. By continuity of x and F it is clear from (24) that x is C^1 on $(0, t_0)$. The uniqueness follows from (25). $\square$

Since (23) is fulfilled by (21) and (22), we apply Lemma 8 with $F(s, x) = G(s, \gamma)$, $x_0 = \gamma_0$ and $N = 1$ to get a unique local solution of (20) with $\gamma(0) = \gamma_0$. So (17) is locally solvable. Since (19) can be reduced to the equation of ρ of the form (24) with

$$F(s, \rho) = 1 - \frac{\Delta + D(-\rho)}{\alpha - \rho + sC(\rho)/q} \qquad \text{and} \qquad x_0 = \rho_0$$

satisfying (23), again Lemma 8 applies to get a local solution of (19).

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