The determinant of a hypergeometric period matrix

A. Douai and H. Terao

Series #345. September 1996

HOKKAIDO UNIVERSITY
PREPRINT SERIES IN MATHEMATICS

#321 M. Ohnuma, Axisymmetric solutions and singular parabolic equations in the theory of viscosity solutions, 26 pages. 1995.

#322 T. Nakazi, An outer function and several important functions in two variables, 12 pages. 1995.

#323 N. Kawazumi, An infinitesimal approach to the stable cohomology of the moduli of Riemann surfaces, 22 pages. 1995.

#324 A. Arai, Factorization of self-adjoint operators by abstract Dirac operators and its application to second quantizations on Boson Fermion Fock spaces, 15 pages. 1995.

#325 K. Sugano, On strongly separable Frobenius extensions, 11 pages. 1995.

#326 D. Lehmann and T. Suwa, Residues of holomorphic vector fields on singular varieties, 21 pages. 1995.

#327 K. Tsutaya, Local regularity of non-resonant nonlinear wave equations, 23 pages. 1996.

#328 T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, 25 pages. 1996.

#329 O. Ogurisu, Anticommutativity and spin 1/2 Schrödinger operators with magnetic fields, 12 pages. 1996.

#330 Y. Kurokawa, Singularities for projections of contour lines of surfaces onto planes, 24 pages. 1996.

#331 M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, 94 pages. 1996.

#332 M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the p-laplace diffusion equation, 20 pages. 1996.

#333 T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.

#334 B. Khanedani and T. Suwa, First variation of horomorphic forms and some applications, 11 pages. 1996.

#335 J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations[1], 28 pages. 1996.

#336 Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.

#337 I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.

#338 F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.

#339 N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.

#340 A. Inoue, Abel-Tauber theorems for Fourier-Stieltjes coefficients, 17 pages. 1996.

#341 G. Ishikawa, Topological classification of the tangent developables of space curves, 19 pages. 1996.

#342 Y. Shimizu, A remark on estimates of bilinear forms of gradients in Hardy space, 8 pages. 1996.

#343 N. kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, 11 pages. 1996.

#344 M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, 18 pages. 1996.

The determinant of a hypergeometric period matrix

ANTOINE DOUAI
Unité associée au CNRS 168, Université de Nice, Parc Valrose, F-06108 Nice Cedex 2 FRANCE
HIROAKI TERAO *
Department of Mathematics, University of Wisconsin, Madison, WI 53706 USA

1 Introduction

Let $f_1, \dots, f_p$ be polynomials with real coefficients of degree one which define an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $\alpha_1, \dots, \alpha_p$ complex numbers. Let $U_\alpha = f_1^{\alpha_1} \cdots f_p^{\alpha_p}$. Varchenko ([V1, Theorem 1.1], [V2]) calculated, for arrangements of hyperplanes in general position, the determinant of a (period) matrix $\mathsf{PM}(\mathcal{A}, \alpha)$ whose entries are (hypergeometric) integrals $\int_\Delta U_\alpha \phi$, where Δ runs over the set $\mathsf{Ch}(\mathcal{A})$ of bounded connected components of $\mathbb{R}^n - \bigcup_{i=1}^p \{f_i = 0\}$ and the n-form ϕ runs over the set $\Phi^p(\mathcal{A}) = \{\alpha_{i_1} \cdots \alpha_{i_n} df_{i_1}/f_{i_1} \wedge \cdots \wedge df_{i_n}/f_{i_n} \mid 1 < i_1 < \cdots < i_n \leq p\}$. Since $|\mathsf{Ch}(\mathcal{A})| = |\Phi^p(\mathcal{A})| = \binom{p-1}{n}$, the period matrix $\mathsf{PM}(\mathcal{A}, \alpha)$ is of size $\binom{p-1}{n} \times \binom{p-1}{n}$. The formula by Varchenko expresses the determinant of $\mathsf{PM}(\mathcal{A}, \alpha)$ by the product of critical values and a certain function, called the Beta function $B(\mathcal{A}, \alpha)$ of the arrangement. The Beta function $B(\mathcal{A}, \alpha)$ is explicitly given as an alternating product of Gamma functions whose arguments are appropriate linear combinations of the parameters α_i. In the general case, Varchenko conjectured an analogous explicit formula [V2, 6.3 Fundamental conjecture] and proved it for normal-crossing arrangements [V1, Theorem 1.4] and arrangements in general position at infinity [V2, Theorem 6.1]. Note that this determinant can also regarded as "Wronskian" of a certain system of partial differential equations (cf. for example [Ki]).

More recently, F. Loeser and C. Sabbah ([LS]), gave a general formula for such a determinant. In this formula enters the characteristic polynomial of some monodromies associated with the family $f_1, \cdots, f_p$ of polynomials. Since the bases of cycles and forms are not specified, there are some unknowns in their result and our purpose is to remove them. In order to do that, we use the β **nbc**-bases of the cohomology defined in [FT, Theorem 3.6]. The calculation of the determinant of the period matrix in a β **nbc**-basis can be done by using recurrence (deletion-restriction) and by studying zeros and poles of the determinant in terms

*second author partially supported by NSF Grant DMS9504457

of the non-resonance set. In the end, we show that the conjecture announced by Varchenko is true for any arrangement for an arbitrary β **nbc**-basis.

2 Arrangements

In this section we review results from [FT].

2.1

Let $f_1, \cdots, f_p$ be linear polynomials defined on the n-dimensional complex affine space V. Let I denote $\{1, \cdots, p\}$ and $\mathcal{A}$ be the arrangement $\{H_i\}_{i\in I}$ where $H_i = \ker f_i$ is the hyperplane defined by f_i.

Definition 2.1.1 *An **edge** of $\mathcal{A}$ is a nonempty intersection of some of its hyperplanes.*

Let $L(\mathcal{A})$ denote the set of all these edges. $L(\mathcal{A})$ is partially ordered by reverse inclusion. The maximal elements of $L(\mathcal{A})$ all have the same codimension, which we denote by r. Agree that $V \in L(\mathcal{A})$ which is the unique minimum element.

Definition 2.1.2 *(i) $\mathcal{A}$ is said to be **essential** if $r = n$ (in particular $| I | \geq n$). (ii) $\mathcal{A}$ is said to be **real** if the polynomials f_i all have real coefficients.*

Definition 2.1.3 *An arrangement $\mathcal{A}$ is said to be in **general position** if, for all subarrangement $\{H_{i_1}, \cdots, H_{i_k}\}$ of $\mathcal{A}$ codim $(H_{i_1} \cap \cdots \cap H_{i_k}) = k$ if $1 \leq k \leq n$ and $H_{i_1} \cap \cdots \cap H_{i_k} = \emptyset$ if $k > n$. An arrangement $\mathcal{A}$ is said to be in **general position at infinity** if, for all subarrangement $\{H_{i_1}, \cdots, H_{i_k}\}$ of $\mathcal{A}$, $H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset$ for $k \leq n$. It is said to be **normal** if $\cup H$ is a normal crossing divisor in V.*

Notation 2.1.4 *(i) Let*

$$M(\mathcal{A}) = V - \cup_{i\in I} H_i$$

and, if $\mathcal{A}$ is real,

$$M_{\mathbb{R}}(\mathcal{A}) = M(\mathcal{A}) \cap V_{\mathbb{R}}$$

where $V_{\mathbb{R}}$ denote the real part of V.
(ii) If $\mathcal{A}$ is real, let $\mathrm{Ch}(\mathcal{A})$ denote the set of all n-dimensional bounded components of $M_{\mathbb{R}}(\mathcal{A})$ and $\beta(\mathcal{A})$ its cardinality.

Until the end of this paper we suppose $\mathcal{A}$ real and essential.

2.2 Linear orders

Let $i_0 \in I$. We define a linear order $<_{i_0}$ in $\mathcal{A}$ putting $H_i <_{i_0} H_j$ if $i < j$, $i, j \neq i_0$, and $H_i <_{i_0} H_{i_0}$ for all $i \in I - \{i_0\}$.

Remark 2.2.1 *(i)* $<_p$ *is the standard order defined in* [OT, page 67].
(ii) If $\mathcal{B} \subset \mathcal{A}$ *is a subarrangement which do not contains* H_{i_0}*, it inherits the standard order.*

2.3 Non-resonant weights

Let $F \in L(\mathcal{A}) - \{V\}$ be an edge. Define

$$I(F) = \{i \in I \mid F \subseteq H_i\}.$$

Definition 2.3.1 *The edge* F *is called* **dense** *if the arrangement*

$$\{H_i \mid i \in I(F)\}$$

is not decomposable [STV, Section 2], *that is, the arrangement is not a product of two nonempty arrangements.*

Let $\mathbb{P}^n$ be complex projective space, which is a compactification of $V = \mathbb{C}^n$. Define the arrangement $\mathcal{A}_\infty$ of projective hyperplanes by

$$\mathcal{A}_\infty = \{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_p, \overline{H}_\infty\},$$

where $\overline{H}_i$ is the projective closure of H_i $(1 \leq i \leq p)$ and $\overline{H}_\infty = \mathbb{P}^n - \mathbb{C}^n$. Let $L(\mathcal{A}_\infty)$ be the collection of nonempty intersections of projective hyperplanes in $\mathcal{A}_\infty$. Define $L_-(\mathcal{A}_\infty)$ (resp. $L_+(\mathcal{A}_\infty)$) to be the set of edges of $L(\mathcal{A}_\infty)$ contained (resp. not contained) in $\overline{H}_\infty$. Then $L(\mathcal{A}_\infty) = L_-(\mathcal{A}_\infty) \cup L_+(\mathcal{A}_\infty)$ (disjoint). Cover $\mathbb{P}^n$ by the standard affine opens $U_0, U_1, \ldots, U_n$, each of which is isomorphic to $\mathbb{C}^n$. Let $\mathcal{A}_i$ $(0 \leq i \leq n)$ be the arrangement in $U_i \simeq \mathbb{C}^n$ obtained by restricting each projective hyperplane in $\mathcal{A}_\infty$ to U_i. Let $F \in L(\mathcal{A}_\infty) - \{\mathbb{P}^n\}$. We say that F is **dense** if $F \cap U_i$ is dense in $\mathcal{A}_i$ for $0 \leq i \leq n$ with $F \cap U_i \neq \emptyset$. (cf. [STV, Section 3].)

Let $\overline{I} = \{1, \ldots, p, \infty\}$. For $F \in L(\mathcal{A}_\infty)$ define

$$\overline{I}(F) = \{i \in \overline{I} \mid F \subseteq \overline{H}_i\}.$$

To each polynomial f_i (and therefore to each hyperplane H_i), one associates a complex number α_i. These numbers are called **weights**. Define $\alpha_\infty = -\sum_{i=1}^{p} \alpha_i$. For $F \in L(\mathcal{A}_\infty) - \{\mathbb{P}^n\}$, let $\alpha(F)$ be the sum of α_i with $i \in \overline{I}(F)$. In other words, for $F \in L_+(\mathcal{A}_\infty)$,

$$\alpha(F) = \sum_{i \in I(F)} \alpha_i,$$

and, for $F \in L_-(\mathcal{A}_\infty)$,

$$\alpha(F) = \alpha_\infty + \sum_{j \in I \cap \bar{I}(F)} \alpha_i = -\sum_{i \in I} \alpha_i + \sum_{j \in I \cap \bar{I}(F)} \alpha_i = -\sum_{j \in I - \bar{I}(F)} \alpha_i.$$

Define the **resonance set** of $\mathcal{A}$ by

$$\mathrm{Rsn}(\mathcal{A}) = \{\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbb{C}^p \mid \alpha(F) \in \mathbb{Z} \text{ for some dense edge } F \in L(\mathcal{A}_\infty)\}.$$

It is the union of a locally finite infinite family of hyperplanes.

Definition 2.3.2 *We say that the weights $\alpha = (\alpha_1, \cdots, \alpha_p)$ are* ***non-resonant*** *if $\alpha \notin \mathrm{Rsn}(\mathcal{A})$. In other words, the weights α are non-resonant if $\alpha(F) \notin \mathbb{Z}$ for each dense edge $F \in L(\mathcal{A}_\infty)$.*

2.4 βnbc-bases

Let $i_0 \in I$. We define the linear order $<_{i_0}$ in $\mathcal{A}$.

A subset $\{H_i\}_{i \in J}$ of $\mathcal{A}$ is **dependent** if $\cap_{i \in J} H_i \neq \emptyset$ and $\mathrm{codim}(\cap_{i \in J} H_i) < |J|$. A subset of $\mathcal{A}$ which has nonempty intersection and is not dependent is called **independant**. Maximal independent sets are called **bases**. Every base has cardinality n.
A k-tuple $S = (H_1, \cdots, H_k)$ is a **circuit** if $(H_1, \cdots, H_k)$ is dependent and if, for each l, $1 \leq l \leq k$, the $(k-1)$-tuple $(H_1, \cdots, \widehat{H_l}, \cdots, H_k)$ is independent. A k-tuple S is a **broken circuit** if there exists $H <_{i_0} min(S)$ such that $\{H\} \cup S$ is a circuit, where $min(S)$ denotes the minimal element of S for $<_{i_0}$.

The collection of subsets of $\mathcal{A}$ having nonempty intersection and containing no broken circuits is denoted by **BC**. **BC** consists of independent sets. Maximal (with respect to inclusion) elements of **BC** are bases of $\mathcal{A}$ called **nbc**-bases.
A **nbc**-basis $B = (H_{i_1}, \cdots, H_{i_n})$ is **ordered** if $H_{i_1} <_{i_0} H_{i_2} <_{i_0} \cdots <_{i_0} H_{i_n}$.
We denote by $\mathbf{nbc}_{i_0}(\mathcal{A})$ the set of all ordered **nbc**-bases of $\mathcal{A}$. We introduce a linear order in $\mathbf{nbc}_{i_0}(\mathcal{A})$ using the lexicographic order on the hyperplanes read from right to left.

Definition 2.4.1 *A basis B is called β* **nbc**-*basis if B is a* **nbc**-*basis and if, for every $H \in B$, there exists $H' <_{i_0} H$ such that $(B - \{H\}) \cup \{H'\}$ is a base.*

Notation 2.4.2 *Denote by β* $\mathbf{nbc}_{i_0}(\mathcal{A})$ *the set of all β* **nbc**-*bases, ordered by $<_{i_0}$.*

Remark 2.4.3 *(i) β* $\mathbf{nbc}_{i_0}(\mathcal{A})$ *inherits the order defined on* $\mathbf{nbc}_{i_0}(\mathcal{A})$.
(ii) In what follows, we omit the index i_0 when the linear order on $\mathcal{A}$ is the standard order.

Definition 2.4.4 *If $B = (H_{i_1}, \cdots, H_{i_n}) \in \beta$* $\mathbf{nbc}_{i_0}(\mathcal{A})$, *let*

$$F_j = \bigcap_{k=j+1}^{n} H_{i_k}$$

for $0 \leq j \leq n-1$ *and* $F_n = V$. *Define*

$$\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n),$$

which is a flag of affine subspaces of V *with* $\dim F_j = j$ $(0 \leq j \leq n)$. *This flag* $\xi(B)$ *is called the* β **nbc-flag** *associated with* B.

Notation 2.4.5 *For* $(i_1, \cdots, i_k) \subseteq I$, $\omega_{i_1 \cdots i_k} = df_{i_1}/f_{i_1} \wedge \cdots \wedge df_{i_k}/f_{i_k}$. *If* $F \in L(\mathcal{A}) - \{V\}$, *define*

$$I(F) = \{i \in I \mid F \subseteq H_i\},$$

and

$$\omega_\alpha(F, \mathcal{A}) = \sum_{i \in I(F)} \alpha_i \omega_i.$$

For $B = (H_{i_1}, \cdots, H_{i_n}) \in \beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$, *let* $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ *be the associated flag. Define*

$$\Xi(B, \mathcal{A}) = \omega_\alpha(F_0, \mathcal{A}) \wedge \cdots \wedge \omega_\alpha(F_{n-1}, \mathcal{A}).$$

Definition 2.4.6 *If* $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) = \{B_1, \cdots, B_{\beta(\mathcal{A})}\}$ *and* $\phi_i^{i_0} = \phi_i^{i_0}(\mathcal{A}) = \Xi(B_i, \mathcal{A})$, *define*

$$\Phi^{i_0}(\mathcal{A}) = \{\phi_1^{i_0}, \cdots, \phi_{\beta(\mathcal{A})}^{i_0}\}.$$

Define

$$I'_{i_0} = I - \{i_0\}$$

and

$$\mathcal{A}'_{i_0} = \{H'_i\}_{i \in I'_{i_0}} \text{with } H'_i = H_i.$$

Let

$$\mathcal{A}''_{i_0} = \{H_i \cap H_{i_0} \mid i \in I'_{i_0}, H_i \cap H_{i_0} \neq \emptyset\}.$$

Then $\mathcal{A}''_{i_0}$ is an arrangement of hyperplanes in H_{i_0}. The linear order on $\mathcal{A}'_{i_0}$ is inherited from $\mathcal{A}$. If $H'' \in \mathcal{A}''_{i_0}$, let $\nu(H'')$ be the smallest hyperplane of $\mathcal{A}'_{i_0}$ containing H''. We order $\mathcal{A}''_{i_0}$ setting $H'' < K''$ if and only if $\nu(H'') <_{i_0} \nu(K'')$.

Remark 2.4.7 $\mathcal{A}'_{i_0}$ *and* $\mathcal{A}''_{i_0}$ *are equipped with standard orders.*

Let

$$I''_{i_0} = \{i \in I'_{i_0} \mid H_i = \nu(H'') \text{ for some } H'' \in \mathcal{A}''_{i_0}\}.$$

To each $i \in I'_{i_0}$ (or $i \in I''_{i_0}$), we associate the weights $\alpha'_i := \alpha_i$ (resp. $\alpha''_i := \sum \alpha_k$ where the sum runs over all the $H'_k \in \mathcal{A}'_{i_0}$ such that $H''_i \subset H'_k$).

There is an inductive definition (deletion-restriction) of $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$:

Proposition 2.4.8 (Ziegler[Z, Theorem 1.5] [FT, Theorem 2.4]) *Let $(\mathcal{A}, \mathcal{A}'_{i_0}, \mathcal{A}''_{i_0})$ be the triple defined above. Suppose that $\mathcal{A}'_{i_0}$ is essential. Then*

$$\beta\ \mathbf{nbc}_{i_0}(\mathcal{A}) = \beta\ \mathbf{nbc}(\mathcal{A}'_{i_0}) \cup \{(\nu(B''), H_{i_0}) \mid B'' \in \beta\ \mathbf{nbc}(\mathcal{A}''_{i_0})\}$$

where $\nu(B'') = (\nu(H''_1), \cdots, \nu(H''_k))$ if $B'' = (H''_1, \cdots, H''_k)$.

Remark 2.4.9 *Write $\overline{\beta\ \mathbf{nbc}}(\mathcal{A}''_{i_0}) = \{(\nu(B''), H_{i_0}) \mid B'' \in \beta\ \mathbf{nbc}(\mathcal{A}''_{i_0})\}$. Elements of $\beta\ \mathbf{nbc}(\mathcal{A}'_{i_0})$ are always less than elements of $\overline{\beta\ \mathbf{nbc}}(\mathcal{A}''_{i_0})$.*

Lemma 2.4.10 *If $B \in \beta\ \mathbf{nbc}(\mathcal{A}'_{i_0})$, then $\Xi(B, \mathcal{A})_{|\alpha_{i_0}=0} = \Xi'(B, \mathcal{A}'_{i_0})$.*

Proof. Let $F \in L(\mathcal{A})$. If $i_0 \notin I(F)$, then $\omega_\alpha(F, \mathcal{A}) = \omega_\alpha(F, \mathcal{A}'_{i_0})$ and if $i_0 \in I(F)$, then $\omega_\alpha(F, \mathcal{A}) = \omega_\alpha(F, \mathcal{A}'_{i_0}) + \psi\alpha_{i_0}\omega_{i_0}$ for some form ψ. $\square$

Lemma 2.4.11 *Let $B'' \in \beta\ \mathbf{nbc}(\mathcal{A}''_{i_0})$ and $B = (\nu(B''), H_{i_0}) \in \overline{\beta\ \mathbf{nbc}}(\mathcal{A}''_{i_0})$. Then the residue of $\Xi(B, \mathcal{A})$ along H_{i_0} is equal to $\Xi(B'', \mathcal{A}''_{i_0})$.*

Proof. The last factor of $\Xi(B, \mathcal{A})$ is $\alpha_{i_0}\omega_{i_0}$. Since the product is the exterior one, it follows that $\alpha_{i_0}\omega_{i_0}$ may be removed as a summand from all the other factors of the product without changing its value. Taking residue of this rewritten product removes the factor $\alpha_{i_0}\omega_{i_0}$ and restricts the remaining terms to H_{i_0}. The residue is now just $\Xi(B'', \mathcal{A}''_{i_0})$. $\square$

Example 2.4.12 *Let $\mathcal{A}$ be an arrangement in general position. Consider the standard order $<_p$. Then $\beta(\mathcal{A}) = \binom{p-1}{n}$.*

$$\beta\ \mathbf{nbc}_p(\mathcal{A}) = \{(H_{i_1}, \dots, H_{i_n}) \mid 1 < i_1 < \cdots < i_n \leq p\},$$

and

$$\Phi^p(\mathcal{A}) = \{\alpha_{i_1} \cdots \alpha_{i_n}\omega_{i_1 \cdots i_n} \mid 1 < i_1 < \cdots < i_n \leq p\}.$$

Example 2.4.13 *Let $\mathcal{A} = \{H_i\}_{1 \leq i \leq 4}$ with*

$$f_1(x,y) = x - y, f_2(x,y) = 1 - x, f_3(x,y) = y, f_4(x,y) = 1 - y, f_5(x,y) = x.$$

(*i*) *Consider the order $<_1$:*

$$H_2 <_1 H_3 <_1 H_4 <_1 H_5 <_1 H_1.$$

- *The circuits are (H_2, H_4, H_1) and (H_3, H_5, H_1) and the broken circuits are (H_4, H_1) and (H_5, H_1). Thus*

$$\mathbf{nbc}_1(\mathcal{A}) = \{(H_2, H_1), (H_3, H_1), (H_2, H_3), (H_2, H_4), (H_3, H_5), (H_4, H_5)\}$$

and

$$\beta\,\mathbf{nbc}_1(\mathcal{A}) = \{(H_4, H_5), (H_3, H_1)\}.$$

- *We have*

$$\mathcal{A}'_1 = \{H_2 < H_3 < H_4 < H_5\},$$
$$\mathcal{A}''_1 = \{H_2 \cap H_1, H_3 \cap H_1\},$$

$\beta\,\mathbf{nbc}(\mathcal{A}'_1) = \{(H_4, H_5)\}$, $\beta\,\mathbf{nbc}(\mathcal{A}''_1) = \{H_{351}\}$ $(H_{351} := H_3 \cap H_5 = H_3 \cap H_5 \cap H_1, \nu(H_{351}) = min(H_3, H_5, H_1) = H_3)$ *and*

$$\beta\,\mathbf{nbc}_1(\mathcal{A}) = \beta\,\mathbf{nbc}(\mathcal{A}''_1) \cup \overline{\beta\,\mathbf{nbc}}(\mathcal{A}''_1).$$

- $$\Phi^1 = \{\alpha_4\alpha_5\omega_{45}, \alpha_3\alpha_1\omega_{31} + \alpha_5\alpha_1\omega_{51}\}.$$

(*ii*) *In the same way,*

$$\Phi^2 = \{\alpha_4\alpha_5\omega_{45}, \alpha_3\alpha_2\omega_{32}\}$$
$$\Phi^3 = \{\alpha_4\alpha_5\omega_{45}, \alpha_2\alpha_3\omega_{23}\}$$
$$\Phi^4 = \{\alpha_2\alpha_3\omega_{23}, \alpha_5\alpha_4\omega_{54}\}$$
$$\Phi^5 = \{\alpha_2\alpha_3\omega_{23}, \alpha_4\alpha_5\omega_{45}\}.$$

3 Hypergeometric period matrix

3.1 The β nbc-ordered homology basis

Recall that $\mathsf{Ch}(\mathcal{A})$ is the set of real bounded chambers of $\mathcal{A}$. Let $\beta = \beta(\mathcal{A}) = |\mathsf{Ch}(\mathcal{A})|$. In order to define the period matrix, we label $\mathsf{Ch}(\mathcal{A})$ by $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$.

Definition 3.1.1 *Let $\xi = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be a flag of affine subspaces $F_i \in L(\mathcal{A})$ with* $\dim F_i = i$ $(i = 0, \ldots, n)$. *Let $\Delta \in \mathsf{Ch}(\mathcal{A})$ and $\overline{\Delta}$ be its closure in $\mathbb{R}^n$. We say that ξ is* ***adjacent to*** Δ *if* $\dim(F_i \cap \overline{\Delta}) = i$ *for* $i = 0, \ldots, n$.

For $B = (H_{i_1}, \ldots, H_{i_n}) \in \beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$, recall the associated β **nbc**-flag $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ from Definition 2.4.4.

Proposition 3.1.2 *There exists a unique bijection*

$$C : \beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) \longrightarrow \mathsf{Ch}(\mathcal{A})$$

with the property that $\xi(B)$ is adjacent to the bounded chamber $C(B)$ for any $B \in \beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$.

Proof. If $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) = \emptyset$, then $\mathsf{Ch}(\mathcal{A}) = \emptyset$. Suppose $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) \neq \emptyset$. We will prove by induction on $|\mathcal{A}|$. Assume that the maps C' and C'' already exist for $\mathcal{A}'_{i_0}$ and $\mathcal{A}''_{i_0}$. There are the following four kinds of bounded chambers of $\mathcal{A}$:

(i) $\Delta \in \mathsf{Ch}(\mathcal{A})$ is called *undivided* if $\Delta \in \mathsf{Ch}(\mathcal{A}'_{i_0})$, i.e., Δ does not intersect H_{i_0}.

(ii) $\Delta \in \mathsf{Ch}(\mathcal{A})$ is called *newborn* if there exists an unbounded chamber of $\mathcal{A}'_{i_0}$ which contains Δ.

(iii) Suppose that a bounded chamber Δ' of $\mathcal{A}'_{i_0}$ is divided in two by H_{i_0}. (In this case, a bounded chamber Δ' of $\mathcal{A}'_{i_0}$ is called *divided.*) Let $B' \in \beta\,\mathbf{nbc}(\mathcal{A}'_{i_0})$ with $\Delta' = C'(B')$. Then $\xi' := \xi(B')$ is adjacent to Δ'. The two new chambers of $\mathcal{A}$ inside Δ' are denoted by Δ^+ and Δ^-. We can easily observe that ξ' is adjacent to exactly one of the two chambers Δ^+ and Δ^-, say, Δ^+. The chamber $\Delta^+ \in \mathsf{Ch}(\mathcal{A})$ is called the *heir* of Δ'. The other chamber $\Delta^- \in \mathsf{Ch}(\mathcal{A})$ is called the *cutoff* of Δ'.

Define $C : \beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) \longrightarrow \mathsf{Ch}(\mathcal{A})$ as follows:

(a) If $B \in \beta\,\mathbf{nbc}(\mathcal{A}'_{i_0})$ and $C'(B) \in \mathsf{Ch}(\mathcal{A})$, then let $C(B) = C'(B)$.

(b) If $B \in \beta\,\mathbf{nbc}(\mathcal{A}'_{i_0})$ and $C'(B)$ is divided in two by H_{i_0}, then define $C(B) = \Delta^+$, where Δ^+ is the heir of $C'(B)$.

(c) If $B = (\nu B'', H_{i_0})$ with $B'' \in \beta\,\mathbf{nbc}(\mathcal{A}''_{i_0})$ and $C''(B'')$ is inside an unbounded chamber of $\mathcal{A}'_{i_0}$, then $C''(B'')$ is a wall of a unique newborn chamber Δ. Define $C(B) = \Delta$.

(d) If $B = (\nu B'', H_{i_0})$ with $B'' \in \beta\,\mathbf{nbc}(\mathcal{A}''_{i_0})$ and $C''(B'')$ is inside a bounded chamber Δ' of $\mathcal{A}'_{i_0}$, then $C''(B'')$ is a wall of a unique cutoff chamber Δ^- of Δ'. Define $C(B) = \Delta^-$.

By construction, C is bijective and satisfies the condition. The uniqueness is obvious from the construction. □

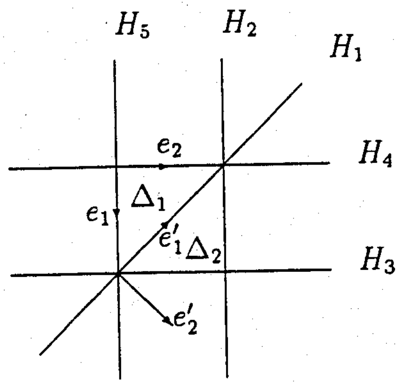
Let $\beta\,\mathbf{nbc}_{i_0}(\mathcal{A}) = \{B_1, \ldots, B_\beta\}$ be linearly ordered as in section 2.4. Define $\Delta_i := C(B_i) \in \mathsf{Ch}(\mathcal{A})$ for $i = 1, \ldots, \beta$. We call $\Delta_1, \ldots, \Delta_\beta$ the **$\beta\,\mathbf{nbc}_{i_0}$-ordered chambers** of $\mathcal{A}$. If Δ_i is either undivided or an heir and Δ_j is either newborn or a cutoff, then $i < j$.

We give an orientation to each $\Delta \in \mathsf{Ch}(\mathcal{A})$ as follows: Let $\Delta = C(B)$ with $B \in \beta\,\mathbf{nbc}_{i_0}(\mathcal{A})$. Let $\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n)$ be the associated $\beta\,\mathbf{nbc}$-flag. Choose the intrinsic orientation [V2, 6.2] of Δ obtained from $\xi(B)$. In other words, an orthonormal frame $\{e_1, \ldots, e_n\}$ is chosen so that each e_i is a unit vector originating from the point F_0 in the direction of $F_i \cap \overline{\Delta}$. This orientation is called the **$\beta\,\mathbf{nbc}$-orientation** of Δ.

Example 3.1.3 *Recall Example 2.4.13 with the order $<_1$:*

$$H_2 <_1 H_3 <_1 H_4 <_1 H_5 <_1 H_1.$$

Let $B_1 = (H_4, H_5)$ and $B_2 = (H_3, H_1)$. Then $\beta\,\mathbf{nbc}_1(\mathcal{A}) = \{B_1, B_2\}$.



The bijection map

$$C : \beta\ \mathbf{nbc}_1(\mathcal{A}) \longrightarrow \mathsf{Ch}(\mathcal{A})$$

*described in Proposition 3.1.2 is given by $C(B_i) = \Delta_i$ for $i = 1, 2$. The β **nbc**-flags are*

$$\xi(B_1) = (H_4 \cap H_5 \subset H_5 \subset V)$$

and

$$\xi(B_2) = (H_3 \cap H_1 \subset H_1 \subset V).$$

*The β **nbc**-orientations of Δ_1 and Δ_2 are given by orthonormal frames $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ respectively.*

Let $\mathcal{L}_\alpha$ be the rank one local system on $M = M(\mathcal{A})$ (cf. 2.1.4) defined by the kernel of an integral connection

$$\nabla_\alpha : \mathcal{O}_M \longrightarrow \Omega^1_M$$

where $\nabla_\alpha(f) = df + f \sum_{i \in I} \alpha_i \omega_i$. Then its monodromy around H_k is $e^{-2i\pi\alpha_k}$.

Proposition 3.1.4 (cf. [Ko], [AK, 4.1.1]) *Suppose that the weights $\alpha = (\alpha_1, \cdots, \alpha_p)$ are non-resonant* (cf. Definition 2.3.2). *We have*

(*i*) *$H_j(M(\mathcal{A}), \mathcal{L}_\alpha) = H^{lf}_j(M(\mathcal{A}), \mathcal{L}_\alpha) = 0$ pour $j \neq n$. Here H^{lf} stands for the locally finite homology.*

(*ii*) *The natural map*

$$H_n(M(\mathcal{A}), \mathcal{L}_\alpha) \longrightarrow H^{lf}_n(M(\mathcal{A}), \mathcal{L}_\alpha)$$

is an isomorphism.

(*iii*) *$\{[\Delta_j] \mid \Delta_j \in \mathsf{Ch}(\mathcal{A})\}$ forms a basis for $H^{lf}_n(M(\mathcal{A}), \mathcal{L}_\alpha)$.*

Proof. The following proof is similar to Kohno's proof [Ko] in the case that the arrangement is generic to infinity.

For (i) and (ii), we use the exactly same argument as [Ko, Theorem 1] except that we blow up $\mathbb{P}^n$ along all the dense edges of codimension of codimension > 1.

(iii): Write $M = M(\mathcal{A})$ and $\Delta = \bigcup_{j=1}^{\beta} \Delta_j$. In order to apply the argument in [Ko], it is enough to show $H^q(M - \Delta, \mathcal{L}_\alpha) = 0$ for all q. Let W be a small tubular neighborhood in $\mathbb{P}^n$ of the hyperplane at infinity $\overline{H}_\infty$. Note that the inclusion map $W \cap M \hookrightarrow M - \Delta$ is homotopy equivalent.

Let $j : W \cap M \hookrightarrow W$. Let $x \in W$. If $x \in W \cap M$, since there is no hyperplane in $\mathcal{A}_\infty$ going through x we have $(R^q j_* \mathcal{L}_\alpha)_x = 0$ for $q \neq 0$ and $(R^0 j_* \mathcal{L}_\alpha)_x \simeq \mathbb{C}$. If $x \in W - M$, then $W - M$ is locally a central arrangement near x. In this case, since the (local) Euler characteristic of M (intersected with a small open ball centered at x) is zero, we have $(R^q j_* \mathcal{L}_\alpha)_x = 0$ for all q by (i). Therefore we have $H^q(W \cap M, \mathcal{L}_\alpha) \simeq H^q(W, j_* \mathcal{L}_\alpha)$. For each $x \in \overline{H}_\infty$, there exists a small neighborhood W_x of x in W such that

$$H^q(W_{x_1} \cap \cdots \cap W_{x_k}, j_* \mathcal{L}_\alpha) = 0$$

as long as $W_{x_1} \cap \cdots \cap W_{x_k} \neq \emptyset$. Since $\overline{H}_\infty \simeq \mathbb{P}^{n-1}$ is compact, we may choose $W_{x_1}, \ldots, W_{x_m}$ which cover $\overline{H}_\infty$. Let $W_0 = W_{x_1} \cup \cdots \cup W_{x_m}$. By applying the Mayer-Vietoris theorem repeatedly, we have $H^q(W_0, j_* \mathcal{L}_\alpha) = 0$ for all q. By the Poincaré duality, we have

$$H^q(M - \Delta, \mathcal{L}_\alpha) \simeq H^q(W_0 \cap M, \mathcal{L}_\alpha) \simeq H^q(W_0, j_* \mathcal{L}_\alpha) = 0. \ \square$$

3.2 The β nbc-ordered cohomology basis

Proposition 3.2.1 *Suppose that the weights $\alpha = (\alpha_1, \cdots, \alpha_p)$ are non-resonant. We have*

(i) $H^j(M(\mathcal{A}), \mathcal{L}_\alpha) = H^j_c(M(\mathcal{A}), \mathcal{L}_\alpha) = 0$ pour $j \neq n$. Here H_c stands for the compact support cohomology.

(ii) The natural map

$$H^n_c(M(\mathcal{A}), \mathcal{L}_\alpha) \longrightarrow H^n(M(\mathcal{A}), \mathcal{L}_\alpha)$$

is an isomorphism.

(iii) The set $\Phi^{i_0}(\mathcal{A})$ (cf. Definition 2.4.6) *forms a basis for $H^n(M(\mathcal{A}), \mathcal{L}_\alpha)$.*

Proof. Let $\mathcal{L}^\vee_\alpha$ be the dual local system of $\mathcal{L}_\alpha$. (i) and (ii) are obtained from the Poincaré dualities

$$H^q(M(\mathcal{A}), \mathcal{L}_\alpha) \simeq H^{lf}_{2n-q}(M(\mathcal{A}), \mathcal{L}^\vee_\alpha), \ H^q_c(M(\mathcal{A}), \mathcal{L}_\alpha) \simeq H_{2n-q}(M(\mathcal{A}), \mathcal{L}^\vee_\alpha),$$

and Proposition 3.1.4. (iii) is [FT, Theorem 3.7]. $\square$

If $j_0 \in I$ such that $j_0 \neq i_0$, it may happens that $\Phi^{j_0} \neq \Phi^{i_0}$ (cf. Example 2.4.13). However we have the following

Proposition 3.2.2 [FT, Proposition 3.10] *For all $j_0 \in I$ the transition matrix between the bases Φ^{i_0} and Φ^{j_0} is an integral unimodular matrix independent of α.*

3.3 The definition of hypergeometric period matrix

Definition 3.3.1 *Let $\beta = \beta(\mathcal{A})$. Assume that* $\mathsf{Ch}(\mathcal{A}) = \{\Delta_1, \ldots, \Delta_\beta\}$ *is the β* nbc*-ordered chambers in 3.1.4 (iii) and $\Phi^{i_0}(\mathcal{A}) = \{\phi_1^{i_0}, \ldots, \phi_\beta^{i_0}\}$ is the β* nbc*-ordered basis of $H^n(M, \mathcal{L}_\alpha)$ in 3.2.1 (iii). Choose a branch of $f_j^{\alpha_j}$ on each chamber Δ_i. Let*

$$U_\alpha := f_1^{\alpha_1} \cdots f_p^{\alpha_p}.$$

Also choose the β nbc*-orientation of each chamber Δ_i. Define the* ***hypergeometric period matrix*** $\mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$ *by*

$$\mathsf{PM}_{i_0}(\mathcal{A}, \alpha) = \left[\int_{\Delta_j} U_\alpha \phi_i^{i_0}\right].$$

If $\Re\alpha_j > 0$, $\forall j \in I$, then each entry of $\mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$ can be regarded as an holomorphic function of $\alpha = (\alpha_1, \cdots, \alpha_p)$ and it can be analytically continued to be a meromorphic function on the entire $\mathbb{C}^p$. In order to precise this point, first introduce the following

Definition 3.3.2 *Let Δ_j^* be a twisted cycle representing the homology class in $H_n(M(\mathcal{A}), \mathcal{L}_\alpha^\vee)$ which is sent to the homology class of Δ_j in $H_n^{lf}(M(\mathcal{A}), \mathcal{L}_\alpha^\vee)$ via the isomorphism in Proposition 3.1.4 (ii). Define*

$$\mathsf{PM}_{i_0}^*(\mathcal{A}, \alpha) = \left[\int_{\Delta_j^*} U_\alpha \phi_i^{i_0}\right].$$

Remark 3.3.3 *It is known* (e.g., see [LS, 4.2]) *that each entry of* $\mathsf{PM}_{i_0}^*(\mathcal{A}, \alpha)$ *can be regarded as a meromorphic function on $\mathbb{C}^p$ whose poles lie on some hypersurfaces defined by equations $e^{2i\pi L(\alpha)} - \lambda = 0$, where L is a linear form of $\alpha_1, \cdots, \alpha_p$ and λ is a nonzero complex number. Since the twisted de Rham pairing*

$$H_n(M(\mathcal{A}), \mathcal{L}_\alpha^\vee) \times H^n(M(\mathcal{A}), \mathcal{L}_\alpha) \longrightarrow \mathbb{C},$$

which is given by the hypergeometric integrals $(\Delta^*, \phi) \mapsto \int_{\Delta^*} U_\alpha \phi$, is a nondegenerate pairing (e.g., see [Ki, 1.4]), *we may write $L(\alpha) = \alpha(F)$ for a dense edge F and $\lambda = 1$ by Propositions 3.1.4 (iii) and 3.2.1 (iii). In other words,* $\det \mathsf{PM}_{i_0}^*(\mathcal{A}, \alpha)$ *takes a finite nonzero value at each $\alpha \notin$* $\mathrm{Rsn}(\mathcal{A})$. *Moreover, if $\Re\alpha_j > 0$, $\forall j \in I$, then*

$$\int_{\Delta_j^*} U_\alpha \phi_i^{i_0} = \int_{\Delta_j} U_\alpha \phi_i^{i_0}.$$

In particular, if $\Re\alpha_j > 0$ for all $j \in I$,

$$\mathsf{PM}_{i_0}^*(\mathcal{A}, \alpha) = \mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$$

and the analytic continuation of the determinant of $\mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$ *is equal to the determinant of* $\mathsf{PM}_{i_0}^*(\mathcal{A}, \alpha)$.

Remark 3.3.4 *We formally define* $\det \mathsf{PM}_{i_0}(\mathcal{A}, \alpha) = 1$ *if $\beta(\mathcal{A}) = 0$.*

Proposition 3.3.5 *The determinant of* $\mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$ *is independent (up to sign) of the choice of i_0.*

Proof. Obvious from Proposition 3.2.2. □

4 Beta function of an arrangement

We keep the notations of Section 2.

4.1

Let $\mathcal{A}$ be an affine arrangement and $\mathcal{A}_\infty$ the arrangement of projective hyperplanes defined in 2.3. Recall $\overline{I} = I \cup \{\infty\} = \{1, \dots, p, \infty\}$ and $\alpha_\infty = -(\alpha_1 + \cdots + \alpha_p)$. We note that $L_-(\mathcal{A}_\infty)$ (or $L_+(\mathcal{A}_\infty)$) is the set of edges of $L(\mathcal{A}_\infty)$ contained (resp. not contained) in $\overline{H}_\infty$. For $F \in L(\mathcal{A}_\infty)$, let

$$\begin{aligned} I(F) &= \{i \in I \mid F \subset H_i\}, \\ \overline{I}(F) &= \{i \in \overline{I} \mid F \subset \overline{H}_i\}, \\ \mathcal{A}_\infty^F &= \{\overline{H}_i \mid i \in \overline{I}(F)\}, \\ \mathcal{A}_F^\infty &= \{\overline{H}_i \cap F \mid F \not\subseteq \overline{H}_i\}, \end{aligned}$$

$$\alpha(F) = \sum_{i \in \overline{I}(F)} \alpha_i = \begin{cases} \sum_{i \in I(F)} \alpha_i & \text{if } F \in L_+(\mathcal{A}_\infty), \\ -\sum_{i \in I - \overline{I}(F)} \alpha_i & \text{if } F \in L_-(\mathcal{A}_\infty). \end{cases}$$

Following Varchenko [V1, 1.5], we associate to each edge F of codimension r a projective arrangement $\mathbb{P}\mathcal{A}_\infty^F$ in the $r-1$ dimensional projective space. If $\mathcal{A}_\infty$ is a projective arrangement, let $\chi(\mathcal{A}_\infty)$ denote the Euler characteristic of $\mathbb{P}^n - \cup_{i \in \overline{I}} \overline{H}_i$. If F is an edge of $\mathcal{A}_\infty$ defined above, we put

$$\mu(F, \mathcal{A}_\infty) = \mid \chi(\mathcal{A}_F^\infty)\chi(\mathbb{P}\mathcal{A}_\infty^F) \mid .$$

Proposition 4.1.1 *If an edge $F \in L(\mathcal{A}_\infty)$ is not dense, then $\mu(F, \mathcal{A}_\infty) = 0$.*

Proof. Recall that F is dense if and only if $\chi(\mathbb{P}\mathcal{A}_\infty^F) \neq 0$ [STV, Proposition 7]. □

Definition 4.1.2 *Let $j \in I$ and $i \in \{1, \dots, \beta(\mathcal{A})\}$. Choose a branch of $f_j^{\alpha_j}$ on each Δ_i. Define*

$$R(f_j, \Delta_i)^{\alpha_j} = \{f_j^{\alpha_j}(x) \mid \left|f_j^{\alpha_j}(x)\right| \geq \left|f_j^{\alpha_j}(y)\right|, \forall y \in \overline{\Delta_i}\},$$

$$R(\mathcal{A})^\alpha = \prod_{j=1}^{p} \prod_{i=1}^{\beta} R(f_j, \Delta_i)^{\alpha_j}$$

and

$$B(\mathcal{A}, \alpha) = \prod_{F \in L_+(\mathcal{A}_\infty)} \Gamma(\alpha(F) + 1)^{\mu(F, \mathcal{A}_\infty)} \prod_{F \in L_-(\mathcal{A}_\infty)} \Gamma(-\alpha(F) + 1)^{-\mu(F, \mathcal{A}_\infty)}.$$

Remark 4.1.3 *Define $B(\mathcal{A}, \alpha) = 1$ and $R(\mathcal{A})^\alpha = 1$ if $\beta(\mathcal{A}) = 0$.*

Remark 4.1.4 *By Proposition 4.1.1 $B(\mathcal{A}, \alpha)$ involves only with $\alpha(F)$ for dense edges F.*

4.2 Recursion formulas for $B(\mathcal{A},\alpha)$ and $R(\mathcal{A})^\alpha$

Let $i_0 \in I$. Define the order $<_{i_0}$ in $\mathcal{A}$, and orders in $\mathcal{A}'_{i_0}$ and $\mathcal{A}''_{i_0}$ as in 2.4.7. Define

$$I^*_{i_0} = \{i \in I'_{i_0} \mid H_i \cap H_{i_0} = \emptyset\}.$$

Theorem 4.2.1 *We have*

$$\begin{aligned} B(\mathcal{A},\alpha) = {} & B(\mathcal{A}'_{i_0},\alpha')B(\mathcal{A}''_{i_0},\alpha'') \\ & \times \prod_{F\in L_+(\mathcal{A}_\infty), F\subseteq H_{i_0}} [\Gamma(\alpha(F)+1)/\Gamma(\alpha(F)-\alpha_{i_0}+1)]^{\mu(F,\mathcal{A}_\infty)} \\ & \times \prod_{F\in L_-(\mathcal{A}_\infty), F\not\subseteq H_{i_0}} [\Gamma(-\alpha(F)-\alpha_{i_0}+1)/\Gamma(-\alpha(F)+1)]^{\mu(F,\mathcal{A}_\infty)} \end{aligned}$$

and

$$R(\mathcal{A})^\alpha = R(\mathcal{A}'_{i_0})^{\alpha'} R(\mathcal{A}''_{i_0})^{\alpha''} \prod_{i=1}^{\mu} R(f_{i_0},\Delta_i)^{\alpha_{i_0}} [\prod_{i\in I^*_{i_0}} f_i^{\alpha_i} \mid_{H_{i_0}}]^{\beta(\mathcal{A}''_{i_0})}.$$

Proof. Analogous to the proof of Proposition 6.3 of [L]. □

Corollary 4.2.2 *Assume the arrangement $\mathcal{A}$ normal. Then*

$$B(\mathcal{A},\alpha) = B(\mathcal{A}'_{i_0},\alpha')B(\mathcal{A}''_{i_0},\alpha'')\Gamma(\alpha_{i_0}+1)^{\beta(\mathcal{A}''_{i_0})}$$

$$\times \prod_{F\in L_-(\mathcal{A}_\infty), F\not\subseteq H_{i_0}} [\Gamma(-\alpha(F)-\alpha_{i_0}+1)/\Gamma(-\alpha(F)+1)]^{\mu(F,\mathcal{A}_\infty)}.$$

This result was proved first by Varchenko ([V1, Theorem 2.5]).

Corollary 4.2.3 *We have*

$$B(\mathcal{A},\alpha)\mid_{\alpha_{i_0}=0} = B(\mathcal{A}'_{i_0},\alpha')B(\mathcal{A}''_{i_0},\alpha'')$$

and

$$R(\mathcal{A})^\alpha \mid_{\alpha_{i_0}=0} = R(\mathcal{A}'_{i_0})^{\alpha'} R(\mathcal{A}''_{i_0})^{\alpha''} [\prod_{j\in I^*_{i_0}} f_j^{\alpha_j} \mid_{H_{i_0}}]^{\beta(\mathcal{A}''_{i_0})}.$$

Example 4.2.4 1. *If $\mathcal{A}$ is in general position*

$$B(\mathcal{A},\alpha) = [\prod_{i=1}^{p}\Gamma(\alpha_i+1)/\Gamma(\sum_{i=1}^{p}\alpha_i+1)]^{\binom{p-2}{n-1}}.$$

2. *If $\mathcal{A}$ is the arrangement defined in Example 2.4.13,*

$$\begin{aligned} B(\mathcal{A},\alpha) = {} & \prod_{i=1}^{5}\Gamma(\alpha_i+1)\Gamma(\alpha_1+\alpha_3+\alpha_5+1)\Gamma(\alpha_1+\alpha_2+\alpha_4+1) \\ & \times [\Gamma(\sum_{i=1}^{5}\alpha_i+1)\Gamma(\alpha_1+\alpha_3+\alpha_4+1)\Gamma(\alpha_1+\alpha_2+\alpha_5+1)]^{-1}. \end{aligned}$$

5 The main theorem and its proof

5.1 The main theorem

The main result of this paper is the following

Theorem 5.1.1 *Suppose $\Re\alpha_i > 0$ for all $i \in I$. Then, for all $i_0 \in I$, we have*

$$\det \mathsf{PM}_{i_0}(\mathcal{A}, \alpha) = R(\mathcal{A})^\alpha B(\mathcal{A}, \alpha),$$

$\mathcal{A}$ being equipped with the order $<_{i_0}$ (cf. 2.2.1).

Remark 5.1.2 *$R(\mathcal{A})^\alpha B(\mathcal{A}, \alpha)$ is independent of i_0. Thus so is the determinant.*

Let $B = (H_{i_1}, \ldots, H_{i_n}) \in \beta\, \mathbf{nbc}_{i_0}(\mathcal{A})$. Recall the associated β **nbc**-flag

$$\xi(B) = (F_0 \subset F_1 \subset \cdots \subset F_n).$$

Let $\beta\, \mathbf{nbc}_{i_0}(\mathcal{A}) = \{B_1, B_2, \ldots, B_\beta\}$. It is shown [BV] that the set $\{\xi(B_1), \ldots, \xi(B_\beta)\}$ of the β **nbc**-flags gives a $\mathbb{Z}$-basis for the flag complex cohomology $H^n(\mathcal{F})$ which is studied by Schechtman and Varchenko in [SV, sections 2, 3] [V3, 10.1]. (It is also known that $H^n(\mathcal{F})$ is naturally isomorphic to the reduced cohomology $\tilde{H}^{n-1}(K(\hat{L}), \mathbb{Z})$ where $\hat{L} = L(\mathcal{A}) - \{V\}$ and $K(\hat{L})$ is the order complex of $\hat{L}$ [FT, Remark 3.8].) For an arbitrary flag $\xi = (F_0 \subset F_1 \subset \cdots \subset F_n)$, $F_i \in L(\mathcal{A})$, $\dim F_i = i$ $(i = 0, \ldots, n)$, associate a differential n-form

$$\Xi(\xi) = \omega_\alpha(F_0, \mathcal{A}) \wedge \cdots \wedge \omega_\alpha(F_{n-1}, \mathcal{A})$$

(cf. Notation 2.4.5). Consider the homomorphism

$$\pi_\alpha : H^n(\mathcal{F}) \otimes \mathbb{C} \longrightarrow H^n(M(\mathcal{A}), \mathcal{L}_\alpha)$$

such that $\pi_\alpha([\xi] \otimes 1) = [\Xi(\xi)]$. Then $\pi_\alpha([\xi(B_i)] \otimes 1) = [\phi_i^{i_0}(\mathcal{A})]$ for $1 \le i \le \beta$. The map π_α is an isomorphism when α is non-resonant.

Corollary 5.1.3 *Suppose $\Re\alpha_i > 0$ for all $i \in I$. Choose a branch of $f_j^{\alpha_j}$ on each $\Delta_i \in \mathsf{Ch}(\mathcal{A})$. Let $\xi_1, \ldots, \xi_\beta$ be flags of length $n+1$ such that their cohomology classes in the flag complex cohomology $H^n(\mathcal{F})$ form a $\mathbb{Z}$-basis. Let $\psi_i(\alpha) = \Xi(\xi_i)$ for $1 \le i \le \beta$. Then we have*

$$\det \left[\int_{\Delta_j} U_\alpha \psi_i(\alpha) \right] = \pm R(\mathcal{A})^\alpha B(\mathcal{A}, \alpha).$$

Proof. Since $\{\xi(B_1), \ldots, \xi(B_\beta)\}$ and $\{\xi_1, \ldots, \xi_\beta\}$ are connected by a unimodular integral constant matrix in $H^n(\mathcal{F})$, so are $\Phi^{i_0}(\mathcal{A})$ and $\{\psi_1(\alpha), \ldots, \psi_\beta(\alpha)\}$ in $H^n(M(\mathcal{A}), \mathcal{L}_\alpha)$. Apply Theorem 5.1.1. □

Remark 5.1.4 *Theorem 5.1.1 shows that the conjecture by Varchenko* ([V2, 6.3 Fundamental conjecture]) *is true for the β* nbc*-bases and the β* nbc*-orientations. If Varchenko's flags F_{Δ_j} $(1 \le j \le \beta)$ in* [V2, 6.2] *form a $\mathbb{Z}$-basis for $H^n(\mathcal{F})$, then the affirmative answer (up to sign) to the original conjecture follows from Corollary 5.1.3. (In general, the original conjecture by Varchenko is always true up to a constant integral multiple by Theorem 5.1.1. Especially, for 2-dimensional arrangements, M. Neergaard and the second author have verified the original conjecture by studying the relationship between the β* nbc*-flags and Varchenko's flags.) Note that the basis Φ^p coincides (up to sign) with Varchenko's basis $\{\Xi(F_{\Delta_j})\}$ when $\mathcal{A}$ is normal or in general position at infinity.*

5.2 A theorem by Loeser-Sabbah

Recall, at first, one of the main results of [LS].

Theorem 5.2.1 *We have*

$$\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha) = c_1^{\alpha_1} \dots c_p^{\alpha_p} B(\mathcal{A}, \alpha) h_{i_0}(\alpha),$$

where $c_1, \cdots, c_p$ are nonzero constants and $h_{i_0} \in \mathbb{C}(\alpha_1, \cdots, \alpha_p)^*$.

Proof. By [LS, 4.2.10], we have

$$\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha) = \varphi_{i_0}(e^{2i\pi\alpha_1}, \cdots, e^{2i\pi\alpha_p}) c_1^{\alpha_1} \dots c_p^{\alpha_p} B(\mathcal{A}, \alpha) \hat{h}_{i_0}(\alpha),$$

where φ_{i_0} is a periodic function of $\alpha = (\alpha_1, \cdots, \alpha_p)$, $c_1, \cdots, c_p$ are nonzero constants and $\hat{h}_{i_0} \in \mathbb{C}(\alpha_1, \cdots, \alpha_p)^*$. Since the polynomials f_i take real values on each Δ_i and $\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha)$ is holomorphic if $\Re\alpha_i > 0$ for all $i \in I$ by Remark 3.3.3, $\varphi_{i_0}(e^{2i\pi\alpha_1}, \cdots, e^{2i\pi\alpha_p})$ is constant by [LS, final remark] (see also Remark 5.4.3). Denote $\varphi_{i_0}\hat{h}_{i_0}(\alpha)$ by $h_{i_0}(\alpha)$. $\square$

Remark 5.2.2 *The constants c_k are defined as critical values (counted with multiplicities) of the polynomials f_k.*

Recall that $\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha) = \det \mathsf{PM}_{i_0}(\mathcal{A}, \alpha)$ if $\Re\alpha_i > 0$ for all $i \in I$. We shall show that h_{i_0} equals $+1$ and $c_k^{\alpha_k} = \prod_{j=1}^{\beta} R(f_k, \Delta_j)^{\alpha_k}$.

5.3 A recursion formula for hypergeometric period determinants

Let $i_0 \in I$. Give $\mathcal{A}$ the linear order $<_{i_0}$. Recall $\mathcal{A}'_{i_0}$, $\mathcal{A}''_{i_0}$, I'_{i_0}, and I''_{i_0} from 2.4. To each $i \in I'_{i_0}$ (or $i \in I''_{i_0}$), we associate the weights $\alpha'_i := \alpha_i$ (resp. $\alpha''_i := \sum \alpha_k$ where the sum runs over all the $H'_k \in \mathcal{A}'_{i_0}$ such that $H''_i \subset H'_k$). Let $\alpha' = (\alpha'_i)_{i \in I'_{i_0}}$ and $\alpha'' = (\alpha''_i)_{i \in I''_{i_0}}$. Denote the restriction of f_j to H_{i_0} by $\overline{f}_j$. Define

$$U'_\alpha = \prod_{j \in I'_{i_0}} f_j^{\alpha'_j}, U''_\alpha = \prod_{j \in I''_{i_0}} \overline{f}_j^{\alpha''_j}.$$

Recall $I_{i_0}^{*} = \{i \in I'_{i_0} \mid H_i \cap H_{i_0} = \emptyset\}$. Fix a branch of $\prod_{j \in I_{i_0}^{*}} (f_j^{\alpha_j})_{|H_{i_0}}$ and call it c_{i_0}. Then c_{i_0} is a constant number. Choose a branch of U'_α and a branch of U''_α on each bounded chamber of $\mathrm{Ch}(\mathcal{A}'_{i_0})$ and on each bounded chamber of $\mathrm{Ch}(\mathcal{A}''_{i_0})$ respectively. Also choose a branch of $f_{i_0}^{\alpha_{i_0}}$ on each bounded chamber $\Delta \in \mathrm{Ch}(\mathcal{A})$. Define a branch U_Δ of U_α on $\Delta \in \mathrm{Ch}(\mathcal{A})$ as follows (we use the terminology from the proof of Proposition 3.1.2):

(i) if Δ is undivided, then $\Delta \in \mathrm{Ch}(\mathcal{A}'_{i_0})$. Define $U_\Delta = (f_{i_0}^{\alpha_{i_0}} \text{ on } \Delta)(U'_\alpha \text{ on } \Delta)$.

(ii) if Δ is the heir of $\Delta' \in \mathrm{Ch}(\mathcal{A}'_{i_0})$, then define $U_\Delta = (f_{i_0}^{\alpha_{i_0}} \text{ on } \Delta)(U'_\alpha \text{ on } \Delta')_{|\Delta}$.

(iii) if Δ is either a cutoff or newborn, then let $\Delta'' \in \mathrm{Ch}(\mathcal{A}''_{i_0})$ be the wall of Δ. Choose a unique branch U'_Δ of U'_α on Δ such that $U'_{\Delta|\Delta''} = c_{i_0}(U''_\alpha \text{ on } \Delta'')$. Let $U_\Delta = (f_{i_0}^{\alpha_{i_0}} \text{ on } \Delta)U'_\Delta$.

Recall that we are using the β **nbc**-orientation for every chamber of $\mathrm{Ch}(\mathcal{A}'_{i_0})$, $\mathrm{Ch}(\mathcal{A}''_{i_0})$ and $\mathrm{Ch}(\mathcal{A})$. Then

(i) if $\Delta \in \mathrm{Ch}(\mathcal{A})$ is the heir of $\Delta' \in \mathrm{Ch}(\mathcal{A}'_{i_0})$, then the corresponding β **nbc**-flags are equal. So the orientation of Δ is the induced one from the orientation of Δ', and

(iii) if $\Delta \in \mathrm{Ch}(\mathcal{A})$ is either a cutoff or newborn, then the orthonormal frame for Δ is given by the orthonormal frame for $\Delta'' := \overline{\Delta} \cap H_{i_0} \in \mathrm{Ch}(\mathcal{A}''_{i_0})$ together with the unit vector in the direction of Δ as the last vector of the frame.

Define $\mathrm{PM}_{i_0}(\mathcal{A}, \alpha)$, $\mathrm{PM}(\mathcal{A}'_{i_0}, \alpha')$ and $\mathrm{PM}(\mathcal{A}''_{i_0}, \alpha'')$ using these branches and orientations.

We analytically continue the determinant $\det \mathrm{PM}_{i_0}(\mathcal{A}, \alpha)$ onto the hyperplane $\alpha_{i_0} = 0$.

Proposition 5.3.1 *Suppose that the real part of α_i is positive for all $i = 1, \ldots, p$, $i \neq i_0$. Then*

$$\det \mathrm{PM}_{i_0}(\mathcal{A}, \alpha)_{|\alpha_{i_0}=0} = \det \mathrm{PM}(\mathcal{A}'_{i_0}, \alpha') \det \mathrm{PM}(\mathcal{A}''_{i_0}, \alpha'')\, c_{i_0}^{\beta(\mathcal{A}''_{i_0})}.$$

Proof. Let $\Delta' \in \mathrm{Ch}(\mathcal{A}'_{i_0})$ be a divided chamber. Let Δ^+ and Δ^- be its heir and cutoff respectively. Let U^+ and U^- be the branches of U_α on Δ^+ and Δ^-. Choose a constant number $c_{\Delta'}$ such that $U^+ = c_{\Delta'}U^-$ on $\Delta' \cap H_{i_0}$. Let M be the matrix obtained from $\mathrm{PM}_{i_0}(\mathcal{A}, \alpha)$ by adding for each divided chamber $\Delta' \in \mathrm{Ch}(\mathcal{A}'_{i_0})$ the column corresponding to the cutoff Δ^- of Δ' multiplied by $c_{\Delta'}$ to the column corresponding to the heir Δ^+ of Δ' and setting $\alpha_{i_0} = 0$. Then $\det \mathsf{M} = \det \mathrm{PM}_{i_0}(\mathcal{A}, \alpha)_{|\alpha_{i_0}=0}$. Note that the column of M corresponding to Δ^+ is

$$\left[\int_{\Delta'} U'_\alpha f_{i_0}^{\alpha_{i_0}} \phi_1, \ldots, \int_{\Delta'} U'_\alpha f_{i_0}^{\alpha_{i_0}} \phi_\beta\right]^t_{|\alpha_{i_0}=0}.$$

Write

$$\mathsf{M} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

where P is a square matrix of size $\beta(\mathcal{A}'_{i_0})$ and S is a square matrix of size $\beta(\mathcal{A}''_{i_0})$. Since the first $\beta(\mathcal{A}'_{i_0})$ columns of M are labelled by $\mathrm{Ch}(\mathcal{A}'_{i_0})$, it follows from Lemma 2.4.10 that $P = \mathrm{PM}(\mathcal{A}'_{i_0}, \alpha')$.

When computing R and S we may take $\phi = \Xi(B)$ where $B = (\nu B'', H_{i_0}) \in \overline{\beta\ \mathbf{nbc}}(\mathcal{A}''_{i_0})$ with $B'' \in \beta\ \mathbf{nbc}(\mathcal{A}''_{i_0})$. Write $\phi = \psi(\alpha_{i_0}\omega_{i_0})$. Let $\Delta' \in \mathrm{Ch}(\mathcal{A}'_{i_0})$. Set $\Delta'_t = \Delta' \cap \{f_{i_0} = t\}$

and $F(t) = \int_{\Delta'_t} U'_\alpha \psi$. Define real numbers $a < b$ such that $\Delta'_t \neq \emptyset$ if and only if $a \leq t \leq b$. Using the variable $t = f_{i_0}$, Fubini's theorem and integration by parts give

$$\pm \int_{\Delta'} U'_\alpha f_{i_0}^{\alpha_{i_0}} \phi = \int_a^b \alpha_{i_0} t^{\alpha_{i_0}-1} F(t)dt = [t^{\alpha_{i_0}} F(t)]_a^b - \int_a^b t^{\alpha_{i_0}} F'(t)dt.$$

Taking the limit as $\alpha_{i_0} \to 0$, $\Re\alpha_{i_0} > 0$, we get

$$\lim \left[[t^{\alpha_{i_0}} F(t)]_a^b - \int_a^b t^{\alpha_{i_0}} F'(t)dt \right] = \begin{cases} 0 & 0 \notin \{a, b\} \\ F(0) & 0 = a < b \\ -F(0) & a < b = 0. \end{cases}$$

If Δ' is divided, then we apply the first part to get zero. If H_{i_0} intersects $\overline{\Delta}'$ in a face of codimension > 1, then $F(0) = 0$. If H_{i_0} does not intersect $\overline{\Delta}'$, then the integral is again zero. Thus $\mathsf{M}(\Delta', \phi) = 0$. This shows that $R = 0$.

It remains to compute the entries of S. Let $\Delta \in \mathsf{Ch}(\mathcal{A})$ be either a cutoff or newborn. In this case H_{i_0} is a wall of Δ so $\Delta_0 = C''(B'')$. Let $\Delta'' = C''(B'')$. It follows from Lemma 2.4.11 that $\phi'' := \psi_{|\Delta''} = \Xi(B'', \mathcal{A}''_{i_0})$. Set $\Delta_t = \Delta \cap \{f_{i_0} = t\}$ and $G(t) = \int_{\Delta_t} U'_\Delta \psi$, where U'_Δ is a unique branch of U'_α on Δ such that $U'_{\Delta|\Delta''} = c_{i_0}(U''_\alpha \text{ on } \Delta'')$. Define real numbers $a < b$ such that $\Delta_t \neq \emptyset$ if and only if $a \leq t \leq b$. Then $0 \in \{a, b\}$. Recall the choice of branch of U_α on Δ and orientation of Δ. By the same calculation as above, using the variable $t = f_{i_0}$, Fubini's theorem and integration by parts, we get

$$\mathsf{M}(\Delta, \phi) = \lim \int_\Delta U_\alpha \phi = G(0) = c_{i_0} \int_{\Delta''} U''_\alpha \phi'' = c_{i_0} \mathsf{M}(\Delta'', \phi'')$$

as $\alpha_{i_0} \to 0$, $\Re\alpha_{i_0} > 0$. So $S = c_{i_0} \mathsf{PM}(\mathcal{A}''_{i_0}, \alpha'')$. Thus we have

$$\det \mathsf{PM}_{i_0}(\mathcal{A}, \alpha)_{|\alpha_{i_0}=0} = \det \mathsf{M} = (\det P)(\det S) = \det \mathsf{PM}(\mathcal{A}'_{i_0}, \alpha') \det \mathsf{PM}(\mathcal{A}''_{i_0}, \alpha'')\, c_{i_0}^{\beta(\mathcal{A}''_{i_0})}. \ \square$$

Corollary 5.3.2

$$\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha)_{|\alpha_{i_0}=0} = \det \mathsf{PM}^*(\mathcal{A}'_{i_0}, \alpha') \det \mathsf{PM}^*(\mathcal{A}''_{i_0}, \alpha'')\, c_{i_0}^{\beta(\mathcal{A}''_{i_0})}.$$

Proof. When the real part of α_i is positive for all $i = 1, \ldots, p$, $i \neq i_0$, this functional equality has been proved in Proposition 5.3.1. Therefore this equality holds true everywhere. $\square$

Remark 5.3.3 *The recursion formula 5.3.1, in the case of arrangements in general position, is found in* [V1, p.546].

5.4 Proof of the main theorem

We prove the theorem by induction on (n, p) (equipped with lexicographical order). If $n = 1$ the theorem is well-known. If $\beta(\mathcal{A}) = 0$, then the theorem asserts $1 = 1$. Note that $\beta(\mathcal{A}) = 0$ whenever $p \leq n$. Let $i_0 \in I$. From the induction hypothesis we have

$$\det \mathrm{PM}^*(\mathcal{A}'_{i_0}, \alpha') = R(\mathcal{A}'_{i_0})^{\alpha'} B(\mathcal{A}'_{i_0}, \alpha')$$

and

$$\det \mathrm{PM}^*(\mathcal{A}''_{i_0}, \alpha'') = R(\mathcal{A}''_{i_0})^{\alpha''} B(\mathcal{A}''_{i_0}, \alpha'').$$

First step : we determine the product of critical values.
The induction hypothesis gives, together with Corollary 5.3.2 and Corollary 4.2.3,

$$c_1^{\alpha_1} \dots c_p^{\alpha_p} \mid_{\alpha_{i_0}=0} = R(\mathcal{A})^{\alpha} \mid_{\alpha_{i_0}=0} .$$

Thus we have $c_k^{\alpha_k} = \prod_{j=1}^{\beta} R(f_k, \Delta_j)^{\alpha_k}$ for $k \neq i_0$. By considering another linear order $<_k$ $(k \neq i_0)$, we have

$$c_1^{\alpha_1} \dots c_p^{\alpha_p} \mid_{\alpha_k=0} = R(\mathcal{A})^{\alpha} \mid_{\alpha_k=0}$$

so $c_{i_0}^{\alpha_{i_0}} = \prod_{j=1}^{\beta} R(f_{i_0}, \Delta_j)^{\alpha_{i_0}}$.

Second step : we determine the rational function.
We have

$$\det \mathrm{PM}^*_{i_0}(\mathcal{A}, \alpha) = R(\mathcal{A})^{\alpha} B(\mathcal{A}, \alpha) h_{i_0}(\alpha).$$

First summarize what we know about the rational function h_{i_0}. Let

$$\mathcal{L} = \{\alpha(F) + m \mid F \text{ is a dense edge in } L(\mathcal{A}_\infty), m \in \mathbb{Z}\}.$$

Lemma 5.4.1 *(i) h_{i_0} is independant (up to sign) of i_0.*
(ii) The numerator and the denominator of h are (up to sign) products of linear forms belonging to $\mathcal{L}$.
(iii) For all $i_0 \in I$, $h(\alpha_1, \cdots, \alpha_p) \mid_{\alpha_{i_0}=0}$ is equal to either 1 or -1.

Proof. (i) follows from Proposition 3.3.5 and the fact that both $B(\mathcal{A}, \alpha)$ and $R(\mathcal{A})^{\alpha}$ are independent of i_0.

As for (ii), recall that the determinant of $\mathrm{PM}_{i_0}(\mathcal{A}, \alpha)$ takes a finite nonzero value at each $\alpha \notin \mathrm{Rsn}(\mathcal{A})$ by Remark 3.3.3. Neither of $B(\mathcal{A}, \alpha)$ or $R(\mathcal{A})^{\alpha}$ has a zero or pole at $\alpha \notin \mathrm{Rsn}(\mathcal{A})$ (cf. Remark 4.1.4). Therefore h is a rational function which takes a finite nonzero value at every $\alpha \notin \mathrm{Rsn}$. Since $\mathrm{Rsn}(\mathcal{A})$ is the union of a locally finite infinite family of hyperplanes, we have (ii).

Lastly, (iii) is a consequence of the induction assumption, Corollary 5.3.2 and Corollary 4.2.3. □

Lemma 5.4.2 *h is equal to a constant function which is either 1 or -1 .*

Proof. Suppose that h is not constant. By Lemma 5.4.1(ii), we may write h as a fraction whose denominator and numerator are both products of finitely many elements of

$$\mathcal{L} = \{\alpha(F) + m \mid F \text{ is a dense edge in } L(\mathcal{A}_\infty), m \in \mathbb{Z}\}.$$

Suppose $\alpha(F) + m$ appears in the expression. By Lemma 5.4.1 (iii), $\alpha(F) - \alpha_j + m$ also appears in the expression for each j such that α_j appears in $\alpha(F)$. Also, $\alpha(F) + \alpha_j + m$ also appears in the expression for each j such that α_j does not appear in $\alpha(F)$. Therefore, by repeately using these observations, we finally can conclude that $\sum_{i\in J} \alpha_i + m$ appears for every subset J of I. In particular, $\alpha_1 + \cdots + \alpha_{p-1} + m$ appears in the expression. This implies either (i) $F := H_1 \cap \cdots \cap H_{p-1}$ is dense and $I(F) = \{1, 2, \ldots, p-1\}$, or (ii) $F_\infty := \overline{H}_p \cap \overline{H}_\infty$ is dense and $\overline{I}(F_\infty) = \{p, \infty\}$. Since (ii) is a contradiction, (i) always occurs. In particular, $H_1, \ldots, H_{p-1}$ are dependent and there exists $j_0 \in \{1, \ldots, p-1\}$ such that $F = H_1 \cap \cdots \cap H_{j_0-1} \cap H_{j_0+1} \cap \cdots \cap H_{p-1}$. If $\mathcal{A}$ is central, there is nothing to prove. So we may assume $\emptyset = H_1 \cap \cdots \cap H_p$. Thus

$$\emptyset = H_1 \cap \cdots \cap H_p = F \cap H_p = H_1 \cap \cdots \cap H_{j_0-1} \cap H_{j_0+1} \cap \cdots \cap H_p.$$

This implies that $\alpha_1 + \cdots + \alpha_{j_0-1} + \alpha_{j_0+1} + \cdots + \alpha_p + m$ does not appear in the expression of h, which is a contradiction. This shows that h is a constant. By Lemma 5.4.1 (iii), the constant is equal to either 1 or -1. □

It follows from Lemma 5.4.2 that

$$\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha) = \pm R(\mathcal{A})^\alpha B(\mathcal{A}, \alpha).$$

Let us determine the sign. It is known that the sign is plus when $n = 1$ or $\beta(\mathcal{A}) = 0$. By Corollaries 5.3.2 and 4.2.3, we can inductively show that the sign is always plus:

$$\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha) = R(\mathcal{A})^\alpha B(\mathcal{A}, \alpha).$$

This, together with Remark 3.3.3, proves the main theorem.

Remark 5.4.3 *We could show in the same way that the periodic function φ_{i_0} appearing in the proof of theorem 5.2.1 is constant.*

Remark 5.4.4 *It should be interesting to study the connection between the roots of a Bernstein polynomial of $f = (f_1, \cdots, f_p)$ ([S]) and the poles of* $\det \mathsf{PM}^*_{i_0}(\mathcal{A}, \alpha)$.

Acknowledgements. The first author thanks F. Loeser, C. Sabbah and F. Maaref for fruitful discussions. The second author also thanks K. Aomoto, P. Orlik, R. Silvotti and A. Varchenko for useful and stimulating discussions.

References

[A] Aomoto, K.: Les équations aux différences finies et les intgrales de fonctions multiformes, J. Fac. Sci. Tokyo **22** (1975), 271-297 et **26**, 519-523 (1979)

[AK] Aomoto, K., Kita, M.: Hypergeometric functions (in Japanese). Tokyo: Springer 1994

[BV] Brylawski, T., Varchenko, A.: The determinant formula for a matroid bilinear form (preprint)

[FT] Falk, M. J., Terao, H.: β **nbc**-bases for cohomology of local systems on hyperplanes complements, Trans. Amer. Math. Soc. (to appear)

[Ki] Kita, M.: On hypergeometric functions in several variables II. The wronskian of the hypergeometric functions of type (n+1, m+1), J. Math. Soc. Japan **45**, 645-689 (1993)

[Ko] Kohno, T.: Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. **62**, Ser. A, 144-147 (1986)

[L] Loeser, F.: Arrangements d'hyperplans et somme de Gauss, Ann. Sci. École Norm. Sup. **24**, 379-400 (1991)

[LS] Loeser, F. and Sabbah, C.: Equations aux différences finies et déterminants d'intégrales de fonctions multiformes, Comment. Math. Helv. **66**, 458-503 (1991)

[OT] Orlik, P., Terao, H.: Arrangements of hyperplanes. Grundlehren der Math. Wiss. **300**, Berlin Heildelberg New York: Springer, 1992

[S] Sabbah, C.: Proximité évanescente I, Compositio Math. **62**, 283-328 (1987)

[SV] Schechtman, V., Varchenko, A.: Arrangements of hyperplanes and Lie algebra homology, Invent. math. **106**, 139–194 (1991)

[STV] Schechtman, V., Terao, H., Varchenko, A.: Local systems over complements of hyperplanes and the Kac-Kazhdan conditions for singular vectors, J. Pure Appl. Algebra **100**, 93-102 (1995)

[V1,V2] Varchenko, A.: The Euler Beta-function, the Vandermonde determinant, Legendre's equation, and critical values of linear functions on a configuration of hyperplanes, Math. USSR Izvestija **35** (1990), 543-572 and **36**, 155-168 (1991)

[V3] Varchenko, A.: Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, Advanced Series in Mathematical Physics - **21**, World Scientific Publishers, 1995

[Z] Ziegler, G.: Matroid shellability, β–systems, and affine arrangements, J. Alg. Combinatorics, **1**, 283–300 (1992)