



Title	Weighted Norm Inequalities For Some Singular Integral Operators
Author(s)	Nakazi, T.; Yamamoto, T.
Citation	Hokkaido University Preprint Series in Mathematics, 347, 1-17
Issue Date	1996-9-1
DOI	10.14943/83493
Doc URL	<a href="http://hdl.handle.net/2115/69097">http://hdl.handle.net/2115/69097</a>
Type	bulletin (article)
File Information	pre347.pdf



[Instructions for use](#)

**Weighted Norm Inequalities  
For Some Singular Integral Operators**

**T. Nakazi and T. Yamamoto**

Series #347. September 1996

**HOKKAIDO UNIVERSITY**  
**PREPRINT SERIES IN MATHEMATICS**

- #323 N. Kawazumi, An infinitesimal approach to the stable cohomology of the moduli of Riemann surfaces, 22 pages. 1995.
- #324 A. Arai, Factorization of self-adjoint operators by abstract Dirac operators and its application to second quantizations on Boson Fermion Fock spaces, 15 pages. 1995.
- #325 K. Sugano, On strongly separable Frobenius extensions, 11 pages. 1995.
- #326 D. Lehmann and T. Suwa, Residues of holomorphic vector fields on singular varieties, 21 pages. 1995.
- #327 K. Tsutaya, Local regularity of non-resonant nonlinear wave equations, 23 pages. 1996.
- #328 T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, 25 pages. 1996.
- #329 O. Ogurisu, Anticommutativity and spin 1/2 Schrödinger operators with magnetic fields, 12 pages. 1996.
- #330 Y. Kurokawa, Singularities for projections of contour lines of surfaces onto planes, 24 pages. 1996.
- #331 M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, 94 pages. 1996.
- #332 M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the  $p$ -laplace diffusion equation, 20 pages. 1996.
- #333 T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.
- #334 B. Khanedani and T. Suwa, First variation of holomorphic forms and some applications, 11 pages. 1996.
- #335 J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations<sup>1</sup>, 28 pages. 1996.
- #336 Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.
- #337 I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.
- #338 F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.
- #339 N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.
- #340 A. Inoue, Abel-Tauber theorems for Fourier-Stieltjes coefficients, 17 pages. 1996.
- #341 G. Ishikawa, Topological classification of the tangent developables of space curves, 19 pages. 1996.
- #342 Y. Shimizu, A remark on estimates of bilinear forms of gradients in Hardy space, 8 pages. 1996.
- #343 N. Kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, 11 pages. 1996.
- #344 M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, 18 pages. 1996.
- #345 A. Douai and H. Terao, The determinant of a hypergeometric period matrix, 20 pages. 1996.
- #346 H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of  $\square u = |u|^p$  for supercritical values  $p$  in even space dimensions, 66 pages. 1996.

Weighted Norm Inequalities  
For  
Some Singular Integral Operators

by  
Takahiko Nakazi\*  
and  
Takanori Yamamoto\*

Dedicated to Professor Satoru Igari on his sixtieth birthday

Department of Mathematics  
Faculty of Science  
Hokkaido University  
Sapporo 060, Japan

Department of Mathematics  
Faculty of General Education  
Hokkai-Gakuen University  
Sapporo 060, Japan

\*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education

Abstract. For bounded Lebesgue measurable functions  $\alpha, \beta$  on the unit circle,  $S_{\alpha, \beta} = \alpha P_+ + \beta P_-$  is called a singular integral operator, where  $P_+$  is an analytic projection and  $P_-$  is a co-analytic projection. We study one-weighted norm inequalities of  $S_{\alpha, \beta}$  on  $L^2(W)$ . We introduce a class  $HS_r$  of weights with  $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$  in order to characterize those weights. For example, we show that  $S_{\alpha, \beta}$  is bounded with respect to a weight  $W$  if and only if  $W$  belongs to  $HS_r$  or  $|\alpha - \beta|W \equiv 0$ . If  $r$  is a nonzero constant, then  $HS_r$  is just a well known class of weights due to Helson and Szegő. Moreover we study the Koosis type problem of two weights of  $S_{\alpha, \beta}$  and get very simple necessary and sufficient conditions for such weights.

## §1. Introduction

Let  $m$  denote the normalized Lebesgue measure on the unit circle  $T$ . Let  $A$  be the disc algebra, that is,  $A$  is the algebra of all continuous functions on  $T$  whose negative Fourier coefficients are zero. For  $0 < p < \infty$ , the Hardy space  $H^p$  is the closure of  $A$  in  $L^p = L^p(m)$ , and  $H^\infty$  is the weak\*-closure of  $A$  in  $L^\infty = L^\infty(m)$ . A function  $Q$  in  $H^\infty$  is an inner function if  $|Q| = 1$  a.e.. A function  $h$  is an outer function if there exists a real function  $t$  in  $L^1$  and a real constant  $c$  such that  $h = e^{t+i\tilde{t}+ic}$ , where  $\tilde{t}$  denotes the harmonic conjugate function of  $t$  with  $\tilde{t}(0) = 0$ . Let  $A_0$  be the subspace of all functions in  $A$  whose mean value is zero, and let  $\tilde{A}_0$  be the subspace of all complex conjugates of functions in  $A_0$ . Let  $S$  be the singular integral operator defined by

$$Sf(\zeta) = \frac{1}{\pi i} \int_T \frac{f(z)}{z - \zeta} dz$$

the integral being a Cauchy principal value. If  $f$  is in  $L^1$ , then  $Sf(\zeta)$  exists for almost everywhere  $\zeta$  on  $T$ , and  $Sf(\zeta)$  is a measurable function. We shall define the analytic projection  $P_+$  and co-analytic projection  $P_-$  by

$$P_+ = (I + S)/2, P_- = (I - S)/2,$$

where  $I$  denotes the identity operator. Then

$$(P_+ - P_-)f(\zeta) = Sf(\zeta) = i\tilde{f}(\zeta) + \int_T f dm.$$

For functions  $\alpha$  and  $\beta$  in  $L^\infty$ ,

$$S_{\alpha,\beta} = \alpha P_+ + \beta P_- = \frac{\alpha - \beta}{2} S + \frac{\alpha + \beta}{2} I$$

is called a singular integral operator (cf.[3]). For a nonnegative function  $W$  in  $L^1$ , put

$$L^2(W) = L^2(W dm) \text{ and } \|f\|_W = \left\{ \int_T |f|^2 W dm \right\}^{1/2}.$$

In this paper, we consider the Helson-Szegö type problem of one weight of  $S_{\alpha,\beta}$ . We treat the one-weighted norm inequalities as the following :

$$\|S_{\alpha,\beta} f\|_W \leq C \|f\|_W$$

and

$$\|S_{\alpha,\beta} f\|_W \geq \varepsilon \|f\|_W$$

where  $C$  and  $\varepsilon$  are positive constants. The first one-weighted inequality was studied by H.Helson and G.Szegö [4] when  $\alpha = 1$  and  $\beta = 0$  or  $\alpha = 1$  and  $\beta = -1$ . Then they introduced a class  $HS$  of weights which satisfy such an inequality. In §3 of this paper, we show that the first one-weighted inequality is true for some positive constant  $C$  if and only if the weight  $W$  belongs to a class  $HS_r$  of weights with  $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$  or if  $|\alpha - \beta|W \equiv 0$ .  $HS_r$  contains  $HS$  and if  $r$  is a positive constant, then  $HS_r = HS$  [see §2]. In §2, we study the relation between  $HS_r$  and the  $(A_{2,r})$  condition which is called the weighted  $(A_2)$  condition. The  $(A_2)$  condition was introduced by M.Rosenblum in his study of the weighted continuity for the Poisson transform, and then used by B.Muckenhoupt (before [5]) for the maximal function. R.A.Hunt, B.Muckenhoupt and R.L.Wheeden [5] showed that  $HS$  consists of weights which satisfy the  $(A_2)$  condition. T.Nakazi and T.Yamamoto [8], and T.Yamamoto [12] studied the first one-weighted inequality for  $C = 1$ . They gave necessary and sufficient conditions that are more complicated than  $HS_r$ . A class of weights was introduced in [12]. Our class  $HS_r$  is larger and more simple than that. The second one-weighted inequality was studied by R.Rochberg [10] when the weight  $W$  is in  $HS$ . T.Yamamoto [11] and T.Nakazi [7] studied it in some special cases. In §4 of this paper, we give a simple necessary and sufficient condition for the second one-weighted inequality for some positive constant  $\varepsilon$ , when  $\text{ess. inf } |\alpha - \beta| > 0$ .

In this paper, we consider the Koosis type problem of two weights, that is, given a nonnegative function  $W$  in  $L^1$  find necessary and sufficient conditions on  $W$  such that there exists a nonnegative and nonzero function  $U$  satisfying one of the following two-weighted norm inequalities :

$$\|S_{\alpha,\beta}f\|_U \leq \|f\|_W$$

and

$$\|S_{\alpha,\beta}f\|_W \geq \|f\|_U.$$

The first two-weighted inequality was studied by P.Koosis [6] when  $\alpha = 1$  and  $\beta = 0$  or  $\alpha = 1$  and  $\beta = -1$ . He showed that there exists a nonzero function  $U$  if and only if  $W^{-1}$  belongs to  $L^1$ . In §3, we can show that this type theorem is true for functions  $\alpha$  and  $\beta$  in  $L^\infty$ . The second two-weighted inequality has never been studied before. In §4, we give a very simple necessary and sufficient condition for that there exists a nonzero function  $U$  for the second one. There is a different problem, that is, given nonnegative function  $W$  and  $U$  in  $L^1$  find necessary and sufficient conditions on  $W$  and  $U$  such that one of the two-weighted norm inequalities above holds. This problem was solved in the particular case of  $\alpha = 1$  and  $\beta = -1$  by Cotlar and Sadosky using their lifting theorem. Since this problem becomes very complicated in the general case of  $\alpha, \beta \in L^\infty$ , we do not consider this problem. In this paper, the Cotlar-Sadosky lifting theorem [1] is essential and is used several times.

## §2. Weighted Helson-Szegő or $(A_2)$ condition

In this section, we define a class  $HS_r$  of weights which is called a weighted Helson-Szegő class, and a class  $A_{2,r}$  of weights which satisfy the weighted  $(A_2)$  condition. We will show  $A_{2,r} \supseteq HS_r$ . The weighted Helson-Szegő class will be used later. Let  $r$  be a function in  $L^\infty$  with  $0 \leq r \leq 1$ .

$$W \in HS_r \text{ if and only if } W = e^{u+\tilde{v}}$$

where  $u$  and  $v$  are real functions in  $L^1$  with  $|v| \leq \pi/2$ , and there exists a finite constant  $\gamma > 0$  such that

$$r^2 e^u \leq \gamma \cos v \text{ and } e^{-u} \leq \gamma \cos v.$$

$W \in A_{2,r}$  if and only if there exists a finite constant  $\gamma > 0$  such that  $W^{-1} \in L^1$  and

$$r^2 \widehat{W}(a) \widehat{W}^{-1}(a) \leq \gamma \quad (a \in D).$$

$D$  denotes the open unit disc and  $\widehat{V}$  is the harmonic extension to  $D$  of a function  $V$  in  $L^1$ .  $W \in HS$  if and only if  $W = e^{u+\tilde{v}}$  where  $u$  and  $v$  are real functions in  $L^\infty$  with  $\|v\|_\infty < \pi/2$  (see [4]).  $W \in HS_1$  if and only if  $W = e^{u+\tilde{v}}$  where  $u$  and  $v$  are real functions in  $L^1$  with  $|v| \leq \frac{\pi}{2}$ , and there exists a finite constant  $\gamma > 0$  such that

$$2 \leq e^{-u} + e^u \leq \gamma \cos v.$$

Hence, if  $r \equiv 1$  then  $HS_r = HS$  and  $A_{2,r} = A_2$ . The set  $A_2$  is the well known class of weights which satisfy the  $(A_2)$  condition and  $HS = A_2$  (see [5, Theorem 2]).

Theorem 1. Let  $r$  and  $r'$  be functions in  $L^\infty$  with  $0 \leq r \leq 1$  and  $0 \leq r' \leq 1$ .

(1)  $A_{2,r} \supseteq HS_r \supseteq HS$ . In particular, if  $W \in HS_r$  then both  $r^2 W$  and  $W^{-1}$  belong to  $L^1$ .

(2) If  $r/r'$  is bounded, then  $HS_{r'} \subseteq HS_r$  and hence if both  $r/r'$  and  $r'/r$  are bounded, then  $HS_{r'} = HS_r$ .

(3) If  $r^{-1}$  is bounded, then  $A_{2,r} = A_2 = HS = HS_r$ .

(4) If  $r \equiv 0$ , then  $A_{2,r} \cap L^1 = HS_r \cap L^1$ .

(5) If  $r^{-2} \notin \bigcup_{p>1} L^p$ , then  $HS_r \neq HS$  and hence  $A_{2,r} \neq A_2$ .

(6)  $W \in HS_r$  if and only if there exists a nonzero function  $V \in L^1$  and a real constant  $c$  such that

$$\frac{1}{\gamma} r^2 W \leq V \leq \gamma W \text{ and } (\tilde{V} + c)^2 \leq \gamma V W$$

for some positive constant  $\gamma$ .

Proof. (1) If  $W = e^{u+\tilde{v}}$ ,  $u \in L^\infty$  and  $\|v\|_\infty < \pi/2$ , then  $\cos v \geq \delta > 0$  for some constant  $\delta$ ,  $e^u \in L^\infty$  and  $e^{-u} \in L^\infty$ . Hence there exists a finite constant  $\gamma \geq 1$  such that



$$r^2 e^u \leq \gamma \cos v \text{ and } e^{-u} \leq \gamma \cos v.$$

Therefore  $HS_r \supseteq HS$ . If  $W \in HS_r$ , then  $W = e^{u+\tilde{v}}$ ,  $r^2 e^u \leq \gamma \cos v$ ,  $e^{-u} \leq \gamma \cos v$  and  $\gamma \geq 1$ . Hence

$$r^2 W \leq \gamma e^{\tilde{v}} \cos v \text{ and } W^{-1} \leq \gamma e^{-\tilde{v}} \cos v.$$

Put  $f = e^{\tilde{v}-iv}$ , then both  $f$  and  $f^{-1}$  belongs to  $H^p$  for some  $p < 1$  because  $\|v\|_\infty \leq \pi/2$ . Hence for any  $a \in D$

$$\widehat{r^2 W}(a) \leq \gamma \operatorname{Re} f(a) \leq \gamma |f(a)|$$

and

$$\widehat{W^{-1}}(a) \leq \gamma \operatorname{Re} f^{-1}(a) \leq \gamma |f^{-1}(a)|.$$

This implies that  $W \in A_{2,r}$  and  $A_{2,r} \supseteq HS_r$ . (2) If  $r \leq Cr'$  for some positive constant  $C$  and  $W \in HS_{r'}$  and  $W = e^{u+\tilde{v}}$  with  $\gamma' \geq 1$ , then  $r^2 e^u \leq (Cr')^2 e^u \leq C^2 \gamma' \cos v$  and  $e^{-u} \leq \gamma' \cos v$ . This implies that  $W \in HS_r$  and  $HS_{r'} \subseteq HS_r$ . (3) follows trivially from (2). (4) When  $r \equiv 0$ ,  $W \in A_{2,r}$  if and only if  $W^{-1} \in L^1$ , and  $W \in HS_r$  if and only if  $W = e^{u+\tilde{v}}$  and  $e^{-u} \leq \gamma \cos v$ . If  $V = W^{-1} \in L^1$ , then  $V + i\tilde{V} = e^{c-\tilde{v}+iv}$  and  $\|v\|_\infty \leq \pi/2$  for some real constant  $c$ . Hence  $V = e^{c-\tilde{v}} \cos v$ . If  $W$  is in  $L^1$ , then  $u = -c - \log(\cos v)$  belongs to  $L^1$ ,  $W = e^{u+\tilde{v}}$  and  $e^{-u} = e^c \cos v$ . (5) Suppose  $r^{-2} \notin \bigcup_{p>1} L^p$ . When  $\log r \in L^1$ ,

if  $W = r^{-2}$ , then we can write  $W = e^{u+\tilde{v}}$  where  $v = 0$ ,  $u = -2 \log r \in L^1$ ,  $r^2 e^u = \cos v$  and  $e^{-u} \leq \cos v$ . Hence  $W \in HS_r$ . If  $HS_r = HS$ , then  $r^{-2} \in HS$  and hence  $r^{-2} \in \bigcup_{p>1} L^p$ .

This contradicts the hypothesis on  $r$ . When  $\log r \notin L^1$ , there exists  $r'$  such that  $r \leq r' \leq 1$ ,  $\log r' \in L^1$  but  $(r')^{-2} \notin \bigcup_{p>1} L^p$ . By (1) and (2),  $HS_r \supseteq HS_{r'} \supseteq HS$ . If  $HS_r = HS$ , then  $HS_{r'} = HS$  and hence  $(r')^{-2} \in HS$ . This contradicts that  $(r')^{-2} \notin \bigcup_{p>1} L^p$ . Thus

$HS_r \neq HS$ .

(6) If  $W \in HS_r$ , then by the proof of (1) there exists a function  $f$  in  $H^p$  with  $\operatorname{Re} f \in L^1$  such that

$$r^2 W \leq \gamma \operatorname{Re} f \text{ and } W^{-1} \leq \gamma \operatorname{Re} f^{-1}.$$

If we put  $V = \operatorname{Re} f$ , then  $\operatorname{Re} f^{-1} = V / \{V^2 + (\tilde{V} + c)^2\}$  for some real constant  $c$ . Hence  $r^2 W \leq \gamma V$  and

$$W^{-1} \leq \gamma \frac{V}{V^2 + (\tilde{V} + c)^2} \leq \gamma V^{-1}$$

and  $(\tilde{V} + c)^2 \leq V^2 + (\tilde{V} + c)^2 \leq \gamma VW$ . Conversely if  $\frac{1}{\gamma} r^2 W \leq V \leq \gamma W$  and  $(\tilde{V} + c)^2 \leq \gamma VW$ , then  $r^2 W \leq \gamma V$  and  $V^2 + (\tilde{V} + c)^2 \leq 2\gamma VW$ . Hence  $W^{-1} \leq 2\gamma V / \{V^2 + (\tilde{V} + c)^2\}$ . If  $f = V + i(\tilde{V} + c)$ , then  $f = e^{\tilde{v}+s-iv}$  for some real constant  $s$  where  $|v| \leq \pi/2$ , and hence

$$r^2W \leq \gamma e^{\tilde{v}+s} \cos v \quad \text{and} \quad W^{-1} \leq 2\gamma e^{-(\tilde{v}+s)} \cos v.$$

Put  $e^u = We^{-\tilde{v}}$ , then  $W \in HS_r$ .

(6) of Theorem 1 with  $r \equiv 1$  implies a result of V.F.Gapokin (cf.[4]). That is,  $W \in HS$  if and only if there exists a nonzero function  $V \in L^1$  and a real constant  $c$  such that

$$\frac{1}{\gamma}W \leq V \leq \gamma W \quad \text{and} \quad (\tilde{V} + c)^2 \leq \gamma V^2$$

for some positive constant  $\gamma \geq 1$ . Hence  $W \in HS$  if and only if there is a bounded function  $u_1$  and a real constant  $c$  such that

$$|\tilde{V} + c| < \gamma V,$$

where  $V = e^{-u_1}W$ . We could not show that  $c = 0$ . We can ask two natural questions. Is the converse of (3) of Theorem 1 true? Is (4) of Theorem 1 true in general? The structure of  $HS_r$  is not simple. We define two more simple classes  $B_r^0$  and  $B_r^1$  of weights. Put

$B_r^0 = \{W > 0 ; W = W_0W_1, \log W \in L^1, r^2W_0 \in L^\infty, W_0^{-1} \in L^\infty, W_1 = e^{\tilde{v}}$   
and  $|v| \leq \pi/2 - \varepsilon$  for some  $\varepsilon > 0\}$  and

$B_r^1 = \{W > 0 ; W = W_0W_1, \log W \in L^1, r^2W_0 \in L^\infty, W_0^{-1} \in L^\infty, W_1 = e^{\tilde{v}}$   
and  $|v| \leq \pi/2\}$ . Then  $B_r^1 \supseteq HS_r \supseteq B_r^0 \supseteq HS$ . There exists  $v$  such that  $|v| \leq \frac{\pi}{2}$  and  $e^{-\tilde{v}} \notin L^1$ . Suppose  $W = W_0W_1 = 1 \times e^{\tilde{v}}$ , then  $W \in B_r^1$  and  $W^{-1} \notin L^1$ . This and (1) of Theorem 1 implies that  $W \notin HS_r$ . Hence  $B_r^1 \not\supseteq HS_r$  for arbitrary  $r$ . If  $W = r^{-2}$  with  $r^{-2} \notin \bigcup_{p>1} L^p$ , then  $W \in B_r^0$  and  $W \notin HS$ . Hence  $B_r^0 \not\supseteq HS$  for some  $r$ . If  $r^{-1} \in L^\infty$ , then

$HS_r = B_r^0 = HS$ . It is very nice if  $HS_r = B_r^0$  for arbitrary  $r$ . Unfortunately it is easy to see that it is not true. In fact, if  $W \in B_r^0$  then by definition  $W^{-1}$  belongs to  $L^p$  for some  $p > 1$ . Then the following characterization of  $W$  with  $W^{-1} \in L^1$  shows that  $HS_r \neq B_r^0$  for some  $r \geq 0$ . Both  $W$  and  $W^{-1}$  belong to  $L^1$  and  $W > 0$  if and only if there exists a nonzero  $r$  such that  $W \in HS_r \cap L^1$  if and only if  $W \in HS_0 \cap L^1$ . Suppose  $W > 0$  and  $W, W^{-1} \in L^1$ . Let  $k = W^{-1} + i(W^{-1})^\sim$ , then  $k$  is an outer function, since  $Re k > 0$ . Since  $|k^{-1}| \leq W, k^{-1}$  is in  $H^1$ . Let  $V = Re(k^{-1})$ , then there exists a real constant  $c$  such that  $\tilde{V} + c = Im(k^{-1})$ . Hence,

$$\frac{W^{-2}}{W^{-2} + (W^{-1})^{\sim 2}} W = V \leq W$$

and

$$\begin{aligned} (\tilde{V} + c)^2 &= \{Im(k^{-1})\}^2 = \frac{(W^{-1})^{\sim 2}}{\{W^{-2} + (W^{-1})^{\sim 2}\}^2} \\ &\leq \frac{1}{W^{-2} + (W^{-1})^{\sim 2}} = VW. \end{aligned}$$

By (6) of Theorem 1,  $W \in HS_r \cap L^1$ , where

$$r = \sqrt{\frac{W^{-2}}{W^{-2} + (W^{-1})^2}} > 0 \quad a.e.$$

Conversely, if  $W \in HS_r \cap L^1$ , then by (1) of Theorem 1,  $W, W^{-1} \in L^1$ . Then we can choose  $r$  such that  $r > 0$  a.e.. Now it is easy to see that  $W \in L^p$  and  $W^{-1} \in L^p$  for some  $p > 1$  and  $W > 0$  if and only if there exists  $r$  such that  $r > 0$  a.e.,  $W \in B_r^0 \cap L^p$  for some  $p > 1$ , since if  $W \in B_r^1$  and  $0 < \varepsilon < 1$  then  $W^{1-\varepsilon} \in B_r^0$ .

The following two lemmas will be used in §3. The first one was used by the second author in [11] and its proof is clear.

Lemma 1. Let  $u, v$  and  $r$  be real functions. Then  $|1 - e^{-u-iv}|^2 \leq 1 - r^2$  if and only if  $r^2 e^u + e^{-u} \leq 2 \cos v$ .

Lemma 2. Let  $u, v$  and  $r$  be real functions, and  $0 \leq r \leq 1$ .

$$r^2 e^u \leq \gamma \cos v \quad \text{and} \quad e^{-u} \leq \gamma \cos v$$

for some constant  $\gamma > 0$  if and only if there exists a constant  $\varepsilon > 0$  such that

$$|1 - \varepsilon e^{-u-iv}|^2 \leq 1 - (\varepsilon r)^2.$$

Proof. If  $r^2 e^u \leq \gamma \cos v$  and  $e^{-u} \leq \gamma \cos v$  for some  $\gamma > 0$ , then

$$\frac{1}{\gamma^2} r^2 e^{u+\log \gamma} \leq \cos v \quad \text{and} \quad e^{-(u+\log \gamma)} \leq \cos v.$$

Put  $\varepsilon = 1/\gamma$ , then

$$(\varepsilon r)^2 e^{u+\log \gamma} + e^{-(u+\log \gamma)} \leq 2 \cos v.$$

By Lemma 1,  $|1 - \varepsilon e^{-u-iv}|^2 \leq 1 - (\varepsilon r)^2$ . Conversely, if  $|1 - \varepsilon e^{-u-iv}|^2 \leq 1 - (\varepsilon r)^2$ , then by Lemma 1

$$(\varepsilon r)^2 e^s + e^{-s} \leq 2 \cos v \quad \text{and} \quad s = u - \log \varepsilon.$$

Hence  $(\varepsilon r)^2 e^s \leq 2 \cos v$  and  $e^{-s} \leq 2 \cos v$ . Therefore if  $\gamma = 2/\varepsilon$ , then  $r^2 e^u \leq \gamma \cos v$  and  $e^{-u} \leq \gamma \cos v$ .

### §3. Bounded singular integral operators

In this section, the first one-weighted norm inequality of Helson-Szegő type and the first two-weighted norm inequality of Koosis type are studied.

Theorem 2. Suppose  $\alpha, \beta$  are in  $L^\infty$  and  $W$  is a nonnegative function in  $L^1$ . Then

$$\|S_{\alpha, \beta} f\|_W \leq C \|f\|_W \quad (f \in A + \bar{A}_0)$$

for some positive constant  $C$  if and only if  $\alpha - \beta \not\equiv 0$  and  $W$  belongs to  $HS_r$  with  $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$ , or  $|\alpha - \beta|W \equiv 0$ .

Proof.  $\|S_{\alpha, \beta}\|_W < \infty$  if and only if  $\|S_{\alpha - \beta, 0}\|_W < \infty$  because  $S_{\alpha, \beta} f = (\alpha - \beta)P_+ f + \beta f$  ( $f \in A + \bar{A}_0$ ). We shall prove that  $\|S_{\alpha - \beta, 0}\|_W < \infty$  if and only if  $\alpha - \beta \not\equiv 0$  and  $W \in HS_r$  with  $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$  or  $|\alpha - \beta|W \equiv 0$ . Suppose  $\|S_{\alpha - \beta, 0}\|_W < \infty$  and  $C \geq \|S_{\alpha - \beta, 0}\|_W$ . Then, for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$

$$\|(\alpha - \beta)f_1\|_W^2 \leq C^2 \|f_1 + f_2\|_W^2.$$

Let  $W_1 = (C^2 - |\alpha - \beta|^2)W$ ,  $W_2 = W_3 = C^2W$ , then

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\operatorname{Re}(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], there exists a  $k$  in  $H^1$  such that  $|W_3 - k|^2 \leq W_1 W_2$  and hence  $|C^2W - k|^2 \leq C^2(C^2 - |\alpha - \beta|^2)W^2$ . Suppose  $|\alpha - \beta|W \not\equiv 0$ . Then  $0 < |k| \leq 2C^2W$  and hence  $\log W \in L^1$ . Then

$$\left|1 - \frac{k}{C^2W}\right|^2 \leq 1 - \frac{|\alpha - \beta|^2}{C^2}.$$

Put  $r' = |\alpha - \beta|/C$  and  $e^{-u-iv} = k/C^2W$  where  $u$  and  $v$  are real functions,  $u \in L^1$ ,  $|v| \leq \pi/2$ , then  $0 \leq r' \leq 1$  and

$$|1 - e^{-u-iv}|^2 \leq 1 - (r')^2.$$

By Lemma 2,  $e^{u+\bar{v}}$  belongs to  $HS_{r'}$  and hence by (1) of Theorem 1,  $e^{-u-\bar{v}} \in L^1$ . Therefore  $C^2W e^{-u-\bar{v}} = k e^{-\bar{v}+iv}$  is a positive function in  $H^{1/2}$  and hence by the Neuwirth-Newman theorem [9], it is a positive constant  $C'$ . Thus  $W = C' C^{-2} e^{u+\bar{v}}$  and  $W$  belongs to  $HS_{r'}$ . Put  $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$ , then by (2) of Theorem 1,  $W \in HS_r$ . For the converse, if  $|\alpha - \beta|W \equiv 0$ , then  $\|S_{\alpha - \beta, 0}\|_W = 0$  and hence we may assume  $W \in HS_r$ . By definition of  $HS_r$  and Lemma 2,  $W = e^{u+\bar{v}}$  and

$$|1 - \varepsilon e^{-u-iv}|^2 \leq 1 - (\varepsilon r)^2$$

for some positive constant  $\varepsilon$ . If  $k = \varepsilon W e^{-u-iv}$ , then  $k = \varepsilon e^{\bar{v}-iv}$  is an outer function and

$$|W - k|^2 \leq \{1 - (\varepsilon r)^2\} W^2.$$

This implies  $k \in H^1$ . Let  $W_1 = \{1 - (\varepsilon r)^2\} W$ ,  $W_2 = W_3 = W$ , then

$$|W_3 - k|^2 \leq W_1 W_2.$$

By the Cotlar-Sadosky lifting theorem [1], for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2 \operatorname{Re}(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

Hence,

$$\|\varepsilon r f_1\|_W^2 \leq \|f_1 + f_2\|_W^2.$$

Thus  $\|S_{\alpha-\beta,0} f\|_W^2 \leq \|\alpha - \beta\|_\infty^2 \varepsilon^{-2} \|f\|_W^2$ .

If  $\operatorname{ess. inf} |\alpha - \beta| > 0$  and  $W \not\equiv 0$  in Theorem 2, then by (3) of Theorem 1,  $\|S_{\alpha,\beta}\|_W < \infty$  if and only if  $W \in HS$  (see [11]). Hence Theorem 2 shows the Helson-Szegő theorem [4]. The following theorem for  $\alpha = 1$  and  $\beta = 0$  is the Koosis theorem [6].

**Theorem 3.** Suppose  $\alpha, \beta$  are in  $L^\infty$  with  $\alpha - \beta \not\equiv 0$  and  $W$  is a nonnegative function in  $L^1$ . There exists a nonnegative function  $U$  with  $|\alpha - \beta|U \not\equiv 0$  such that

$$\|S_{\alpha,\beta} f\|_U \leq \|f\|_W \quad (f \in A + \bar{A}_0)$$

if and only if  $W^{-1}$  belongs to  $L^1$ .

*Proof.* Suppose  $|\alpha - \beta|U \not\equiv 0$  and  $\|S_{\alpha,\beta} f\|_U \leq \|f\|_W$ . Since  $\max\{|\alpha|^2, |\beta|^2\}U \leq W$ , we have  $\{|\alpha - \beta|U > 0\} \subseteq \{W > 0\} \subseteq T$ . Put  $d = (U/W + \varepsilon)^{1/2}$  for some  $\varepsilon > 0$ , then  $d\alpha - d\beta \not\equiv 0$  and  $\|S_{d\alpha,d\beta} f\|_W \leq \|f\|_W$ . By Theorem 2,  $W \in HS_r$  for  $r = d|\alpha - \beta|/\|d(\alpha - \beta)\|_\infty$ . By (1) of Theorem 1,  $W^{-1}$  belongs to  $L^1$ . Conversely, if  $W^{-1} \in L^1$ , by the remark after Theorem 1 there exists  $r' > 0$  a.e. such that  $W \in HS_{r'}$ . There exists  $d \in L^\infty$  such that  $d > 0$  a.e. and  $d|\alpha - \beta| \leq r'$ . By (2) of Theorem 1,  $W \in HS_r$  with  $r = d|\alpha - \beta|/\|d(\alpha - \beta)\|_\infty$ . By Theorem 2,  $\|S_{d\alpha,d\beta} f\|_W \leq C\|f\|_W$ . Put  $U = C^{-2}d^2W$ , then  $\|S_{\alpha,\beta} f\|_U \leq \|f\|_W$  and  $|\alpha - \beta|U \not\equiv 0$ , since  $U > 0$  a.e. and  $\alpha - \beta \not\equiv 0$ .

#### §4. Bounded below singular integral operators

In this section, the second one-weighted norm inequality and the second two-weighted norm inequality are studied. We could not give a necessary and sufficient condition for the one-weighted inequality, in general. The second author [11] gave a necessary and sufficient condition for that  $S_{\alpha, \pm\beta}$  are bounded below with respect to  $W$ . When  $W \in HS$ , if  $S_{\alpha, \beta}$  is bounded below *w.r.t.*  $W$ , then  $S_{\alpha, -\beta}$  is also bounded below *w.r.t.*  $W$ . Hence his result covers the Rochberg theorem for  $p = 2$ . The first author [7] generalized the above result for arbitrary  $p, 1 < p < \infty$ . In fact, Rochberg [10] gave the necessary and sufficient condition for the invertibility of the Toeplitz operators on weighted  $H^p$  spaces for  $p, 1 < p < \infty$ . In Theorem 4,  $HS_r \supseteq B_r^0$  (see §2) but unfortunately  $HS_r \neq B_r^0$ , in general. If  $r^{-1}$  is bounded, then  $HS_r = B_r^0$  (see §2).

Lemma 3. Let  $r$  be a function in  $L^\infty$  with  $0 \leq r \leq 1$  and  $\text{ess. inf } |\alpha\beta| > 0$ . The following (1) and (2) are equivalent.

(1) For any enough small positive constant  $\varepsilon$ , there exist an inner function  $Q$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{it} \text{ and } We^{-t} \in B_r^0.$$

(2) There exist an inner function  $Q$  and a real function  $\ell$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{i\ell} \text{ and } We^{-\ell} \in B_r^0.$$

Proof. Put  $e^{is} = (\alpha\bar{\beta})|\alpha\bar{\beta} - \varepsilon^2|/|\alpha\bar{\beta}|(\alpha\bar{\beta} - \varepsilon^2)$  and  $\ell = t - \tilde{s}$ , then

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{it}e^{is} = e^{ic}Qe^{i\ell}$$

for some real constant  $c$  and

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = e^{ic}Qe^{i\ell}e^{-is} = Qe^{i\ell}.$$

Since  $We^{-\ell} = (We^{-t})e^{\tilde{s}}$  and  $\|s\|_\infty \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , for enough small  $\varepsilon > 0$  it is easy to show that  $We^{-\ell} \in B_r^0$  if and only if  $We^{-t} \in B_r^0$  by definition of  $B_r^0$  in §2. These imply the equivalence of (1) and (2).

Theorem 4. Suppose  $\alpha, \beta$  are in  $L^\infty$  and  $W$  is a nonnegative function in  $L^1$  satisfying  $|\alpha - \beta|W \neq 0$ .

(1) If there exists a positive constant  $\delta$  such that

$$\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W \quad (f \in A + \bar{A}_0),$$

then for each positive constant  $\varepsilon < \delta$  there exist an inner function  $Q$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{it} \text{ and } We^{-t} \in HS_r$$

where  $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$ , and  $\alpha\bar{\beta} - \varepsilon^2$ ,  $\alpha$  and  $\beta$  are invertible in  $L^\infty$ .

(2) If for each enough small positive constant  $\varepsilon$  there exist an inner function  $Q$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Qe^{it} \text{ and } We^{-t} \in B_r^0$$

where  $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$ , and  $\alpha\bar{\beta} - \varepsilon^2$ ,  $\alpha$  and  $\beta$  are invertible in  $L^\infty$ , then

$$\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W \quad (f \in A + \bar{A}_0)$$

for some positive constant  $\delta$ .

Proof. (1) Suppose  $\|S_{\alpha,\beta}f\|_W \geq \delta\|f\|_W$  ( $f \in A + \bar{A}_0$ ). Let  $W'_1 = (|\alpha|^2 - \delta^2)W$ ,  $W'_2 = (|\beta|^2 - \delta^2)W$ ,  $W'_3 = (\alpha\bar{\beta} - \delta^2)W$ . Then, for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$ ,

$$\int_T \{|f_1|^2 W'_1 + |f_2|^2 W'_2 + 2\operatorname{Re}(f_1 \bar{f}_2 W'_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1],  $W'_1 \geq 0$ ,  $W'_2 \geq 0$ , and there exists a  $k'$  in  $H^1$  such that  $|W'_3 - k'|^2 \leq W'_1 W'_2$ . Hence,  $(|\alpha|^2 - \delta^2)W \geq 0$ ,  $(|\beta|^2 - \delta^2)W \geq 0$ , and  $|(\alpha\bar{\beta} - \delta^2)W - k'|^2 \leq (|\alpha|^2 - \delta^2)(|\beta|^2 - \delta^2)W^2$ . If  $k' \equiv 0$ , then  $\delta^2|\alpha - \beta|^2 W^2 \leq 0$ . This contradiction implies  $k' \equiv 0$ . Since  $(|\alpha|^2 - \delta^2)(|\beta|^2 - \delta^2) \leq |\alpha\bar{\beta} - \delta^2|^2$ , we have  $0 < |k'| \leq 2|\alpha\bar{\beta} - \delta^2|W$  and hence  $W > 0$ . Hence  $|\alpha| \geq \delta$  and  $|\beta| \geq \delta$ . If  $0 < \varepsilon < \delta$ , then

$$\|S_{\alpha,\beta}f\|_W \geq \varepsilon\|f\|_W \quad (f \in A + \bar{A}_0).$$

Let  $W_1 = (|\alpha|^2 - \varepsilon^2)W$ ,  $W_2 = (|\beta|^2 - \varepsilon^2)W$ ,  $W_3 = (\alpha\bar{\beta} - \varepsilon^2)W$ . Then, for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$ ,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\operatorname{Re}(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1], there exists a  $k$  in  $H^1$  such that  $|W_3 - k|^2 \leq W_1 W_2$ . Hence,

$$|(\alpha\bar{\beta} - \varepsilon^2)W - k|^2 \leq (|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2)W^2.$$

Since  $|\alpha| \geq \delta$ ,  $|\beta| \geq \delta$  and  $0 < \varepsilon < \delta$ , we have  $|\alpha\bar{\beta} - \varepsilon^2| \geq |\alpha\beta| - \varepsilon^2 \geq \delta^2 - \varepsilon^2 > 0$ . Hence  $\alpha\bar{\beta} - \varepsilon^2$ ,  $\alpha$  and  $\beta$  are invertible in  $L^\infty$ . If  $k \equiv 0$ , then  $\varepsilon^2|\alpha - \beta|^2 W^2 \leq 0$ . This contradiction implies  $k \neq 0$ . Since  $(|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2) \leq |\alpha\bar{\beta} - \varepsilon^2|^2$ , we have  $0 < |k| \leq 2|\alpha\bar{\beta} - \varepsilon^2|W$  and hence  $\log(|\alpha\bar{\beta} - \varepsilon^2|W) \in L^1$ . Then

$$\left| 1 - \frac{k}{(\alpha\bar{\beta} - \varepsilon^2)W} \right|^2 \leq \frac{(|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2)}{|\alpha\bar{\beta} - \varepsilon^2|^2}.$$

If  $e^{-u-iv} = k/(\alpha\bar{\beta} - \varepsilon^2)W$  and  $q = \varepsilon|\alpha - \beta|/|\alpha\bar{\beta} - \varepsilon^2|$  where  $u$  and  $v$  are real functions, then  $|1 - e^{-u-iv}|^2 \leq 1 - q^2$ ,  $(\alpha\bar{\beta} - \varepsilon^2)W = ke^{u+iv}$  and  $|\alpha\bar{\beta} - \varepsilon^2|W = |k|e^u$ . Hence we can take  $u, v$  in  $L^1$  and  $|v| \leq \frac{\pi}{2}$ . By the inner-outer factorization of  $k$ ,  $k = Q \exp\{\log |k| + i(\log |k|)^\sim\}$  where  $Q$  is an inner function. Put  $t = -\tilde{v} + \log |k|$ , then

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = \frac{k}{|k|} e^{iv} = Q' e^{i\tilde{t}}$$

where  $Q' = Qe^{ic}$  for some real constant  $c$  and by Lemma 2

$$|\alpha\bar{\beta} - \varepsilon^2|W e^{-t} = e^{u+\tilde{v}} \in HS_q.$$

Let  $u_0 = \log |\alpha\bar{\beta} - \varepsilon^2|$ , then  $u_0 \in L^\infty$  and  $W e^{-t} = e^{(u-u_0)+\tilde{v}}$ . Since  $q^2 e^u \leq \gamma \cos v$  and  $e^{-u} \leq \gamma \cos v$ , we have  $q^2 e^{u-u_0} \leq (\gamma e^{\|u_0\|_\infty}) \cos v$  and  $e^{-(u-u_0)} \leq (\gamma e^{\|u_0\|_\infty}) \cos v$ . Hence  $W e^{-t} \in HS_q$ . If  $r = |\alpha - \beta|/\|\alpha - \beta\|_\infty$ , then both  $r/q$  and  $q/r$  are bounded. Hence by (2) of Theorem 1,  $W e^{-t}$  belongs to  $HS_r$ .

(2) If for each enough small positive constant  $\varepsilon$  there exist an inner function  $Q$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Q e^{i\tilde{t}} \quad \text{and} \quad W e^{-t} \in B_r^0,$$

then by Lemma 3

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Q e^{i(c+\tilde{t})} \quad \text{and} \quad W e^{-\ell} \in B_r^0$$

and  $\ell = t - \tilde{s}$  where  $e^{i\tilde{s}} = (\alpha\bar{\beta})|\alpha\bar{\beta} - \varepsilon^2|/|\alpha\bar{\beta}|(\alpha\bar{\beta} - \varepsilon^2)$  and  $c$  is a real constant. Note  $\|\tilde{s}\|_\infty \rightarrow 0$  as  $\varepsilon \rightarrow 0$  since  $\text{ess. inf } |\alpha\bar{\beta}| > 0$ . By definition of  $B_r^0$ ,  $W e^{-\ell} = e^{u_0+\tilde{v}_0}$  where  $u_0 \in L^1$ ,  $\|v_0\|_\infty < \frac{\pi}{2}$ ,

$$r^2 e^{u_0} \leq \gamma_0 \cos v_0 \quad \text{and} \quad e^{-u_0} \leq \gamma_0 \cos v_0.$$

Since  $t - \ell = \tilde{s}$  and  $\|\tilde{s}\|_\infty \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,

$$W e^{-t} = W e^{-\ell} e^{-\tilde{s}} = e^{u_0+(v_0-\tilde{s})^\sim}$$

and  $\|v_0 - \tilde{s}\|_\infty < \pi/2$ . Put  $u_1 = u_0$  and  $v_1 = v_0 - \tilde{s}$ , then for small  $\varepsilon > 0$

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Q e^{i\tilde{t}} \quad \text{and} \quad W e^{-t} = e^{u_1+\tilde{v}_1} \in B_r^0$$



where

$$r^2 e^{u_1} \leq \gamma_\varepsilon \cos v_1 \text{ and } e^{-u_1} \leq \gamma_\varepsilon \cos v_1.$$

Since  $v_1 = v_0 - s$  and  $\|s\|_\infty \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ,  $\gamma_\varepsilon$  is bounded. Put  $u = u_1 + \log \gamma_\varepsilon$ ,  $v = v_1$  and  $\rho = r/\gamma_\varepsilon$ , then  $\gamma_\varepsilon W e^{-t} = e^{u+\bar{v}}$ ,  $\rho^2 e^u \leq \cos v$  and  $e^{-u} \leq \cos v$ , and hence

$$\rho^2 e^u + e^{-u} \leq 2 \cos v.$$

Put  $W' = \gamma_\varepsilon W / |\alpha\bar{\beta} - \varepsilon^2|$ , then  $|\alpha\bar{\beta} - \varepsilon^2| W' e^{-t} = \gamma_\varepsilon W e^{-t}$ . Put  $k = (\alpha\bar{\beta} - \varepsilon^2) W' e^{-u-iv}$ , then  $k = Q e^{t+i\bar{t}} e^{\bar{v}-iv}$  is in  $H^p$  for some  $p > 0$  and by Lemma 1,

$$\left| 1 - \frac{k}{(\alpha\bar{\beta} - \varepsilon^2) W'} \right|^2 = |1 - e^{-u-iv}|^2 \leq 1 - \rho^2.$$

Hence  $|k| \leq C W'$  for some positive constant  $C$  and so  $k \in H^1$ . If we put  $q = \varepsilon |\alpha - \beta| / |\varepsilon^2 - \alpha\bar{\beta}|$ , then

$$\frac{q}{\rho} = \gamma_\varepsilon \frac{\|\alpha - \beta\|_\infty}{|\alpha - \beta|} \frac{\varepsilon |\alpha - \beta|}{|\alpha\bar{\beta} - \varepsilon^2|} = \frac{\varepsilon \gamma_\varepsilon \|\alpha - \beta\|_\infty}{|\alpha\bar{\beta} - \varepsilon^2|}$$

and hence there exists  $\varepsilon > 0$  such that  $q \leq \rho$  because  $\gamma_\varepsilon$  and  $\|(\alpha\bar{\beta} - \varepsilon^2)^{-1}\|_\infty$  are bounded. Therefore

$$|(\alpha\bar{\beta} - \varepsilon^2) W' - k|^2 \leq |\alpha\bar{\beta} - \varepsilon^2|^2 (1 - q^2) W'^2 = (|\alpha|^2 - \varepsilon^2)(|\beta|^2 - \varepsilon^2) W'^2$$

Since  $\alpha$  and  $\beta$  are invertible in  $L^\infty$  and  $\varepsilon$  is enough small, we have  $|\alpha| \geq \varepsilon$  and  $|\beta| \geq \varepsilon$ . Let  $W_1 = (|\alpha|^2 - \varepsilon^2) W'$ ,  $W_2 = (|\beta|^2 - \varepsilon^2) W'$  and  $W_3 = (\alpha\bar{\beta} - \varepsilon^2) W'$ . Then,  $|W_3 - k|^2 \leq W_1 W_2$ . By the Cotlar-Sadosky lifting theorem, for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$ ,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 + 2 \operatorname{Re}(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

Hence  $\|S_{\alpha,\beta} f\|_{W'} \geq \varepsilon \|f\|_{W'}$  ( $f \in A + \bar{A}_0$ ) and so  $\|S_{\alpha,\beta} f\|_W \geq \delta \|f\|_W$  for some positive constant  $\delta$  because  $W/W', W'/W \in L^\infty$ .

**Corollary.** Suppose  $\alpha, \beta$  are in  $L^\infty$  with  $\operatorname{ess. inf} |\alpha - \beta| > 0$  and  $W$  is a nonnegative function in  $L^1$ . Then the following (1) ~ (3) are equivalent.

- (1)  $\|S_{\alpha,\beta} f\|_W \geq \delta \|f\|_W$  ( $f \in A + \bar{A}_0$ ) for some positive constant  $\delta$ .
- (2) There exist a positive constant  $\varepsilon < \delta$ , an inner function  $Q$  and a real function

$\ell$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta} - \varepsilon^2}{|\alpha\bar{\beta} - \varepsilon^2|} = Q e^{i\bar{\ell}} \text{ and } W e^{-\ell} \in HS$$

where  $\alpha\bar{\beta} - \varepsilon^2$ ,  $\alpha$  and  $\beta$  are invertible in  $L^\infty$ .

(3) There exist an inner function  $Q$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = Qe^{it} \text{ and } We^{-t} \in HS$$

and both  $\alpha$  and  $\beta$  are invertible in  $L^\infty$ .

Proof. If  $\text{ess. inf } |\alpha - \beta| > 0$  and  $r = |\alpha - \beta| / \|\alpha - \beta\|_\infty$ , then by (3) of Theorem 1 and the remark in §2,  $HS_r = B_r^0 = HS$ . Hence the equivalence of (1) and (2) follows from Theorem 4. The equivalence of (2) and (3) follows from Lemma 3.

Theorem 5. Suppose  $\alpha, \beta$  are in  $L^\infty$  with  $\alpha - \beta \neq 0$  and  $W$  is a nonnegative function in  $L^1$ . There exists a nonnegative function  $U$  with  $|\alpha - \beta|U \neq 0$  such that

$$\|S_{\alpha, \beta} f\|_W \geq \|f\|_U \quad (f \in A + \bar{A}_0)$$

if and only if there exist a real constant  $c$  and a real function  $t$  in  $L^1$  such that

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = e^{i(c+t)} \text{ and } \frac{e^t}{|\alpha\bar{\beta}|W} \in L^1$$

Proof. Suppose  $\|S_{\alpha, \beta} f\|_W \geq \|f\|_U$  ( $f \in A + \bar{A}_0$ ). Let  $W_1 = |\alpha|^2 W - U$ ,  $W_2 = |\beta|^2 W - U$ ,  $W_3 = \alpha\bar{\beta}W - U$ . Then, for  $f_1 \in A$  and  $f_2 \in \bar{A}_0$ ,

$$\int_T \{|f_1|^2 W_1 + |f_2|^2 W_2 + 2\text{Re}(f_1 \bar{f}_2 W_3)\} dm \geq 0.$$

By the Cotlar-Sadosky lifting theorem [1],  $W_1 \geq 0$ ,  $W_2 \geq 0$ , and there exists a  $k$  in  $H^1$  such that  $|W_3 - k|^2 \leq W_1 W_2$ . Hence,  $|\alpha|^2 W - U \geq 0$ ,  $|\beta|^2 W - U \geq 0$ , and

$$|(\alpha\bar{\beta}W - U) - k|^2 \leq (|\alpha|^2 W - U)(|\beta|^2 W - U).$$

By a calculation,

$$(|\alpha|^2 W - U)^{1/2} (|\beta|^2 W - U)^{1/2} \leq |\alpha\bar{\beta}|W - U$$

and hence

$$|\alpha\bar{\beta}W - k| \leq (|\alpha|^2 W - U)^{1/2} (|\beta|^2 W - U)^{1/2} + U \leq |\alpha\bar{\beta}|W.$$

If  $k \equiv 0$ , then  $|\alpha\bar{\beta}W - U|^2 \leq (|\alpha|^2 W - U)(|\beta|^2 W - U)$  and hence  $|\alpha - \beta|^2 W U \equiv 0$ . Hence  $|\alpha - \beta|U = 0$  on  $\{W > 0\}$ . Since  $U \leq \min\{|\alpha|^2, |\beta|^2\}W$ , we have  $U = 0$  on  $\{W = 0\}$ , and hence  $|\alpha - \beta|U \equiv 0$ . This contradiction implies that  $k$  is a nonzero function. Hence  $\log |\alpha\bar{\beta}W| \in L^1$  because  $|k| \leq 2|\alpha\bar{\beta}W|$ . There exists an outer function  $g$  in  $H^1$  such that  $|\alpha\bar{\beta}W| = |g|$ . Hence  $|\alpha\bar{\beta}W - k| \leq |g|$ . By a lemma of Koosis (cf. [2, p161]) there exists an outer function  $f$  in  $H^1$  such that

$$\frac{\alpha\bar{\beta}}{g}W = \frac{f}{|f|}.$$

$gf$  is an outer function in  $H^1$  and hence  $gf = e^{t+i(\bar{i}+c)}$  where  $t = \log|gf|$  and  $c$  is a real constant. Thus

$$\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} = \frac{gf}{|gf|} = e^{i(\bar{i}+c)}$$

and

$$\frac{e^t}{|\alpha\bar{\beta}|W} = \frac{e^t}{|g|} = |f| \in L^1.$$

We will show the converse. Put  $G = e^t/|\alpha\bar{\beta}|W$ , then  $G \in L^1$ . Put  $k = e^{t+i(c+\bar{i})}/(G+i\tilde{G})$ , then  $|k| \leq e^t/G = |\alpha\bar{\beta}|W$  and hence  $k \in H^1$ . Since  $\alpha\bar{\beta}/|\alpha\bar{\beta}| = e^{i(c+\bar{i})}$ ,

$$\begin{aligned} |\alpha\bar{\beta}W - k| &= \left| \alpha\bar{\beta}W - \frac{e^{t+i(c+\bar{i})}}{G+i\tilde{G}} \right| = \left| |\alpha\bar{\beta}|W - \frac{e^t}{G+i\tilde{G}} \right| \\ &= |\alpha\bar{\beta}|W \left| 1 - \frac{G}{G+i\tilde{G}} \right| = |\alpha\bar{\beta}|W \left| \frac{\tilde{G}}{G+i\tilde{G}} \right|. \end{aligned}$$

Let  $U = G^2W/(G^2 + \tilde{G}^2)$ , then  $U > 0$  and

$$|\alpha\bar{\beta}W - k|^2 = |\alpha\bar{\beta}|^2W(W - U)$$

We may assume  $|\alpha| \leq 1$  and  $|\beta| \leq 1$ . For all  $f = f_1 + f_2 \in A + \bar{A}_0$ ,

$$\begin{aligned} &\int |S_{\alpha,\beta}f|^2W dm - \int |P_+f|^2|\alpha\bar{\beta}|^2U dm \\ &= \int |f_1|^2(|\alpha|^2W - |\alpha|^2|\beta|^2U) dm + \int |f_2|^2|\beta|^2W dm + 2\operatorname{Re} \int f_1\bar{f}_2(\alpha\bar{\beta}W - k) dm \\ &\geq \int |f_1|^2|\alpha|^2(W - U) dm + \int |f_2|^2|\beta|^2W dm - 2 \int |f_1\bar{f}_2||\alpha\bar{\beta}|W^{1/2}(W - U)^{1/2} dm \\ &= \int \{|f_1||\alpha|(W - U)^{1/2} - |f_2||\beta|W^{1/2}\}^2 dm \geq 0 \end{aligned}$$

Similarly,

$$\begin{aligned} &\int |S_{\alpha,\beta}f|^2W dm - \int |P_-f|^2|\alpha\bar{\beta}|^2U dm \\ &= \int |f_1|^2|\alpha|^2W dm + \int |f_2|^2(|\beta|^2W - |\alpha|^2|\beta|^2U) dm + 2\operatorname{Re} \int f_1\bar{f}_2(\alpha\bar{\beta}W - k) dm \\ &\geq \int |f_1|^2|\alpha|^2W dm + \int |f_2|^2|\beta|^2(W - U) dm - 2 \int |f_1\bar{f}_2||\alpha\bar{\beta}|W^{1/2}(W - U)^{1/2} dm \\ &\geq 0 \end{aligned}$$

Suppose  $U' = |\alpha\bar{\beta}|^2U/4$ , then  $U' > 0$ . Since  $\alpha - \beta \neq 0$ ,  $|\alpha - \beta|U' \neq 0$  and

$$\|S_{\alpha,\beta}f\|_W \geq \|P_+f\|_{U'} + \|P_-f\|_{U'} \geq \|f\|_{U'}.$$

## References

1. M.Cotlar and C.Sadosky, On the Helson-Szegő theorem and a related class of modified Toeplitz kernels, Proc. Sym. Pure Math. AMS 35(1979), 383-407.
2. J.B.Garnett, Bounded analytic functions, Academic Press, 1981.
3. I.Gohberg and N.Krupnik, One-dimensional linear singular integral equations, Vol. I,II, Birkhäuser, 1992.
4. H.Helson and G.Szegő, A problem in prediction theory, Ann. Mat. Pura Appl. 51(1960), 107-138.
5. R.Hunt, B.Muckenhoupt and R.Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176(1973), 227-251.
6. P.Koosis, Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques, C.R.Acad.Sci. Paris 291(1980), 255-257.
7. T.Nakazi, Toeplitz operators and weighted norm inequalities, Acta Sci. Math. (Szeged) 58(1993), 443-452.
8. T.Nakazi and T.Yamamoto, Some singular integral operators and Helson-Szegő measures, J.Funct. Anal. 88(1990), 366-384.
9. J.Neuwirth and D.J.Newman, Positive  $H^{1/2}$  functions are constants, Proc. Amer. Math. Soc. 18(1967), 958.
10. R.Rochberg, Toeplitz operators on weighted  $H^p$  spaces, Indiana Univ. Math.J. 26(1977), 291-298.
11. T.Yamamoto, Invertibility of some singular integral operators and a lifting theorem, Hokkaido Math. J. 22(1993), 181-198.
12. T.Yamamoto, Boundedness of some singular integral operators in weighted  $L^2$  spaces, J.of Operator Theory 32(1994), 243-254.