<table>
<thead>
<tr>
<th>項目</th>
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</tr>
</thead>
<tbody>
<tr>
<td>本の名前</td>
<td>Hodge Theory and Algebraic Geometry</td>
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Théorie de Hodge I, II, III pour cohomologies $p$-adiques.

Yukiyoshi Nakkajima *

**Resumé:** Dans ce rapport, nous présentons quelques résultats démontrés dans [NS] et [Nak2]. Nous avons construit deux objets filtrés fondamentaux: un *complexe cristallin filtré par le prépoids* et un *complexe zariskian par le prépoids*. Nous avons construit aussi la filtration par le poids sur la cohomologie rigide d’un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$.

**Mots-clefs:** cohomologies rigides, cohomologies log-cristallines, suites $p$-adiques spectrales des poids.

**1 Introduction.**

Motivated by a conjectural theory of motives due to A. Grothendieck, P. Deligne has given a translation of concepts on mixed Hodge structures into those on $l$-adic cohomologies and vice versa in [D1], and he has proved his conjectures on mixed Hodge structures in [D2] and [D3].

Let $U$ be a separated scheme of finite type over a field $\kappa$. Then there is the following translation [D1], [D2], [D3]:

*2000 Mathematics subject classification number: 14F30.*
Objects of $\mathbb{F}_q$ (for simplicity), $(l,q) = 1$

\[
\begin{array}{|l|}
\hline
l\text{-adic objects }/\mathbb{F}_q & \text{objects }/\mathbb{C} \\
\hline
H^h_{et}(U_{\mathbb{F}_q},\mathbb{Q}_l) (h \in \mathbb{Z}) & H^h(U^{an},\mathbb{Q}) (h \in \mathbb{Z}) \\
\hline
\end{array}
\]

$F$: geometric Frobenius $\curvearrowright H^h_{et}(U_{\mathbb{F}_q},\mathbb{Q}_l)$; there exists a unique finite increasing filtration $\{P_k\}_{k \in \mathbb{Z}}$ characterized by the following: $P_k H^h_{et}(U_{\mathbb{F}_q},\mathbb{Q}_l)$ is the principal subspace of $H^h_{et}(U_{\mathbb{F}_q},\mathbb{Q}_l)$ where the eigenvalues $\alpha$ of $F$ satisfy the following:

$|\sigma(\alpha)| \leq q^{k/2} \quad (\forall \sigma: \overline{\mathbb{Q}} \hookrightarrow \mathbb{C})$ (cf. [dJ]).

A morphism commuting $F$'s is strictly compatible with $P_k$'s.

Let us consider a more special case as in [D2]. Let $(X,D)$ be a smooth scheme over $\kappa$ with a SNCD (=simple normal crossing divisor) over $\kappa$. Put $U := X \setminus D$ and let $j: U \hookrightarrow X$ be the natural open immersion. Put $M_D := \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\}$. Let $\epsilon: \tilde{X}_{et} \longrightarrow \tilde{X}_{et}$ be the forgetting log morphism induced by a natural morphism $(X,M_D) \longrightarrow (X,\mathcal{O}_X^\times)$ of log schemes in the sense of Fontaine-Illusie-Kato [K]. Let $Y$ be an analytic variety over $\mathbb{C}$, and let us also denote by $\epsilon$ the real blow up $Y^{log} \longrightarrow Y$ [KN] which is denoted by $\tau$ in [loc. cit.]. Then we have the following translation:

\[
\begin{array}{l}
\text{(1.0.2)} \\
\hline
\text{objects }/\mathbb{C} & l\text{-adic objects} \\
\hline
U^{an}, (X^{an})^{log} & U_{et}, X^{log}_{et} \\
X^{an} & \tilde{X}_{et} \\
\epsilon: (X^{an})^{log} \longrightarrow X^{an} & \epsilon: \tilde{X}_{et} \longrightarrow \tilde{X}_{et} \\
R_j^{an}Z = R\epsilon_*Z & R\epsilon_{et,*}(\mathbb{Z}/l^n) = R\epsilon_{et,*}(\mathbb{Z}/l^n) \\
X^{an} \longrightarrow X & ? \\
(\Omega^{\bullet}_{X/\mathbb{C}}(log D), P_k) & (\Omega^{\bullet}_{\tilde{X}_{et}/\mathbb{C}}(log D^{an}), \tau_k) = (\Omega^{\bullet}_{X^{an}/\mathbb{C}}(log D^{an}), P_k) \\
(\Omega^{\bullet}_{X^{an}/\mathbb{C}}(log D^{an}, \tau_k)) = (\Omega^{\bullet}_{X^{an}/\mathbb{C}}(log D^{an}), P_k) & \?
\hline
\end{array}
\]

The purpose of this report is to inform the reader that we have succeeded in making two translations (1.0.1) and (1.0.2) for $p$-adic cohomologies.

This report came out of two preprints [NS], [Nak2] and my lecture entitled with Théorie de Hodge III pour cohomologies $p$-adiques at a symposium Hodge theory and algebraic geometry at Hokkaido-University in October in 2002. We give no proof in this report; we have given proofs in [NS] and [Nak2].
Acknowledgment. I am very grateful to L. Illusie for letting me take an interest in the construction of the weight filtration on the rigid cohomology. I would like to express my sincere thanks to A. Shiho for doing a joint work in [NS] and for giving a comment on an earlier version of this report, and to N. Tsuzuki for explaining the main result of his preprint [T].

Conventions. Let $\mathcal{A}$ be an abelian category.

1. For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer $n$, $(E^{\bullet+n}, d^{\bullet+n})$ or $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the following complex: $\cdots \longrightarrow E_{q+n} \xrightarrow{d_q} E_{q+n+1} \xrightarrow{d_{q+1}} \cdots$. Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

2. For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer $n$, $(E^\bullet[n], d^\bullet[n])$ denotes the following complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphisms $d^\bullet[n] = (-1)^n d^\bullet+n$.

3. For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$, the identity $E^q \longrightarrow E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $\mathcal{H}^q((E^\bullet, -d^\bullet)) \xrightarrow{\sim} \mathcal{H}^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies.

2 Constant simplicial open schemes.

In this section we state some results which Shiho and I have obtained in [NS].

Let $(S, I, \gamma)$ be a PD-scheme with quasi-coherent ideal $I \subset \mathcal{O}_S$. Put $S_0 := \text{Spec}_S(\mathcal{O}_S/I)$. Let $f : (X, D) \longrightarrow S_0$ be a smooth scheme with a relative SNCD (=simple normal crossing divisor) over $S_0$. By abuse of notation, we denote the composite morphism $(X, D) \longrightarrow S_0 \xrightarrow{\iota} S$ also by $f$. Then $(X, D)$ gives an fs log structure $M_D$ on $X$ [NS] (cf. [K, p. 222–223], [Fa, §2 c)]). Let $j : X \setminus D \xrightarrow{\iota} X$ be the natural open immersion. Note that $M_D \not\subset \mathcal{O}_X \cap j_!(\mathcal{O}_U^{\log})$ in general; indeed, the sections of $(\mathcal{O}_X \cap j_!(\mathcal{O}_U^{\log}))/\mathcal{O}_X^{\log}$ over an affine open subscheme of $X$ is not even finitely generated in general [NS].

Let $\Delta := \{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of $D$ by smooth components of $D$, where $\Lambda$ is a set: $D = \bigcup_{\lambda \in \Lambda} D_\lambda$ and each $D_\lambda$ is smooth over $S_0$. Let $k$ be a positive integer. Put $D^{(k)} := \bigcap_{\{\lambda_1, \ldots, \lambda_k \mid \lambda_i \neq \lambda_j \ (i \neq j)\}} D_{\lambda_1} \cap \cdots \cap D_{\lambda_k}$. Then $D^{(k)}$ has been shown to be independent of the choice of the decomposition of $D$ by smooth components of $D$ [NS]. Put $D^{(0)} := X$.

For simplicity of notation, we denote the log crystalline topos $((X, M_D)/S)^{\log \text{crys}}$ simply by $(X/S)^{\log \text{crys}}$. Let $(X/S)^{\text{crys}}$ be the usual crystalline topos. Let $\mathcal{O}_{X/S}$ (resp. $\mathcal{O}_{X/S}^{\log \text{crys}}$) be the structure sheaf in $(X/S)^{\log \text{crys}}$ (resp. $(X/S)^{\text{crys}}$).

**Definition 2.1.** ([NS]) Let $\iota : (X, D) \xrightarrow{\iota} (X, D)$ be an exact closed immersion into a smooth scheme with a relative SNCD over $S$. Then we call $\iota$ an **admissible immersion with respect to $\Delta$** if $D$ has a decomposition $\Delta = \{D_\lambda\}_{\lambda \in \Lambda}$ by smooth components of $D$ and if $\iota$ induces an isomorphism $D_\lambda \xrightarrow{\sim} X \times_{X'} D_\lambda$ of schemes for all $\lambda \in \Lambda$. 


Next let us recall and define orientation sheaves for \( D^{(k)}/S_0 \) and \( D^{(k)}/S \) \((k \in \mathbb{N})\) (cf. [D2, (3.1.4)]).

Let \( E \) be a non-empty finite set with cardinality \( k \). Put \( \varpi_E := \wedge^k \mathbb{Z}^E \). Let \( k \) be a positive integer. Let \( P \) be a point of \( D^{(k)} \). Let \( \{D_{\lambda}\}_{\lambda \in \Lambda} \) be a decomposition of \( D \) by smooth components of \( D \). Let \( D_{\lambda_0}, \ldots, D_{\lambda_k} \) be distinct smooth components of \( D \) such that \( D_{\lambda_0} \cap \cdots \cap D_{\lambda_k} \) contains \( P \). Then, the set \( \{D_{\lambda_0}, \ldots, D_{\lambda_k}\} \) gives an orientation sheaf \( \varpi_{\lambda_0\cdots\lambda_k,zar}(D/S_0) \) on a local neighborhood of \( P \) in \( D^{(k)} \).

The sheaf \( \varpi_{\lambda_0\cdots\lambda_k,zar}(D/S_0) \) is globalized, and we denote this globalized sheaf by the same symbol \( \varpi_{\lambda_0\cdots\lambda_k,zar}(D/S_0) \). Put \( \varpi^{(k)}_{zar}(D/S_0) := \oplus_{\{\lambda_0, \ldots, \lambda_k\}} \varpi_{\lambda_0\cdots\lambda_k,zar}(D/S_0) \).

Put \( \varpi^{(k)}_{zar}(D/S_0) := \mathbb{Z}_X \) for \( k = 0 \). The sheaf \( \varpi^{(k)}_{zar}(D/S_0) \) is extended to a sheaf \( \varpi^{(k)}_{crys}(D/S) \) in the usual crystalline topos \((D^{(k)}/S)^{crys}\).

**Definition 2.2.** We call \( \varpi^{(k)}_{zar}(D/S_0) \) (resp. \( \varpi^{(k)}_{crys}(D/S) \)) the zariskian orientation sheaf (resp. crystalline orientation sheaf) of \( D^{(k)}/S_0 \) (resp. \( D^{(k)}/(S, \mathcal{I}, \gamma) \)).

The sheaves \( \varpi^{(k)}_{zar}(D/S_0) \), \( \varpi^{(k)}_{crys}(D/S) \) are defined by the local nature of \( D \); they are independent of the choice of the decomposition of \( D \) by smooth components of \( D \).

If \( S_0 \) is of characteristic \( p > 0 \), we define the Frobenius action on the orientations sheaves above. Let \( F : (X, D) \longrightarrow (X', D') \) be the relative Frobenius morphism over \( S_0 \). The morphism \( F \) induces the relative Frobenius morphism \( F^{(k)} : D^{(k)} \longrightarrow D'^{(k)} = D^{(k)'} \). Let \( a^{(k)} : D^{(k)} \longrightarrow X \) and \( a^{(k)'} : D^{(k)'} \longrightarrow X' \) be the natural morphisms. Then we define the following two Frobenius morphisms

\[
(2.2.1) \quad \Phi^{(k)} : a^{(k)'}_{crys} \varpi^{(k)}_{crys}(D'/S) \longrightarrow F^{(k)}_{crys} \varpi^{(k)}_{crys}(D'/S)
\]

and

\[
(2.2.2) \quad \Phi^{(k)} : a^{(k)}_{zar} \varpi^{(k)}_{zar}(D'/S_0) \longrightarrow F^{(k)}_{zar} \varpi^{(k)}_{zar}(D'/S_0)
\]

by the multiplication by \( p^k \) under the natural identifications

\[
\varpi^{(k)}_{crys}(D'/S) \sim F^{(k)}_{crys} \varpi^{(k)}_{crys}(D'/S), \quad \varpi^{(k)}_{zar}(D'/S) \sim F^{(k)}_{zar} \varpi^{(k)}_{zar}(D'/S).
\]

In [NS] we have proved the following:

**Theorem-Definition 2.3.** ([NS]) Let \( (S, \mathcal{I}, \gamma) \) and \( (X, D)/S_0 \) be as in the beginning of this section. Then there exist two filtered objects

\[
(\mathcal{C}^{\text{crys}}(\mathcal{O}_{X/S}), P_k) = (\mathcal{C}^{\text{crys}}(\mathcal{O}_{X/S}), P_k \mathcal{C}^{\text{crys}}(\mathcal{O}_{X/S}))_{k \in \mathbb{Z}}
\]

and

\[
(\mathcal{C}^{\text{zar}}(\mathcal{O}_{X/S}), P_k) = (\mathcal{C}^{\text{zar}}(\mathcal{O}_{X/S}), P_k \mathcal{C}^{\text{zar}}(\mathcal{O}_{X/S}))_{k \in \mathbb{Z}}
\]
in the filtered derived categories $\text{D}^+\text{F}(\mathcal{O}_{X/S})$ of bounded below complexes of $\mathcal{O}_{X/S}$-modules and $\text{D}^+\text{F}(f^{-1}(\mathcal{O}_S))$ of bounded below complexes of $f^{-1}(\mathcal{O}_S)$-modules, respectively. We have called $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ the preweight-filtered crystalline complex of $(X, D)/(S, \mathcal{I}, \gamma)$ and the preweight-filtered zariskian complex of $(X, D)/(S, \mathcal{I}, \gamma)$, respectively. They enjoy the following properties:

(1;c): $\{P_k\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S})\}_{k \in \mathbb{Z}}$ is an increasing “filtration” on $\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S})$ which is finite locally on $(X/S)_{\text{crys}}$ such that $P_{-1}\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) = 0$, $P_0\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) \overset{\sim}{\rightarrow} \mathcal{O}_{X/S}$, and $\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) \overset{\sim}{\rightarrow} \mathcal{R}\mathcal{e}_{X/S}*\mathcal{O}_{X/S}$.

(2;c): If $(X, D)$ has an admissible immersion $(X, D) \xhookrightarrow{\kappa} (X, \mathcal{D})$ over $S$ with respect to $\Delta$, then

$$(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k) \simeq (\hat{L}(\Omega^*_X/\log\mathcal{D})), \hat{L}(P_k\Omega^*_X/\log\mathcal{D})),$$

where $\hat{L}$ is the classical linearization functor for $\mathcal{O}_X$-modules [Be, Chap. IV, 3].

(3): $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k) = R\mathcal{H}_{X/S}*(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$.

In particular,

(1;z): $\{P_k\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S})\}_{k \in \mathbb{Z}}$ is an increasing “filtration” on $\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S})$ which is finite locally on $X$ such that $P_{-1}\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) = 0$, $P_0\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) \overset{\sim}{\rightarrow} R\mathcal{H}_{X/S}*\mathcal{O}_{X/S}$, and $\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) \overset{\sim}{\rightarrow} R\mathcal{H}_{X/S}*\mathcal{O}_{X/S}$, and

(2;z): If $(X, D)$ has an admissible immersion $(X, D) \xhookrightarrow{\kappa} (X, \mathcal{D})$ over $S$ with respect to $\Delta$,

$$(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k) \simeq (\mathcal{O}_Y \otimes_G \Omega^*_X/\log\mathcal{D}), \mathcal{O}_Y \otimes_G \Omega^*_X P_k\Omega^*_X/\log\mathcal{D}),$$

where $\mathcal{Y}$ is the PD-envelope of the closed immersion $X \xhookrightarrow{\kappa} \mathcal{X}$.

(4;c): Let $\text{gr}_k^p \colon \text{D}^+\text{F}(\hat{\mathcal{O}}_{X/S}) \rightarrow \text{D}^+\text{F}(\hat{\mathcal{O}}_{X/S})$ ($k \in \mathbb{Z}$) be the gr-functor [NS]. Then $\text{gr}_k^p((\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_l)) = a_{\text{crys}}(\mathcal{O}_{D^{(k)}}) \otimes_{\Delta} a_{\text{crys}}(\mathcal{D}/S)\{k\}$.

(4;z): Let $\text{gr}_k^p \colon \text{D}^+\text{F}(\hat{\mathcal{O}}_{X/S}) \rightarrow \text{D}^+\text{F}(\hat{\mathcal{O}}_{X/S})$ ($k \in \mathbb{Z}$) be the gr-functor. Then $\text{gr}_k^p((\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_l)) = a_{\text{zar}}(\mathcal{O}_{D^{(k)}}) \otimes_{\Delta} a_{\text{zar}}(\mathcal{D}/S)\{k\}$.

(5): $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), \tau_k) \overset{\sim}{\rightarrow} (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$, where $\tau_k$ is the canonical filtration.

(6): If $S_0$ is the spectrum of a perfect field $\kappa$ of characteristic $p$ and if $S$ is the spectrum of the Witt ring $W_n$ of $\kappa$ of finite length $n > 0$, then $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ is canonically isomorphic to the filtered object $(W_n\Omega^*_X/\log\mathcal{D}), P_kW_n\Omega^*_X/\log\mathcal{D}$) in [M1] and [M2].

We give important remarks on (2.3).
(5) is equivalent to the following $p$-adic purity:

\[(2.3.1) \quad R^k\epsilon_{X/S*}\mathcal{O}_{X/S} = a_{\text{crys}}^{(k)}(\mathcal{O}_{D^{(k)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{crys}}^{(k)}(D/S)).\]

Logically speaking, in [NS], we have first proved (2.3.1) by using $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and then we have proved (5).

By (5), $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ are functorial; that is, for another smooth scheme $Y$ with a relative SNCD $E$ over $S_0$ and for a morphism $g: (X, D) \rightarrow (Y, E)$ of log schemes in the sense of Fontaine-Illusie-Kato, we have natural morphisms

\[g_{\text{crys}}^*: (\mathcal{C}_{\text{crys}}(\mathcal{O}_{Y/S}), P_k) \rightarrow Rg_{\text{crys}}^*(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)\]

and

\[g^*: (\mathcal{C}_{\text{zar}}(\mathcal{O}_{Y/S}), P_k) \rightarrow RG_{\text{zar}}(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k).\]

By (1; c) and (4; c), we have the following spectral sequence:

**Corollary-Definition 2.4.** ([NS]) There exists the following functorial spectral sequence

\[(2.4.1) \quad E_1^{-k,h+k} = R^{h-k}f_{D^{(h)}/S*}(\mathcal{O}_{D^{(h)}/S} \otimes_{\mathbb{Z}} \mathcal{O}_{\text{crys}}^{(h)}(D/S)) \Rightarrow R^h f_{X/S*}(\mathcal{O}_{X/S}).\]

We call the spectral sequence (2.4.1) the preweight spectral sequence of $(X, D)/(S, \mathcal{I}, \gamma)$.

**Remark 2.5.** Though we have obtained deep other theorems in [NS], we can only mention them because of lack of spaces: (pre)weight-filtered base change formula, (pre)weight-filtered Künneth formula, weight-filtered Poincaré duality, a weight-filtered Berthelot-Ogus isomorphism, theory for compact support cohomology, the convergence of weight filtrations. See [NS] for details.

If $S$ is a $p$-adic formal $V$-scheme in the sense of [O, §1], we obtain the following translation as a conclusion:

\[(2.5.1) \quad \begin{array}{|c|c|}
\hline
/\mathbb{C} & \text{crystal} \\
\hline
U_{\text{an}}, (X_{\text{an}})^{\log} & (X/S)^{\log} \\
X_{\text{an}} & (X/S)_{\text{crys}} \\
j_{\text{an}}: U_{\text{an}} \rightarrow X_{\text{an}}, \epsilon: (X_{\text{an}})^{\log} \rightarrow X_{\text{an}} & \epsilon_{X/S}: (X/S)^{\log} \rightarrow (X/S)_{\text{crys}} \\
Rf_{\text{an}}Z = Re_*\mathbb{Z} & RF_{X/S*}(\mathcal{O}_{X/S}) \\
X_{\text{an}} \rightarrow X & u_{X/S}: (X/S)^{\text{crys}} \rightarrow X_{\text{zar}} \\
(\Omega^*_{X/\mathbb{C}}(\log D), P_k) & (\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k) \\
(\Omega^*_{X_{\text{an}}/\mathbb{C}}(\log D_{\text{an}}), \tau_k) = (\Omega^*_{X_{\text{an}}/\mathbb{C}}(\log D_{\text{an}}), P_k) & (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), \tau_k) = (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k) \\
\hline
\end{array}\]

6
Remark 2.6. Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics $(0,p)$. Let $\kappa$ (resp. $K$) be the (not necessarily perfect) residue (resp. fraction) field of $V$. Let $\sigma \in \text{Aut} \mathcal{V}$ be a fixed lift of the $p$-th power endomorphism of $\kappa$. Let $(X,D)$ be a smooth scheme with a SNCD over $\kappa$. Fix a decomposition $\{D_\lambda\}_{\lambda \in \Lambda}$ of $D$ by smooth components of $D$ and fix a total order on $\Lambda$. Assume that there exists a closed immersion $X \hookrightarrow \mathcal{P}$ into a formal $V$-scheme such that $\mathcal{P}/\text{Spf} \mathcal{V}$ is formally smooth around $X$. In [CL, (3.8)], Chiarellotto and Le Stum have constructed the following $p$-adic weight spectral sequence by the method of local rigid cohomologies:

$$E_{1,\text{rig}}^{k,h+k} = H_{\text{rig}}^{h-k}(D^{(k)}/K)(-k) \implies H_{\text{rig}}^{h}(U/K).$$

More generally, in (4.3.4) below, we shall construct a $p$-adic weight spectral sequence for the rigid cohomology of a separated scheme of finite type over $\kappa$; our method is a complicated $p$-adic version of that in [D3], and it is different from theirs.

### 3 Simplicial open schemes.

Let $(S, I, \gamma)$ and $S_0$ be as in §2. Let $f : (X_\bullet, D_\bullet)_{\bullet \in \mathbb{N}} \to S_0$ be a simplicial smooth scheme with a simplicial relative SNCD over $S_0$. The morphism $f$ induces a morphism $f_\ast : (X_\bullet, D_\bullet)_{\bullet \in \mathbb{N}} \to S_{0\bullet}$, where $S_{0\bullet}$ is the constant simplicial scheme defined by $S_0$. By abuse of notation, we denote by $f$ (resp. $f_\ast$) the composite morphism $(X_\bullet, D_\bullet) \to S_0 \hookrightarrow S$ (resp. $(X_\bullet, D_\bullet) \to S_{0\bullet} \hookrightarrow S_\ast$). Furthermore, for simplicity of notation, we sometimes denote simply by $f$ the morphism $f_t : (X_t, D_t) \to S_0$ ($t \in \mathbb{N}$) and also by $f$ the composite morphism $(X_t, D_t) \xrightarrow{f_t} S_0 \hookrightarrow S$. We have a log crystalline topos $\left((X_\ast/S)_\text{crys}\right)^{\log} := ((X_\ast/M_{D_\bullet})/S)^{\log \text{crys}}$ and a usual crystalline topos $\left((X_\ast/S)_\text{crys}\right)$. Let $\mathcal{O}_{X_\ast/S}$ (resp. $\mathcal{O}_{X_\ast/S}$) be the structure sheaf in $(X_\ast/S)_\text{crys}$ (resp. $(X_\ast/S)_\text{crys}$). The morphisms $f$ and $f_\ast$ induce morphisms of topoi

$$f_{X_\ast/S} : (X_\ast/S)^{\log \text{crys}} \to \widetilde{S}_{\text{zar}} \quad \text{and} \quad f_{X_\ast/S} : (X_\ast/S)^{\log \text{crys}} \to \widetilde{S}_{\text{zar}},$$

$$f_{X_\ast/S} : (X_\ast/S)^{\text{crys}} \to \widetilde{S}_{\text{zar}} \quad \text{and} \quad f_{X_\ast/S} : (X_\ast/S)^{\text{crys}} \to \widetilde{S}_{\text{zar}}.$$

Then we have a cosimplicial filtered object $Rf_{X_\ast/S}^\circ (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\ast/S}), P_k) \in \mathbb{D}^+\mathbb{F}(f_{X_\ast/S}^{-1}(\mathcal{O}_S))$ by the functoriality of $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\ast/S}), P_k)$. Let $(\mathcal{K}, P_k)$ be a representative of the filtered object $Rf_{X_\ast/S}^\circ (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\ast/S}), P_k)$. The filtered complex $(\mathcal{K}, P_k)$ defines a filtered double complex $(\mathcal{K}_{\bullet \bullet}, P_k)$ (cf. [D3, (5.1.9) (IV)]). Here the first degree is the cosimplicial degree. We make the following convention on the signs of boundary morphisms of the single complex $s \mathcal{K}$ of $\mathcal{K}_{\bullet \bullet}$:

$$\mathcal{K}^{n} = \bigoplus_{i+j=n} \mathcal{K}^{ij}; \quad d(x^{ij}) = \sum_{s=0}^{i+1} (-1)^s \delta^i_s (x^{ij}) + (-1)^i d_{\mathcal{K}}(x^{ij}),$$

$$7$$
where \( \delta : (X_{i+1}, D_{i+1}) \rightarrow (X_i, D_i) \) is a standard face morphism (see, e.g., [D3, (5.1.1)]) and \( d_k : \mathcal{K}_i \rightarrow \mathcal{K}_{i+1} \) is the boundary morphism arising from the boundary morphism of the filtered complex \( \mathcal{C}_{\text{crys}}(\mathcal{O}_{X_i/S}, P_k) \). Our convention on the turn of the degrees is the same as that in [NA, (2.3)] and different from that in [D3, (5.1.9) (IV)] and in [CT, (3.9)]; our convention on the signs of boundary morphisms of \( \mathcal{K} \) is superior to that in [D3, (5.1.9.2)] and in [CT, (3.9)]; if we follow the convention in [D3, (5.1.9.2)], we have to consider distinct signs before \((-1)^s \delta_s^i \) with respect to the degrees arising from the complex \( \mathcal{C}_{\text{crys}}(\mathcal{O}_{X_i/S}, P_k) \); we should eliminate signs \((-1)^p \) in [D3, (5.1.9.2)] by following our convention because these morphisms with various signs are not induced by morphisms of algebro-geometric objects. If we follow the convention in [loc. cit.], it seems to me that it is impossible to give a description of the boundary morphisms in [D3, p. 35]; the diagram in [D3, p. 35] is not a part of a double complex; since it is commutative, it is mistaken. Furthermore, “Gysin” in the diagram in [loc. cit.] is not clear. See [Nak2] for a correction and the explicit formula of “Gysin”.

Let \( L \) be the stupid filtration of \( \mathcal{K} \) with respect to the first degree:

\[
(3.0.2) \quad L^i(s\mathcal{K}) = \bigoplus_{i' \geq i} \mathcal{K}^{i'}. 
\]

Let \( (S, \mathcal{I}, \gamma) \) and \( S_0 \) be as in \( \S 2 \). Now, let us construct the preweight spectral sequence of \( (X, D)/(S, \mathcal{I}, \gamma) \). Let \( \delta(P, L) \) be the diagonal filtration of \( P \) and \( L \) ([D3, (7.1.6.1)]):

\[
(3.0.3) \quad \delta(P, L)_k(s\mathcal{K}) = \bigoplus_{i,j \geq 0} P_{i+k} \mathcal{K}^{i,j} = \sum_{i \geq 0} (sP_{i+k} \mathcal{K}) \cap L^i(s\mathcal{K}).
\]

(Note that the formula [D3, (7.1.6.1)] should be replaced by \( \bigoplus_{p,q} W_{n+p}(K^{a,p}) \).) Then we have \( \text{gr}^{\delta(P, L)}_k(s\mathcal{K}) = \bigoplus_{i \geq 0} \text{gr}^{P}_{i+k} \mathcal{K}^{\bullet} [-i] \). Hence we have the following spectral sequence by the Convention (3) (cf. [D3, (8.1.15)]):

\[
(3.0.4) \quad E_1^{-k,h+k}((X, D))/S = \bigoplus_{i \geq 0} \mathcal{H}^{h-i}(\text{gr}^P_{i+k} \mathcal{K}^{\bullet}) \Rightarrow \mathcal{H}^{h}(s\mathcal{K}).
\]

By (1; c) and (4; c), the spectral sequence (3.0.4) is equal to the following spectral sequence

\[
(3.0.5) \quad E_1^{-k,h+k}((X, D))/S = \bigoplus_{i \geq 0} R^{h-2i-k} f_{D^{[i+k]}/S} (\mathcal{O}_{D^{[i+k]}/S} \otimes_{\mathcal{O}_{X_i}} (D_i/S)) \Rightarrow R^h f_{X_i/S} (\mathcal{O}_{X_i/S}).
\]

**Definition 3.1.** We call (3.0.5) the preweight spectral sequence of \( (X, D)/(S, \mathcal{I}, \gamma) \). If \( S \) is a \( p \)-adic formal \( V \)-scheme in the sense of [O, \S 1], we call (3.0.5) the \( p \)-adic weight spectral sequence of \( (X, D)/(S, \mathcal{I}, \gamma) \). We denote by \( \{ P_k \}_{k \in \mathbb{Z}} \) the induced filtration on \( R^h f_{X_i/S} (\mathcal{O}_{X_i/S}) \). We call \( \{ P_k \}_{k \in \mathbb{Z}} \) the weight filtration on \( R^h f_{X_i/S} (\mathcal{O}_{X_i/S}) \).
Remark 3.2. In [Nak2], we have given an explicit description of the boundary morphism between $E_1$-terms of (3.0.5).

Theorem 3.3. ([NS], [Nak2]) If $S$ is a $p$-adic formal $V$-scheme in the sense of [O, §1] and if $S_0 := \text{Spec}_S(\mathcal{O}_S/p)$, then (3.0.5) degenerates at $E_2$ modulo torsion.

Corollary 3.4. ([NS], [Nak2]) There exists the following spectral sequence of convergent $F$-isocrystals:

$$E_{-k,h+k}((X_\bullet, D_\bullet)/K) = \bigoplus_{i \geq 0} R^{h-2i-k} f_*(\mathcal{O}_{D_i^{(i+k)}/K} \otimes_{\mathbb{Z}} \mathcal{O}_{(i+k)}(D_i/K)) \implies R^h f_*(\mathcal{O}_{X_\bullet/K}).$$

The spectral sequence (3.4.1) degenerates at $E_2$. Here $R^r f_*(\mathcal{O}_{D_i^{(i+k)}/K} \otimes_{\mathbb{Z}} \mathcal{O}_{(i+k)}(D_i/K))$ ($r \in \mathbb{Z}$) is a convergent $F$-isocrystal on $S/V$ whose value at a $p$-adic enlargement $T$ of $S/V$ is $R^r f_*(\mathcal{O}_{D_i^{(i+k)}/T} \otimes_{\mathbb{Z}} \mathcal{O}_{(i+k)}(D_i/T))$.

4 Weight filtration on rigid cohomologies.

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field $\kappa$ of characteristic $p > 0$. Let $K$ (resp. $K_0$) be the fraction field of $\mathcal{V}$ (resp. $\mathcal{W}$). Let $U$ be a separated scheme of finite type over $\kappa$, and let $\iota: U \longrightarrow \overline{U}$ be an open immersion into a proper scheme over $\kappa$ [Nag]. Let $(X_\bullet, D_\bullet)$ be a simplicial proper smooth scheme with a simplicial SNCD over $\kappa$.

First we recall the following:

Definition 4.1. ([T], (cf. [D3, (5.3.8)])) The pair $(U_\bullet, X_\bullet)$ is called a proper hypercovering of $(U, \overline{U})$ if the following conditions are satisfied:

1. $(U_\bullet, X_\bullet)$ is augmented to $(U, \overline{U})$ over $\kappa$,
2. The natural morphism $U_{n+1} \longrightarrow \text{cosh}_n(U_\bullet \leq n)_{n+1}$ is proper and surjective for any $n \in \mathbb{N}$,
3. $U_n = U \times_{\overline{U}} X_n$ for any $n \in \mathbb{N}$.

The following is one of main results of this report:

Theorem 4.2. ([Nak2]) Let the notations be as above. Then the following hold:

1. If $(U_\bullet, X_\bullet)$ is a split proper hypercovering of $(U, \overline{U})$, then there exists a canonical isomorphism

$$(4.2.1) \quad R\Gamma_{\text{rig}}(U/K) \simto R\Gamma((X_\bullet, D_\bullet)/\mathcal{W}) \otimes_{\mathcal{W}} K.$$

2. Let $c$ be an integer such that $H^h_{\text{rig}}(U/K) = 0$ for all $h \geq c$. (We can show the existence of $c$.) Let $N$ be an integer such that there exists a positive integer $r$ satisfying the following two inequalities: $c \leq 2^{-1} r (r - 1)$ and $N \geq 2^{-1} r (r + 1)$. 

9
Assume that there exists a closed immersion \((X_N, D_N) \to (P, M)\) into a fine log smooth scheme over \(\text{Spf} W\) such that the underlying formal scheme \(P\) is also formally smooth over \(\text{Spf} W\).

Then there exists a canonical isomorphism

\[(4.2)\quad R\Gamma_{\text{rig}}(U/K) \simto \tau_N R\Gamma((X_\bullet, D_\bullet)/W) \otimes_W K.\]

**Remark 4.3.** (1) Over the complex number field, we have an equality

\[(4.3.1)\quad R\Gamma(U^\text{an}, \mathbb{Q}) = R\Gamma(U^\text{an}_\bullet, \mathbb{Q})\]

[D3]. In (4.2), the reader should note that we have proved the coincidence of cohomologies in two different cohomology theories; (4.2.1) is harder than (4.3.1).

(2) In the constant simplicial case, the isomorphism (4.2.1) immediately follows from Shiho’s comparison theorem \(h^h_{\text{rig}}(U/K) \simto h^h((X, D)/W) \otimes_W K\) [S, Cor. 2.4.13, Thm. 3.1.1].

(3) The proof of (4.2) essentially uses an argument of (a generalization of) the proof in [T] of the spectral sequence

\[(4.3.2)\quad E_1^{i,h-i} = h^{h-i}(U_1/K) \Rightarrow h^h(U/K)\]

and uses arguments of the proofs for [S, Thm. 2.4.4, Thm. 3.1.1].

(4) The right hand side of (4.2.1) depends only on \(U\) and \(K\); this solves a problem raised in [dJ, Intro.] for the split case. In fact, (4.4) below tells us that the weight filtration on \(h^h((X_\bullet, D_\bullet)/W) \otimes_W K\) depends only on \(U\) and \(K\).

(5) I think that the assumption on the embedding in (4.2) (2) is mild.

(6) By (4.2) and a standard argument in [Bl, III (3.2), (3.4)] (cf. [I2, II (3.2), (3.5)]), we obtain

\[(4.3.3)\quad h^h(U/K) \otimes_W K.\]

In particular, \(h^{h-i}(X_\bullet_\omega_\bullet, \log D_\bullet) \otimes_W K\) depends only on \(U\) and \(K\). Furthermore, we can endow \(h^{h-i}(X_\bullet_\omega_\bullet, \log D_\bullet) \otimes_W K\) with the weight filtration \(P\) [Nak2] (cf. [Nak1, §5]).

(7) We have generalized (4.2) for certain overconvergent \(F\)-isocrystals when \(\mathcal{V} = \mathcal{W}\) (cf. [S, Cor. 2.4.13, Thm. 3.1.1]).

By (3.0.5) and (4.2), we have the following spectral sequence:

\[(4.3.4)\quad E_1^{i,k} = \bigoplus_{i \geq 0} H^{h-2i-k}(D_i^{(i+k)})_{\text{crys}}^{(i+k)}(\mathcal{O}_{D_i^{(i+k)}}/\mathcal{W}) \otimes \mathcal{O}_{D_i^{(i+k)}}(\log D_i) \otimes W K \Rightarrow h^h(U/K).\]

**Theorem-Definition 4.4.** ([Nak2]) There exists a well-defined finite increasing filtration \(\{P_k\}_{k \in \mathbb{Z}}\) on \(h^h(U/K)\) which is calculated by the spectral sequence (4.3.4). We call this filtration the **weight filtration** on \(h^h(U/K)\).
Though we do not know a $p$-adic analogue of the theorem for a morphism in $(\text{MHS}/\mathbb{Q})$ in (1.0.1), we can prove the following by using the specialization argument of Deligne-Illusie [1, 3.10] (cf. [Nak1, §3]).

**Theorem 4.5.** ([Nak2]) Let $f: U \to V$ be a morphism of separated schemes of finite type over $\kappa$. Then the induced morphism $f^*: H^h_{\text{rig}}(V/K) \to H^h_{\text{rig}}(U/K)$ is strictly compatible with the weight filtration.

**Remark 4.6.**

(1) We have proved $p$-adic analogues of theorems in [D3], e.g., [D3, (8.2.4) (ii) (iii), (8.2.5) $\sim (8.2.11)$].

(2) Assume that $\kappa$ is algebraically closed. Assume, also, that $U$ and $X_0$ are connected (for simplicity). In [Nak2], as in [KH], by using dga’s, the bar construction and the Thom-Whitney functor, we have defined a unipotent rigid fundamental group scheme $\pi^\text{rig}_1(U/K, *)$, a simplicial unipotent crystalline fundamental group scheme $\pi^\text{log-crys}_1((X_\bullet, D_\bullet)/K, *_0)$ and a simplicial unipotent de Rham-Witt fundamental group scheme $\pi^\text{dRW}_1((X_\bullet, D_\bullet)/K, *_0)$. We have proved

$$
\pi^\text{rig}_1(U/K, *) = \pi^\text{log-crys}_1((X_\bullet, D_\bullet)/K, *_0) = \pi^\text{dRW}_1((X_\bullet, D_\bullet)/K, *_0)
$$

if $(X_\bullet, D_\bullet, *_0)$ is a pointed split proper hypercovering of $(U, \overline{U}, *)$ (cf. [H, §6]).

**References**


[Nak2] Nakkajima, Y.  *Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic p > 0.* Preprint.


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