Théorie de Hodge I, II, III pour cohomologies $p$-adiques.

Yukiyoshi Nakkajima *

Résumé: Dans ce rapport, nous présentons quelques résultats démontrés dans [NS] et [Nak2]. Nous avons construit deux objets filtrés fondamentaux: un complexe cristallin filtré par le prépoids et un complexe zariskian par le prépoids. Nous avons construit aussi la filtration par le poids sur la cohomologie rigide d’un schéma séparé de type fini sur un corps parfait de caractéristique $p > 0$.

Mots-clefs: cohomologies rigides, cohomologies log-cristallines, suites $p$-adiques spectrales des poids.

1 Introduction.

Motivated by a conjectural theory of motives due to A. Grothendieck, P. Deligne has given a translation of concepts on mixed Hodge structures into those on $l$-adic cohomologies and vice versa in [D1], and he has proved his conjectures on mixed Hodge structures in [D2] and [D3].

Let $U$ be a separated scheme of finite type over a field $\kappa$. Then there is the following translation [D1], [D2], [D3]:

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Let us consider a more special case as in [D2]. Let \((X, D)\) be a smooth scheme over \(\kappa\) with a SNCD\((=\)simple normal crossing divisor\) over \(\kappa\). Put \(U := X \setminus D\) and let \(j : U \subseteq X\) be the natural open immersion. Put \(M_D := \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\}\). Let \(\epsilon : \tilde{X}_{\text{et}} \to \tilde{X}_{\text{et}} \) be the forgetting log morphism induced by a natural morphism \((X, M_D) \to (X, \mathcal{O}_X^*)\) of log schemes in the sense of Fontaine-Illusie-Kato [K]. Let \(Y\) be an analytic variety over \(\mathbb{C}\), and let us also denote by \(\epsilon\) the real blow up \(Y^{\text{log}} \to Y\) [KN] which is denoted by \(\tau\) in [loc. cit.]. Then we have the following translation:

\[ (1.0.2) \]

<table>
<thead>
<tr>
<th>Objects over (\mathbb{C})</th>
<th>(l)-adic over (\mathbb{F}_q)</th>
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<tbody>
<tr>
<td>(U_{\text{an}}^\log, (X_{\text{an}})^{\log})</td>
<td>(j_{\text{an}} : U_{\text{an}}^{\log} \to X_{\text{an}}^\log, \epsilon : (X_{\text{an}})^{\log} \to X_{\text{an}})</td>
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<td>(X_{\text{an}})</td>
<td>(Rj_<em>^\log \mathbb{Z} = R\epsilon_</em> \mathbb{Z}) [KN]</td>
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<td>((\Omega^*_{X/\mathbb{C}}(\log D), P_k))</td>
<td>(X_{\text{an}} \to X)</td>
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<td>((\Omega^<em><em>{X</em>{\text{an}}/\mathbb{C}}(\log D_{\text{an}})^{\log}, \tau_k) = (\Omega^</em><em>{X</em>{\text{an}}/\mathbb{C}}(\log D_{\text{an}}), P_k))</td>
<td>(j_{\text{et}} : \tilde{U}<em>{\text{et}} \to \tilde{X}</em>{\text{et}}, \epsilon : \tilde{X}<em>{\text{et}} \to \tilde{X}</em>{\text{et}})</td>
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<td>(Rj_{\text{et}<em>}(\mathbb{Z}/l^n) = R\epsilon_</em> (\mathbb{Z}/l^n)) [Fu], [FK]</td>
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<td></td>
<td>(\text{id} : \tilde{X}<em>{\text{et}} \to \tilde{X}</em>{\text{et}})</td>
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The purpose of this report is to inform the reader that we have succeeded in making two translations (1.0.1) and (1.0.2) for \(p\)-adic cohomologies.

This report came out of two preprints [NS], [Nak2] and my lecture entitled with \textit{Théorie de Hodge III pour cohomologies \(p\)-adiques} at a symposium \textit{Hodge theory and algebraic geometry} at Hokkaido-University in October in 2002. We give no proof in this report; we have given proofs in [NS] and [Nak2].
Acknowledgment. I am very grateful to L. Illusie for letting me take an interest in the construction of the weight filtration on the rigid cohomology. I would like to express my sincere thanks to A. Shiho for doing a joint work in [NS] and for giving a comment on an earlier version of this report, and to N. Tsuzuki for explaining the main result of his preprint [T].

Conventions. Let $\mathcal{A}$ be an abelian category.

(1) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer $n$, $(E^{\bullet+n}, d^{\bullet+n})$ or $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the following complex: $\cdots \to E_q^{\bullet+n} \xrightarrow{d^q_{\bullet+n}} E_{q+1}^{\bullet+n} \xrightarrow{d^q_{\bullet+n+1}} \cdots$. Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

(2) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer $n$, $(E^\bullet[n], d^\bullet[n])$ denotes the following complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphisms $d^\bullet[n] = (-1)^n d^{\bullet+n}$.

(3) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$, the identity $\text{id} : E^q \to E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $H^q((E^\bullet, -d^\bullet)) \xrightarrow{\sim} H^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies.

2 Constant simplicial open schemes.

In this section we state some results which Shiho and I have obtained in [NS].

Let $(\mathcal{S}, \mathcal{I}, \gamma)$ be a PD-scheme with quasi-coherent ideal $\mathcal{I} \subset \mathcal{O}_S$. Put $S_0 := \text{Spec}_\mathcal{S}(\mathcal{O}_S/\mathcal{I})$. Let $f : (X, D) \to S_0$ be a smooth scheme with a relative SNCD (=simple normal crossing divisor) over $S_0$. By abuse of notation, we denote the composite morphism $(X, D) \to S_0 \xrightarrow{\iota} \mathcal{S}$ also by $f$. Then $(X, D)$ gives an fs log structure $M_D$ on $X$ [NS] (cf. [K, p. 222–223], [Fa, §2 (c)]). Let $j : X \setminus D \xrightarrow{\subset} X$ be the natural open immersion. Note that $M_D \not\subseteq \mathcal{O}_X \cap j_* (\mathcal{O}_U^\text{log})$ in general; indeed, the sections of $(\mathcal{O}_X \cap j_* (\mathcal{O}_U^\text{log}))/\mathcal{O}_X^\text{log}$ over an affine open subscheme of $X$ is not even finitely generated in general [NS].

Let $\Delta := \{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of $D$ by smooth components of $D$, where $\Lambda$ is a set: $D = \bigcup_{\lambda \in \Lambda} D_\lambda$ and each $D_\lambda$ is smooth over $S_0$. Let $k$ be a positive integer. Put $D^{(k)} := \bigsqcup_{\{\lambda_1, \ldots, \lambda_k, |\lambda_i \neq \lambda_j (i \neq j)\}} D_{\lambda_1} \cap \cdots \cap D_{\lambda_k}$. Then $D^{(k)}$ has been shown to be independent of the choice of the decomposition of $D$ by smooth components of $D$ [NS]. Put $D^{(0)} := X$.

For simplicity of notation, we denote the log crystalline topos $((X, M_D)/S)^\text{log}_{\text{crys}}$ simply by $(X/S)^\text{log}_{\text{crys}}$. Let $(X/S)^\text{crys}$ be the usual crystalline topos. Let $\mathcal{O}_{X/S}$ (resp. $\mathcal{O}_{X/S}^\text{log}$) be the structure sheaf in $(X/S)^\text{log}_{\text{crys}}$ (resp. $(X/S)^\text{crys}$).

Definition 2.1. ([NS]) Let $i : (X, D) \xrightarrow{\subset} (\mathcal{X}, \mathcal{D})$ be an exact closed immersion into a smooth scheme with a relative SNCD over $S$. Then we call $i$ an admissible immersion with respect to $\Delta$ if $\mathcal{D}$ has a decomposition $\Delta = \{\mathcal{D}_\lambda\}_{\lambda \in \Lambda}$ by smooth components of $\mathcal{D}$ and if $i$ induces an isomorphism $D_\lambda \xrightarrow{\sim} X \times_{\mathcal{X}} \mathcal{D}_\lambda$ of schemes for all $\lambda \in \Lambda$. 


Next let us recall and define orientation sheaves for $D^{(k)}/S_0$ and $D^{(k)}/S$ ($k \in \mathbb{N}$) (cf. [D2, (3.1.4)]).

Let $E$ be a non-empty finite set with cardinality $k$. Put $\varpi_E := \bigwedge^k \mathbb{Z}^E$. Let $k$ be a positive integer. Let $P$ be a point of $D^{(k)}$. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a decomposition of $D$ by smooth components of $D$. Let $D_{\lambda_0}, \ldots, D_{\lambda_k}$ be distinct smooth components of $D$ such that $D_{\lambda_0} \cap \cdots \cap D_{\lambda_k}$ contains $P$. Then, the set $\{D_{\lambda_0}, \ldots, D_{\lambda_k}\}$ gives an orientation sheaf $\varpi_{\lambda_0 \cdots \lambda_k, \text{zar}}(D/S_0)$ on a local neighborhood of $P$ in $D^{(k)}$.

The sheaf $\varpi_{\lambda_0 \cdots \lambda_k, \text{zar}}(D/S_0)$ is globalized, and we denote this globalized sheaf by the same symbol $\varpi_{\lambda_0 \cdots \lambda_k, \text{zar}}(D/S_0)$. Put $\varpi^{(k)}_{\text{zar}}(D/S_0) := \oplus_{\lambda_0, \ldots, \lambda_k} \varpi_{\lambda_0 \cdots \lambda_k, \text{zar}}(D/S_0)$. Put $\varpi^{(k)}_{\text{zar}}(D/S_0) := \mathbb{Z}_X$ for $k = 0$. The sheaf $\varpi^{(k)}_{\text{zar}}(D/S_0)$ is extended to a sheaf $\varpi^{(k)}_{\text{cris}}(D/S)$ in the usual crystalline topos $(D^{(k)}/S)_{\text{cris}}$.

**Definition 2.2.** We call $\varpi^{(k)}_{\text{zar}}(D/S_0)$ (resp. $\varpi^{(k)}_{\text{cris}}(D/S)$) the zariskian orientation sheaf (resp. crystalline orientation sheaf) of $D^{(k)}/S_0$ (resp. $D^{(k)}/(S, \mathcal{I}, \gamma)$).

The sheaves $\varpi^{(k)}_{\text{zar}}(D/S_0)$, $\varpi^{(k)}_{\text{cris}}(D/S)$ are defined by the local nature of $D$; they are independent of the choice of the decomposition of $D$ by smooth components of $D$.

If $S_0$ is of characteristic $p > 0$, we define the Frobenius action on the orientations sheaves above. Let $F: (X, D) \longrightarrow (X', D')$ be the relative Frobenius morphism over $S_0$. The morphism $F$ induces the relative Frobenius morphism $F^{(k)}: D^{(k)} \longrightarrow D'^{(k)} = D^{(k)'}$. Let $a^{(k)}: D^{(k)} \longrightarrow X$ and $a^{(k)'}: D^{(k)'} \longrightarrow X'$ be the natural morphisms. Then we define the following two Frobenius morphisms

(2.2.1) \[ \Phi^{(k)}: a^{(k)'}_{\text{cris}} \varpi^{(k)}_{\text{cris}}(D'/S) \longrightarrow F^{(k)}_{\text{cris}} \varpi^{(k)}_{\text{cris}}(D/S) \]

and

(2.2.2) \[ \Phi^{(k)}: a^{(k)'}_{\text{zar}} \varpi^{(k)}_{\text{zar}}(D'/S_0) \longrightarrow F^k a^{(k)}_{\text{zar}} \varpi^{(k)}_{\text{zar}}(D/S_0) \]

by the multiplication by $p^k$ under the natural identifications

\[ \varpi^{(k)}_{\text{cris}}(D'/S) \sim F^{(k)}_{\text{cris}} \varpi^{(k)}_{\text{cris}}(D/S), \quad \varpi^{(k)}_{\text{zar}}(D'/S) \sim F^{(k)} \varpi^{(k)}_{\text{zar}}(D/S). \]

In [NS] we have proved the following:

**Theorem-Definition 2.3.** ([NS]) Let $(S, \mathcal{I}, \gamma)$ and $(X, D)/S_0$ be as in the beginning of this section. Then there exist two filtered objects

\[ (\mathcal{C}_{\text{cris}}(\mathcal{O}_X/S), P_k) = (\mathcal{C}_{\text{cris}}(\mathcal{O}_X/S), P_k \mathcal{C}_{\text{cris}}(\mathcal{O}_X/S))_{k \in \mathbb{Z}} \]

and

\[ (\mathcal{C}_{\text{zar}}(\mathcal{O}_X/S), P_k) = (\mathcal{C}_{\text{zar}}(\mathcal{O}_X/S), P_k \mathcal{C}_{\text{zar}}(\mathcal{O}_X/S))_{k \in \mathbb{Z}} \]
in the filtered derived categories $D^+(\mathcal{O}_{X/S})$ of bounded below complexes of $\mathcal{O}_{X/S}$-modules and $D^+(f^{-1}(\mathcal{O}_S))$ of bounded below complexes of $f^{-1}(\mathcal{O}_S)$-modules, respectively. We have called $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ the preweight-filtered crystalline complex of $(X, D)/(S, \mathcal{I}, \gamma)$ and the preweight-filtered zariskian complex of $(X, D)/S$, respectively. They enjoy the following properties:

(1; c): $\{P_k\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S})\}_{k \in \mathbb{Z}}$ is an increasing “filtration” on $\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S})$ which is finite locally on $(X/S)_{\text{crys}}$ such that $P_{-1}\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) = 0$, $P_0\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) \sim \mathcal{O}_{X/S}$, and $\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}) \sim R\epsilon_{X/S*}(\mathcal{O}_{X/S})$.

(2; c): If $(X, D)$ has an admissible immersion $(X, D) \subset (\mathcal{X}, \mathcal{D})$ over $S$ with respect to $\Delta$, then

$$(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k) \simeq (\mathcal{L}(\mathcal{O}^{\bullet}_{X/S}(\log \mathcal{D})), \mathcal{L}(P_k\mathcal{O}^{\bullet}_{X/S}(\log \mathcal{D}))),$$

where $\mathcal{L}$ is the classical linearization functor for $\mathcal{O}_{\mathcal{X}}$-modules [Be, Chap. IV, 3].

(3): $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k) = R\mathcal{H}^0_{X/S*}(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$.

In particular,

(1; z): $\{P_k\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S})\}_{k \in \mathbb{Z}}$ is an increasing “filtration” on $\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S})$ which is finite locally on $X$ such that $P_{-1}\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) = 0$, $P_0\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) \sim R\mathcal{H}^0_{X/S*}\mathcal{O}_{X/S}$, and $\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}) \sim R\epsilon_{X/S*}(\mathcal{O}_{X/S})$, and

(2; z): If $(X, D)$ has an admissible immersion $(X, D) \subset (\mathcal{X}, \mathcal{D})$ over $S$ with respect to $\Delta$,

$$(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k) \simeq (\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}^{\bullet}_{X/S}(\log \mathcal{D}), \mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} P_k\mathcal{O}^{\bullet}_{X/S}(\log \mathcal{D})), $$

where $\mathcal{Y}$ is the PD-envelope of the closed immersion $X \subset \mathcal{X}$.

(4; c): Let $\text{gr}_k^P: D^+(\mathcal{O}_{X/S}) \to D^+(\mathcal{O}_{X/S})$ $(k \in \mathbb{Z})$ be the gr-functor [NS]. Then $\text{gr}_k^P((\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_l)) = d_{\text{crys}}(k)^{(-k)}(\mathcal{O}_{D^{(l)}_{\mathcal{I}*}/S} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}^{(k)}(\mathcal{D}/S))\{(-k)\}$.

(4; z): Let $\text{gr}_k^P: D^+(f^{-1}(\mathcal{O}_S)) \to D^+(f^{-1}(\mathcal{O}_S))$ $(k \in \mathbb{Z})$ be the gr-functor. Then $\text{gr}_k^P((\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_l)) = d^{(k)}(R\mathcal{U}_{D^{(l)*}/S*}(\mathcal{O}_{D^{(l)*}/S}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}^{(k)}(D/S_0))\{(-k)\}$.

(5): $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), \tau_k) \sim (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$, where $\tau_k$ is the canonical filtration.

(6): If $S_0$ is the spectrum of a perfect field $k$ of characteristic $p$ and if $S$ is the spectrum of the Witt ring $W_n$ of $k$ of finite length $n > 0$, then $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ is canonically isomorphic to the filtered object $(W_n\mathcal{O}^{\bullet}_{X}(\log \mathcal{D}), P_kW_n\mathcal{O}^{\bullet}_{X}(\log \mathcal{D}))$ in [M1] and [M2].

We give important remarks on (2,3).
(5) is equivalent to the following $p$-adic purity:

\[ R^k \epsilon_{X/S*} \mathcal{O}_{X/S} = a^{(k)}_{\text{crys}} (\mathcal{O}_{D^{(k)}/S} \otimes \mathcal{O}_{\text{crys}}(D/S)). \]

Logically speaking, in [NS], we have first proved (2.3.1) by using $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and then we have proved (5).

By (5), $(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k)$ and $(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k)$ are functorial; that is, for another smooth scheme $Y$ with a relative SNCD $E$ over $S_0$ and for a morphism $g: (X, D) \longrightarrow (Y, E)$ of log schemes in the sense of Fontaine-Illusie-Kato, we have natural morphisms

\[ g^*: (\mathcal{C}_{\text{crys}}(\mathcal{O}_{Y/S}), P_k) \longrightarrow Rg_{\text{crys}*}(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X/S}), P_k) \]

and

\[ g^*: (\mathcal{C}_{\text{zar}}(\mathcal{O}_{Y/S}), P_k) \longrightarrow Rg_{*}(\mathcal{C}_{\text{zar}}(\mathcal{O}_{X/S}), P_k). \]

By (1; c) and (4; c), we have the following spectral sequence:

**Corollary-Definition 2.4.** ([NS]) There exists the following functorial spectral sequence

\[ E^1_{k,h+k} = R^{h-k} f_{D^{(k)}/S*}(\mathcal{O}_{D^{(k)}/S} \otimes \mathcal{O}_{\text{crys}}(D/S)) \Longrightarrow R^h f_{X/S*}(\mathcal{O}_{X/S}). \]

We call the spectral sequence (2.4.1) the preweight spectral sequence of $(X, D)/(S, \mathcal{I}, \gamma)$.

**Remark 2.5.** Though we have obtained deep other theorems in [NS], we can only mention them because of lack of spaces: (pre)weight-filtered base change formula, (pre)weight-filtered Künneth formula, weight-filtered Poincaré duality, a weight-filtered Berthelot-Ogus isomorphism, theory for compact support cohomology, the convergence of weight filtrations. See [NS] for details.

If $S$ is a $p$-adic formal $V$-scheme in the sense of [O, §1], we obtain the following translation as a conclusion:

\[ (\Omega_{\mathbf{X}/\mathbf{C}}(\log D), P_k) = (\Omega^*_{\mathbf{X}_{\text{an}}/\mathbf{C}}(\log D_{\text{an}}), P_k) \]

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<th>$/\mathbb{C}$</th>
<th>crystal</th>
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<tbody>
<tr>
<td>$U_{\text{an}}$, $(X_{\text{an}})^{\log}$</td>
<td>$(X/S)^{\log}_{\text{crys}}$</td>
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<td>$X_{\text{an}}$</td>
<td>$(X/S)_{\text{crys}}$</td>
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<tr>
<td>$j_{\text{an}}: U_{\text{an}} \hookrightarrow X_{\text{an}}$, $\epsilon: (X_{\text{an}})^{\log} \longrightarrow X_{\text{an}}$</td>
<td>$\epsilon_{X/S}: (X/S)^{\log}<em>{\text{crys}} \longrightarrow (\mathcal{X}</em>{\text{crys}})$</td>
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<tr>
<td>$Rj_{\text{an}}^* \mathcal{O} = R\epsilon_* \mathcal{O}$</td>
<td>$R\epsilon_{X/S*}(\mathcal{O}_{X/S})$</td>
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<tr>
<td>$\mathcal{X}<em>{\text{an}}^{\log} \mathcal{C}</em>{\text{zar}}(\mathcal{O}_{X/S}, P_k)$</td>
<td>$\mathcal{O}_{X/S}^{\log}$</td>
</tr>
<tr>
<td>$(\Omega^*<em>{\mathbf{X}</em>{\text{an}}/\mathbf{C}}(\log D_{\text{an}}), \tau_k) = (\mathcal{O}^{\log}<em>{\mathbf{X}</em>{\text{an}}/\mathbf{C}}(\log D_{\text{an}}), P_k)$</td>
<td>$(\mathcal{C}<em>{\text{crys}}(\mathcal{O}</em>{X/S}, P_k)$</td>
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</table>
Remark 2.6. Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics $(0, p)$. Let $\kappa$ (resp. $K$) be the (not necessarily perfect) residue (resp. fraction) field of $\mathcal{V}$. Let $\sigma \in \text{Aut} \mathcal{V}$ be a fixed lift of the $p$-th power endomorphism of $\kappa$. Let $(X, D)$ be a smooth scheme with a SNCD over $\kappa$. Fix a decomposition $\{D_\lambda\}_{\lambda \in A}$ of $D$ by smooth components of $D$ and fix a total order on $A$. Assume that there exists a closed immersion $X \to \mathcal{P}$ into a formal $V$-scheme such that $\mathcal{P}/\text{Spf} V$ is formally smooth around $X$. In [CL, (3.8)], Chiarellotto and Le Stum have constructed the following $p$-adic weight spectral sequence by the method of local rigid cohomologies:

$$E_{1, \text{rig}}^{−k, h+k} = H_{\text{rig}}^{h−k}(D^{(k)}/K)(−k) \implies H_{\text{rig}}^{h}(U/K).$$

More generally, in (4.3.4) below, we shall construct a $p$-adic weight spectral sequence for the rigid cohomology of a separated scheme of finite type over $\kappa$; our method is a complicated $p$-adic version of that in [D3], and it is different from theirs.

3 Simplicial open schemes.

Let $(S, I, \gamma)$ and $S_0$ be as in §2. Let $f : (X_\bullet, D_\bullet)_{\bullet \in \mathbb{N}} \to S_0$ be a simplicial smooth scheme with a simplicial relative SNCD over $S_0$. The morphism $f$ induces a morphism $f_* : (X_\bullet, D_\bullet)_{\bullet \in \mathbb{N}} \to S_0$, where $S_0$ is the constant simplicial scheme defined by $S_0$. By abuse of notation, we denote by $f$ (resp. $f_*$) the composite morphism $(X_\bullet, D_\bullet) \to S_0 \to S$ (resp. $(X_\bullet, D_\bullet) \to S_0 \to S$). Furthermore, for simplicity of notation, we sometimes denote simply by $f$ the morphism $f_t : (X_t, D_t) \to S_0$ ($t \in \mathbb{N}$) and also by $f$ the composite morphism $(X_t, D_t) \to S_0 \to S$. We have a log crystalline topos $(X_\bullet/S)_{\text{crys}}^{\log} := ((X_\bullet, M_{D_\bullet})/S)_{\text{crys}}^{\log}$ and a usual crystalline topos $(X_\bullet/S)_{\text{crys}}$. Let $\mathcal{O}_{X_\bullet/S}$ (resp. $\mathcal{O}_{X_\bullet/S}$) be the structure sheaf in $(X_\bullet/S)_{\text{crys}}^{\log}$ (resp. $(X_\bullet/S)_{\text{crys}}$). The morphisms $f$ and $f_*$ induce morphisms of topos

$$f_{X_\bullet/S} : (X_\bullet/S)_{\text{crys}} \to \tilde{S}_{\text{zar}} \quad \text{and} \quad f_{X_\bullet/S} : (X_\bullet/S)_{\text{crys}} \to \tilde{S}_{\text{zar}},$$

$$\hat{f}_{X_\bullet/S} : (X_\bullet/S)_{\text{crys}} \to \tilde{S}_{\text{zar}} \quad \text{and} \quad \hat{f}_{X_\bullet/S} : (X_\bullet/S)_{\text{crys}} \to \tilde{S}_{\text{zar}}.$$

Then we have a cosimplicial filtered object $R\hat{f}_{X_\bullet/S_*} (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\bullet/S}, P_k)) \in D^+(\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\bullet/S}, P_k))$ by the functoriality of $\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\bullet/S}, P_k)$. Let $(\mathcal{K}, P_k)$ be a representative of the filtered object $R\hat{f}_{X_\bullet/S_*} (\mathcal{C}_{\text{crys}}(\mathcal{O}_{X_\bullet/S}, P_k))$. The filtered complex $(\mathcal{K}, P_k)$ defines a filtered double complex $(\mathcal{K}^{\bullet\bullet}, P_k)$ (cf. [D3, (5.1.9) (IV)]). Here the first degree is the cosimplicial degree. We make the following convention on the signs of boundary morphisms of the single complex $s\mathcal{K}$ of $\mathcal{K}^{\bullet\bullet}$:

$$(s\mathcal{K})^n = \bigoplus_{i+j=n} \mathcal{K}^{ij}; \quad d(x^{ij}) = \sum_{s=0}^{i+1} (-1)^s \delta^s_s(x^{ij}) + (-1)^i d_K(x^{ij}),$$

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where $\delta_s : (X_{i+1}, D_{i+1}) \rightarrow (X_i, D_i) \quad (0 \leq s \leq i+1)$ is a standard face morphism (see, e.g., [D3, (5.1.1)]) and $d_k : \mathcal{K}^i \rightarrow \mathcal{K}^{i+1}$ is the boundary morphism arising from the boundary morphism of the filtered complex $(\mathcal{C}_{\mathrm{crys}}(\mathcal{O}_{X_{i}/S}), P_k)$. Our convention on the turn of the degrees is the same as that in [NA, (2.3)] and different from that in [D3, (5.1.9) (IV)] and in [CT, (3.9)]; our convention on the signs of boundary morphisms of $\mathcal{K}$ is superior to that in [D3, (5.1.9.2)] and in [CT, (3.9)]; if we follow the convention in [D3, (5.1.9.2)], we have to consider distinct signs before $(-1)^{s}$ with respect to the degrees arising from the complex $(\mathcal{C}_{\mathrm{crys}}(\mathcal{O}_{X_{i}/S}), P_k)$; we should eliminate signs $(-1)^p$ in [D3, (5.1.9.2)] by following our convention because these morphisms with various signs are not induced by morphisms of algebro-geometric objects. If we follow the convention in [loc. cit.], it seems to me that it is impossible to give a description of the boundary morphisms in [D3, p. 35]; the diagram in [D3, p. 35] is not a part of a double complex; since it is commutative, it is mistaken. Furthermore, “Gysin” in the diagram in [loc. cit.] is not clear. See [Nak2] for a correction and the explicit formula of “Gysin”.

Let $L$ be the stupid filtration of $\mathcal{K}$ with respect to the first degree:

$$L^i(\mathcal{K}) = \bigoplus_{i' \geq i} \mathcal{K}^{i'}. \tag{3.0.2}$$

Let $(S, \mathcal{I}, \gamma)$ and $S_0$ be as in §2. Now, let us construct the preweight spectral sequence of $(X, D)/(S, \mathcal{I}, \gamma)$. Let $\delta(P, L)$ be the diagonal filtration of $P$ and $L$ ([D3, (7.1.6.1)]):

$$\delta(P, L)_k(\mathcal{K}) = \bigoplus_{i,j \geq 0} P_{i+k} \mathcal{K}^{i,j} = \sum_{i \geq 0} (sP_{i+k}\mathcal{K}) \cap L^i(\mathcal{K}). \tag{3.0.3}$$

(Note that the formula [D3, (7.1.6.1)] should be replaced by $\bigoplus_{p,q} W_{n+p}(K^{a,p})$.) Then we have $\mathrm{gr}^\delta(P, L)_k(\mathcal{K}) = \bigoplus_{i \geq 0} \mathrm{gr}^P_{i+k} \mathcal{K}^{i}[-i]$. Hence we have the following spectral sequence by the Convention (3) (cf. [D3, (8.1.15)]):

$$E^{i,h+k}_{1}(X_{i}, D_{i})/S) = \bigoplus_{i \geq 0} H^{h-i}(\mathrm{gr}^P_{i+k} \mathcal{K}^{i}) \Rightarrow H^h(\mathcal{K}). \tag{3.0.4}$$

By (1; c) and (4; c), the spectral sequence (3.0.4) is equal to the following spectral sequence

$$E^i_{1, h+k}(X_{i}, D_{i})/S) = \bigoplus_{i \geq 0} R^{h-2i-k} f_{D_{i+k}/S_{i}}(\mathcal{O}_{D_{i+k}/S} \otimes_{\mathcal{O}_{X_{i}/S}}(D_{i}/S)) \Rightarrow R^h f_{X_{i}/S_{i}}(\mathcal{O}_{X_{i}/S}). \tag{3.0.5}$$

**Definition 3.1.** We call (3.0.5) the preweight spectral sequence of $(X, D)/(S, \mathcal{I}, \gamma)$. If $S$ is a $p$-adic formal $V$-scheme in the sense of [O, §1], we call (3.0.5) the $p$-adic weight spectral sequence of $(X, D)/(S, \mathcal{I}, \gamma)$. We denote by $\{P_k\}_{k \in \mathbb{Z}}$ the induced filtration on $R^h f_{X_{i}/S_{i}}(\mathcal{O}_{X_{i}/S})$. We call $\{P_k\}_{k \in \mathbb{Z}}$ the weight filtration on $R^h f_{X_{i}/S_{i}}(\mathcal{O}_{X_{i}/S})$. 

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Remark 3.2. In [Nak2], we have given an explicit description of the boundary morphism between $E_1$-terms of (3.0.5).

Theorem 3.3. ([NS], [Nak2]) If $S$ is a $p$-adic formal $V$-scheme in the sense of [O, §1] and if $S_0 := \text{Spec}_S(O_S/p)$, then (3.0.5) degenerates at $E_2$ modulo torsion.

Corollary 3.4. ([NS], [Nak2]) There exists the following spectral sequence of convergent $F$-isocrystals:

\[
\begin{align*}
E_{1}^{-k,h+k}(\mathcal{X},\mathcal{D})/K &= \bigoplus_{i \geq 0} R^{h-2i-k} f_*(\mathcal{O}_{\mathcal{D}^{i+k}}/K) \otimes \varpi^{(i+k)}(D_i/K) \implies R^h f_*(\mathcal{O}_{\mathcal{X}/K}).
\end{align*}
\]

The spectral sequence (3.4.1) degenerates at $E_2$. Here $R^r f_*(\mathcal{O}_{\mathcal{D}^{i+k}}/K) \otimes \varpi^{(i+k)}(D_i/K)$ $(r \in \mathbb{Z})$ is a convergent $F$-isocrystal on $S/V$ whose value at a $p$-adic enlargement $T$ of $S/V$ is $R^r f_*(\mathcal{O}_{\mathcal{D}^{i+k}})/T \otimes \varpi^{(i+k)}(D_i/T)$.

4 Weight filtration on rigid cohomologies.

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics with perfect residue field $\kappa$ of characteristic $p > 0$. Let $K$ (resp. $K_0$) be the fraction field of $\mathcal{V}$ (resp. $\mathcal{W}$). Let $U$ be a separated scheme of finite type over $\kappa$, and let $\iota: U \hookrightarrow \overline{U}$ be an open immersion into a proper scheme over $\kappa$ [Nag]. Let $(\mathcal{X}, \mathcal{D})$ be a simplicial proper smooth scheme with a simplicial $\text{SNCD}$ over $\kappa$.

Put $U_n := X_n \setminus D_n$.

First we recall the following:

Definition 4.1. ([T], (cf. [D3, (5.3.8)])) The pair $(U_n, X_n)$ is called a proper hypercovering of $(U, \overline{U})$ if the following conditions are satisfied:

1. $(U_n, X_n)$ is augmented to $(U, \overline{U})$ over $\kappa$,
2. The natural morphism $U_{n+1} \longrightarrow \cosk^n U_{n+1}(U \leq n)$ is proper and surjective for any $n \in \mathbb{N}$,
3. $U_n = U \times U X_n$ for any $n \in \mathbb{N}$.

The following is one of main results of this report:

Theorem 4.2. ([Nak2]) Let the notations be as above. Then the following hold:

1. If $(U_n, X_n)$ is a split proper hypercovering of $(U, \overline{U})$, then there exists a canonical isomorphism

\[
R\Gamma_{\text{rig}}(U/K) \cong R\Gamma((X, D)/\mathcal{W}) \otimes_\mathcal{W} K.
\]

2. Let $c$ be an integer such that $H^h_{\text{rig}}(U/K) = 0$ for all $h \geq c$. (We can show the existence of $c$.) Let $N$ be an integer such that there exists a positive integer $r$ satisfying the following two inequalities: $c \leq 2^{-1}r(r-1)$ and $N \geq 2^{-1}r(r+1)$. 

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Assume that there exists a closed immersion \((X_N, D_N) \hookrightarrow (P, \mathcal{M})\) into a fine log smooth scheme over \(\text{Spf} \mathcal{W}\) such that the underlying formal scheme \(P\) is also formally smooth over \(\text{Spf} \mathcal{W}\).

Then there exists a canonical isomorphism
\[
(4.2.2) \quad R\Gamma_{\text{rig}}(U/K) \sim\rightarrow \tau_N R\Gamma((X_\bullet, D_\bullet)/\mathcal{W}) \otimes_{\mathcal{W}} K.
\]

**Remark 4.3.** (1) Over the complex number field, we have an equality
\[
(4.3.1) \quad R\Gamma(U^{\text{an}}, \mathbb{Q}) = R\Gamma(U^{\text{an}}_\bullet, \mathbb{Q})
\]
[D3]. In (4.2), the reader should note that we have proved the coincidence of cohomologies in two different cohomology theories; (4.2.1) is harder than (4.3.1).

(2) In the constant simplicial case, the isomorphism (4.2.1) immediately follows from Shiho’s comparison theorem \(H^h_{\text{rig}}(U/K) \sim\rightarrow H^h((X, D)/\mathcal{W}) \otimes_{\mathcal{W}} K\) [S, Cor. 2.4.13, Thm. 3.1.1].

(3) The proof of (4.2) essentially uses an argument of (a generalization of) the proof in [T] of the spectral sequence
\[
(4.3.2) \quad E^{i,h-i}_1 = H^{h-i}(U_i/K) \Rightarrow H^h_{\text{rig}}(U/K)
\]
and uses arguments of the proofs for [S, Thm. 2.4.4, Thm. 3.1.1].

(4) The right hand side of (4.2.1) depends only on \(U\) and \(K\); this solves a problem raised in [dJ, Intro.] for the split case. In fact, (4.4) below tells us that the weight filtration on \(H^k((X_\bullet, D_\bullet)/\mathcal{W}) \otimes_{\mathcal{W}} K\) depends only on \(U\) and \(K\).

(5) I think that the assumption on the embedding in (4.2) (2) is mild.

(6) By (4.2) and a standard argument in [Bl, III (3.2), (3.4)] (cf. [I2, II (3.2), (3.5)]), we obtain
\[
(4.3.3) \quad H^h_{\text{rig}}(U/K)_{[i,i+1]} = H^{h-i}(X_\bullet, \mathcal{W}\Omega^i_X/(\log D_\bullet)) \otimes_{\mathcal{W}} K.
\]

In particular, \(H^{h-i}(X_\bullet, \mathcal{W}\Omega^i_X/(\log D_\bullet)) \otimes_{\mathcal{W}} K\) depends only on \(U\) and \(K\). Furthermore, we can endow \(H^{h-i}(X_\bullet, \mathcal{W}\Omega^i_X/(\log D_\bullet)) \otimes_{\mathcal{W}} K\) with the weight filtration \(P\) [Nak2] (cf. [Nak1, §5]).

(7) We have generalized (4.2) for certain overconvergent \(F\)-isocrystals when \(\mathcal{V} = \mathcal{W}\) (cf. [S, Cor. 2.4.13, Thm. 3.1.1]).

By (3.0.5) and (4.2), we have the following spectral sequence:
\[
(4.3.4) \quad E^{i,k}_{1} = \bigoplus_{i \geq 0} H^{h-2i-k}(D_i^{(i+k)}) \otimes_{\mathcal{W}} K \Rightarrow H^h_{\text{rig}}(U/K).
\]

**Theorem-Definition 4.4.** ([Nak2]) There exists a well-defined finite increasing filtration \(\{P_k\}_{k \in \mathbb{Z}}\) on \(H^h_{\text{rig}}(U/K)\) which is calculated by the spectral sequence (4.3.4). We call this filtration the weight filtration on \(H^h_{\text{rig}}(U/K)\).
Though we do not know a $p$-adic analogue of the theorem for a morphism in $(\text{MHS}/\mathbb{Q})$ in (1.0.1), we can prove the following by using the specialization argument of Deligne-Illusie [I1, 3.10] (cf. [Nak1, §3]).

**Theorem 4.5.** ([Nak2]) Let $f: U \longrightarrow V$ be a morphism of separated schemes of finite type over $\kappa$. Then the induced morphism $f^*: H^b_{\rig}(V/K) \longrightarrow H^b_{\rig}(U/K)$ is strictly compatible with the weight filtration.

**Remark 4.6.** (1) We have proved $p$-adic analogues of theorems in [D3], e.g., [D3, (8.2.4) (ii) (iii), (8.2.5) $\sim$ (8.2.11)].

(2) Assume that $\kappa$ is algebraically closed. Assume, also, that $U$ and $X_0$ are connected (for simplicity). In [Nak2], as in [KH], by using dga’s, the bar construction and the Thom-Whitney functor, we have defined a unipotent rigid fundamental group scheme $\pi^\rig_1(U/K, *)$, a simplicial unipotent crystalline fundamental group scheme $\pi^\log\text{-}\text{crys}_1((X_\bullet, D_\bullet)/K, *_0)$ and a simplicial unipotent de Rham-Witt fundamental group scheme $\pi^\text{dRW}_1((X_\bullet, D_\bullet)/K, *_0)$. We have proved

$$
\pi^\rig_1(U/K, *) = \pi^\log\text{-}\text{crys}_1((X_\bullet, D_\bullet)/K, *_0) = \pi^\text{dRW}_1((X_\bullet, D_\bullet)/K, *_0)
$$

if $(X_\bullet, D_\bullet, *_0)$ is a pointed split proper hypercovering of $(U, \overline{U}, *)$ (cf. [H, §6]).

**References**


[Nak2] Nakkajima, Y. *Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic p > 0.* Preprint.


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