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<tr>
<th>Author(s)</th>
<th>Matsushita, D.</th>
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</thead>
<tbody>
<tr>
<td>Title</td>
<td>Hodge Theory and Algebraic Geometry</td>
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</tr>
<tr>
<td>Doc URL</td>
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</tr>
</tbody>
</table>

**Instructions for use**

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Note: Hodge Theory and Algebraic Geometry 2002/10/7-11 Department of Mathematics, Hokkaido University 石井志保子 (東工大) 春井岳 (名大多元) 田上健 (京大理) 三浦貴之 (京大理) 大野浩二 (大阪大) 鈴木香織 (東大数理)
We characterise the smooth \( n \)-dimensional hyperquadrics as Fano manifolds of length \( n \).

Given a Fano manifold \( X \) [resp. a pair \((X, x_0)\) of a Fano manifold \( X \) and a closed point \( x_0 \) on it], we define the (global) length \( l(X) \) of \( X \) [resp. the local length \( l(X, x_0) \) of \((X, x_0)\)] to be the positive integer

\[
\min_{C \subset X} \{(C, -K_X)\},
\]

where \( C \) runs through the set of the rational curves contained in \( X \) [resp. the set of the rational curves such that \( x_0 \in C \subset X \)].

The local length \( l(X, x_0) \) is a lower semicontinuous function in \( x_0 \) and the global length \( l(X) \) is by definition equal to \( \inf_{x_0 \in X} l(X, x_0) \). For a given closed point \( x_0 \in X \), it is known that \( l(X, x_0) \leq \dim X + 1 \), the equality holding if and only if \( X \) is projective space \([2]\).

In terms of the notions above, our main result is the following

Theorem 1. Let \( X \) be a smooth Fano variety of dimension \( n \geq 3 \) defined over an algebraically closed field \( k \) of characteristic zero. Then the following three conditions are equivalent:

1. \( X \) is isomorphic to a smooth hyperquadric \( Q_n \subset \mathbb{P}^{n+1} \).
2. The global length \( l(X) \) is \( n \).
3. \( \rho(X) = 1 \) and \( l(X, x_0) = n \) for a sufficiently general point \( x_0 \in X \), where \( \rho(X) \) stands for the Picard number.

This simple numerical result involves the preceding characterisations due to Brieskorn \([1]\), Kobayashi-Ochiai \([6]\), and Cho-Sato \([3][4]\) as immediate corollaries. Namely

Theorem 2. For a smooth \( X \) Fano \( n \)-fold \((n \geq 3)\) over \( \mathbb{C} \), the three conditions in (0.1) are also equivalent to the following four:

1. There is a homotopy equivalence between \( X \) and \( Q_n \) such that the induced cohomology isomorphism \( H^2(Q_n, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \) identifies the anticanonical classes.
2. The anticanonical class \( c_1(X) \) is divisible by \( n \) in \( \text{Pic}(X) \).
3. The tangent bundle \( \Theta_X \) is not ample, but \( \wedge^2 \Theta_X \) is ample.
4. There is a surjective morphism \( Q_m \to X \), \( m \geq n \), and \( X \not\cong \mathbb{P}^n \).
The outline of the proof of Theorem 1 is as follows.

Assume that a smooth Fano $n$-fold $X$ satisfies the condition (3). Because smooth Fano 3-folds with Picard number one are completely classified by Iskovskih [5], we may assume that $n \geq 4$ (this assumption is of course of purely technical nature). Pick up two general points $x_+, x_- \in X$. We consider an (arbitrary) irreducible component $W\langle x_+, x_- \rangle$ of the closed subset

$$\{ C \subset X \mid C \text{ is a connected union of rational curves, } C \supset \{x_+, x_-\}, (C, -K_X) = 2n \}$$

of the Chow scheme Chow($X$).

Under our hypothesis, it is easy to show that $\dim W\langle x_+, x_- \rangle = n - 1$. Each closed point $w \in W\langle x_+, x_- \rangle$ represents either an irreducible rational curve $C \subset X$ or a connected union of two irreducible rational curves $L_+ \cup L_- \subset X$ with $L_\pm \ni x_\pm$.

Let $V\langle x_+, x_- \rangle \subset W\langle x_+, x_- \rangle \times X$ be the associated incidence variety with natural surjective projection $\pi_X : V\langle x_+, x_- \rangle \rightarrow X$. Let $\overline{V}\langle x_+, x_- \rangle$ denote the normalisation of $V\langle x_+, x_- \rangle$ and $\overline{\pi}_X : \overline{V}\langle x_+, x_- \rangle \rightarrow X$ the induced projection. The inverse images of $x_\pm$ via this projection determine distinguished sections $\sigma_\pm \subset \overline{V}\langle x_+, x_- \rangle$ over the normalisation $\overline{W}\langle x_+, x_- \rangle$ of $W\langle x_+, x_- \rangle$.

Given a smooth curve $T$ and a morphism $f : T \rightarrow W\langle x_+, x_- \rangle$, the fibre product $T \times_{\overline{W}\langle x_+, x_- \rangle} \overline{V}\langle x_1, x_2 \rangle$ is a very special conic bundle over $T$ with two specified sections $\sigma_+, \sigma_-$ (strongly symmetric conic bundle).

**Definition.** Let $\pi : C \rightarrow T$ be a normal conic bundle over a smooth projective curve $T$ and $B$ a nef and big Cartier divisor on $C$. The fibre space $\pi : C \rightarrow T$ (or the total space $C$, by abuse of terminology) is said to be an $B$-symmetric conic bundle if $BC_{t_+} = BC_{t_-}$ whenever a closed fibre $C_t$ is a union of two components $C_{t_+}, C_{t_-}$.

Assume that $\pi : C \rightarrow T$ has two distinct sections $\sigma_+, \sigma_-$. The triple $(C; \sigma_+, \sigma_-)$ is said to be strongly $B$-symmetric if the following four conditions are satisfied:

(a) $C$ is $B$-symmetric;
(b) $\sigma_+$ and $\sigma_-$ are mutually disjoint divisors contained in the non-critical locus $C^0 = C \setminus Cr(C)$;
(c) $B\sigma_+ = B\sigma_-;
(d) For any reducible fibre $C_t = C_{t_+} + C_{t_-}$, we have

\[
\sigma_+ C_{t_+} = \sigma_- C_{t_-} = 1,
\]

\[
\sigma_+ C_{t_-} = \sigma_- C_{t_+} = 0,
\]

(possibly after suitable reindexing of the irreducible components $C_{t_\pm}$). □

Given a strongly symmetric conic bundle, we can deduce the following proposition from elementary, but a little complicated, calculation of intersection numbers (the detail will be published elsewhere):

**Proposition.** Let $\pi : C \rightarrow T$ be a normal conic bundle over a smooth projective curve with a nef big divisor $B$ and two sections $\sigma_+, \sigma_-$. Assume that $(C; \sigma_+, \sigma_-)$ is strongly $B$-symmetric and let $s$ denote the number of the singular fibres $C_{t_i} = C_{t_{i_+}} + C_{t_{i_-}}$ and put $\delta_i = C_{t_{i_+}} \cap C_{t_{i_-}}$. Let $\mu : \tilde{C} \rightarrow C$ be the minimal resolution. Then we have:

1. If its strict transform $\tilde{\sigma} \subset \tilde{C}$ has negative self intersection, then a section $\sigma \subset C$ coincides with one of the two specified sections $\sigma_\pm$. In particular,
σ is one of the σ± once a section σ ⊂ C satisfies σ² < 0. If σ ≠ σ± and
its strict transform ˜σ satisfies ˜σ² = 0, then σ is disjoint with σ± and with
Sing(C).

(2) If there are two sections σ₁, σ₂ ≠ σ± ⊂ C such that the strict transforms
˜σ₁, ˜σ₂ are mutually disjoint in ˜C, then σ₁ ∪ σ₂ is away from σ⁺ ∪ σ⁻.

With the aid of this proposition, we show (compare [2], Proof of Theorem 4.2,
Step 3) that the projection pr_X: V⟨x⁺, x⁻⟩ → X has the following remarkable
property.

Lemma. Let C be a general fibre of the conic bundle pr_W: V⟨x⁺, x⁻⟩ → W⟨x⁺, x⁻⟩.
Then pr⁻¹_X(pr_X(C)) is a union of C, σ⁺, σ⁻ and possibly an extra closed subset
which is away from σ±.

In view of this, we can apply a similar argument as in [ibid, Steps 4–8] to prove
that pr_X lifts to an isomorphism between V⟨x⁺, x⁻⟩ and the two-point blowup
Bl₁(x⁺, x⁻)X of X. This induces isomorphisms

\[ W⟨x⁺, x⁻⟩ \simeq \sigma_± \simeq E_± \simeq \mathbb{P}^{n-1}, \]

where E± ⊂ Bl₁(x⁺, x⁻)X is the exceptional divisor over x± ∈ X. The pullback ˜H₀ =
pr⁻¹_WL of the hyperplane divisor L ⊂ W⟨x⁺, x⁻⟩ ≃ \mathbb{P}^{n-1} is a semiample divisor on
V⟨x⁺, x⁻⟩ ≃ Bl₁(x₁, x₂)X. Then we easily see that ˜H₀ contracts to an ample divisor
H₀ on X and that the complete linear system |H₀| defines an isomorphism from X
to a hyperquadric in \mathbb{P}^{n+1}.

The parameter space W⟨x⁺, x⁻⟩ eventually turns out to be the dual projective
space of the complete linear system |μ*H₀ − E⁺ − E⁻| ≃ \mathbb{P}^{n-1} on Bl₁(x⁺, x⁻)X, which
is viewed as the sublinear system |H₀(−x⁺ − x⁻)| ⊂ |H₀| ≃ \mathbb{P}^{n+1} on X. To be
more explicit, for each n − 1-dimensional linear subspace Λ of

\[ H^0(X, I_{x⁺} I_{x⁻}(H₀)) ⊂ H^0(Q_n, O(1)), \]

we associate [C] ∈ W⟨x⁺, x⁻⟩, where C is the plane conic cut out of Q_n by the
n − 1 hyperplanes ∈ Λ through x⁺, x⁻.

References

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