NUMERICAL CHARACTERISATIONS OF HYPERQUADRICS

Yoichi Miyaoka

School of Mathematics, University of Tokyo

We characterise the smooth n-dimensional hyperquadrics as Fano manifolds of length n.

Given a Fano manifold X [resp. a pair (X, x_0) of a Fano manifold X and a closed point x_0 on it], we define the (global) *length* $l(X)$ of X [resp. the *local lenghth* $l(X, x_0)$ of (X, x_0)] to be the positive integer

$$\min_{C \subset X} \{(C, -K_X)\},$$

where C runs through the set of the rational curves contained in X [resp. the set of the rational curves such that $x_0 \in C \subset X$].

The local lenght $l(X, x_0)$ is a lower semiconitinuous function in x_0 and the global lenghth $l(X)$ is by definition equal to $\inf_{x_0 \in X} l(X, x_0)$. For a given closed point $x_0 \in X$, it is known that $l(X, x_0) \leq \dim X + 1$, the equality holding if and only if X is projective space [2].

In terms of the notions above, our main result is the following

Theorem 1. *Let X be a smooth Fano variety of dimension $n \geq 3$ defined over an algebraically closed field k of characteristic zero. Then the following three conditions are equivalent:*

(1) *X is isomorphic to a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$.*
(2) *The global length $l(X)$ is n.*
(3) *$\rho(X) = 1$ and $l(X, x_0) = n$ for a sufficiently general point $x_0 \in X$, where $\rho(X)$ stands for the Picard number.*

This simple numerical result involves the preceding characterisations due to Brieskorn [1], Kobayashi-Ochiai [6], and Cho-Sato [3][4] as immediate corollaries. Namely

Theorem 2. *For a smooth X Fano n-fold $(n \geq 3)$ over $\mathbb{C}$, the three conditions in* (0.1) *are also equivalent to the following four:*

(4) *There is a homotopy equivalence between X and Q_n such that the induced cohomology isomorphism $\mathrm{H}^2(Q_n, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$ identifies the anticanonical classes.*
(5) *The anticanonical class $c_1(X)$ is divisible by n in* $\mathrm{Pic}(X)$.
(6) *The tangent bundle Θ_X is not ample, but $\wedge^2 \Theta_X$ is ample.*
(7) *There is a surjective morphism $Q_m \to X$, $m \geq n$, and $X \not\simeq \mathbb{P}^n$.*

The outline of the proof of Theorem 1 is as follows.

Assume that a smooth Fano n-fold X satisfies the condition (3). Because smooth Fano 3-folds with Picard number one are completely classified by Iskovskih [5], we may assume that $n \geq 4$ (this assumption is of course of purely technical nature). Pick up two general points $x_+, x_- \in X$. We consider an (arbitrary) irreducible component $W\langle x_+, x_-\rangle$ of the closed subset

$$\Big\{C \subset X |\, C \text{ is a connected union of rational curves, } C \supset \{x_+, x_-\}, (C, -K_X) = 2n\Big\}$$

of the Chow scheme $\mathrm{Chow}(X)$.

Under our hypothesis, it is easy to show that $\dim W\langle x_+, x_-\rangle = n - 1$. Each closed point $w \in W\langle x_+, x_-\rangle$ represents either an irreducible rational curve $C \subset X$ or a connected union of two irreducible rational curves $L_+ \cup L_- \subset X$ with $L_\pm \ni x_\pm$, $L_\pm \not\ni x_\mp$.

Let $V\langle x_+, x_-\rangle \subset W\langle x_+, x_-\rangle \times X$ be the associated incidence variety with natural surjective projection $\mathrm{pr}_X \colon V\langle x_+, x_-\rangle \to X$. Let $\overline{V}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$ and $\overline{\mathrm{pr}}_X \colon \overline{V}\langle x_+, x_-\rangle \to X$ the induced projection. The inverse images of $x_\pm$ via this projection determine distinguished sections $\sigma_\pm \subset \overline{V}\langle x_+, x_-\rangle$ over the normalisation $\overline{W}\langle x_+, x_-\rangle$ of $W\langle x_+, x_-\rangle$.

Given a smooth curve T and a morphism $f \colon T \to \overline{W}\langle x_+, x_-\rangle$, the fibre product $T \times_{\overline{W}\langle x_+, x_-\rangle} \overline{V}\langle x_1, x_2\rangle$ is a very special conic bundle over T with two specified sections σ_+, σ_- (*strongly symmetric conic bundle*).

Definition. Let $\pi \colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve T and B a nef and big Cartier divisor on $\mathcal{C}$. The fibre space $\pi \colon \mathcal{C} \to T$ (or the total space $\mathcal{C}$, by abuse of terminology) is said to be an *B-symmetric* conic bundle if $B\mathcal{C}_{t+} = B\mathcal{C}_{t-}$ whenever a closed fibre $\mathcal{C}_t$ is a union of two components $\mathcal{C}_{t+}$, $\mathcal{C}_{t-}$.

Assume that $\pi \colon \mathcal{C} \to T$ has two distinct sections σ_+, σ_-. The triple $(\mathcal{C}; \sigma_+, \sigma_-)$ is said to be *strongly B-symmetric* if the following four conditions are satisfied:

(a) $\mathcal{C}$ is B-symmetric;
(b) σ_+ and σ_- are mutually disjoint divisors contained in the non-critical locus $\mathcal{C}^\circ = \mathcal{C} \setminus \mathrm{Cr}(\mathcal{C})$;
(c) $B\sigma_+ = B\sigma_-$;
(d) For any reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} + \mathcal{C}_{t-}$, we have

$$\sigma_+\mathcal{C}_{t+} = \sigma_-\mathcal{C}_{t-} = 1,$$
$$\sigma_+\mathcal{C}_{t-} = \sigma_-\mathcal{C}_{t+} = 0,$$

(possibly after suitable reindexing of the irreducible components $\mathcal{C}_{t\pm}$). □

Given a strongly symmetric conic bundle, we can deduce the following proposition from elementary, but a little complicated, calculation of intersection numbers (the detail will be published elsewhere):

Proposition. *Let $\pi \colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve with a nef big divisor B and two sections σ_+, σ_-. Assume that $(\mathcal{C}; \sigma_+, \sigma_-)$ is strongly B-symmetric and let s denote the number of the singular fibres $\mathcal{C}_{t_i} = \mathcal{C}_{t_i+} + \mathcal{C}_{t_i-}$ and put $\delta_i = \mathcal{C}_{t_i+} - \mathcal{C}_{t_i-}$. Let $\mu \colon \tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution. Then we have:*

(1) *If its strict transform $\tilde\sigma \subset \tilde{\mathcal{C}}$ has negative self intersection, then a section $\sigma \subset \mathcal{C}$ coincides with one of the two specified sections $\sigma_\pm$. In particular,*

σ is one of the $\sigma_\pm$ once a section $\sigma \subset \mathcal{C}$ satisfies $\sigma^2 < 0$. If $\sigma \neq \sigma_\pm$ and its strict transform $\tilde{\sigma}$ satisfies $\tilde{\sigma}^2 = 0$, then σ is disjoint with $\sigma_\pm$ and with $\mathrm{Sing}(\mathcal{C})$.

(2) *If there are two sections $\sigma_1, \sigma_2 \neq \sigma_\pm \subset \mathcal{C}$ such that the strict transforms $\tilde{\sigma}_1, \tilde{\sigma}_2$ are mutually disjoint in $\tilde{\mathcal{C}}$, then $\sigma_1 \cup \sigma_2$ is away from $\sigma_+ \cup \sigma_-$.*

With the aid of this proposition, we show (compare [2], Proof of Theorem 4.2, Step 3) that the projection $\overline{\mathrm{pr}}_X \colon \overline{V}\langle x_+, x_-\rangle \to X$ has the following remarkable property.

Lemma. *Let C be a general fibre of the conic bundle $\overline{\mathrm{pr}}_W \colon \overline{V}\langle x_+, x_-\rangle \to \overline{W}\langle x_+, x_-\rangle$. Then $\overline{\mathrm{pr}}_X^{-1}(\overline{\mathrm{pr}}_X(C))$ is a union of C, σ_+, σ_- and possibly an extra closed subset which is away from $\sigma_\pm$.*

In view of this, we can apply a similar argument as in [*ibid*, Steps 4–8] to prove that $\overline{\mathrm{pr}}_X$ lifts to an isomophism between $\overline{V}\langle x_+, x_-\rangle$ and the two-point blowup $\mathrm{Bl}_{\{x_+,x_-\}}X$ of X. This induces isomorphisms

$$\overline{W}\langle x_+, x_-\rangle \simeq \sigma_\pm \simeq E_\pm \simeq \mathbb{P}^{n-1},$$

where $E_\pm \subset \mathrm{Bl}_{\{x_+,x_-\}}X$ is the exceptional divisor over $x_\pm \in X$. The pullback $\tilde{H}_0 = \mathrm{pr}_{\overline{W}}^* L$ of the hyperplane divisor $L \subset \overline{W}\langle x_+, x_-\rangle \simeq \mathbb{P}^{n-1}$ is a semiample divisor on $\overline{V}\langle x_+, x_-\rangle \simeq \mathrm{Bl}_{\{x_1,x_2\}}X$. Then we easily see that $\tilde{H}_0$ contracts to an ample divisor H_0 on X and that the complete linear system $|H_0|$ defines an isomorphism from X to a hyperquadric in $\mathbb{P}^{n+1}$.

The parameter space $W\langle x_+, x_-\rangle$ eventually turns out to be the dual projective space of the complete liner system $|\mu^* H_0 - E_+ - E_-| \simeq \mathbb{P}^{n-1}$ on $\mathrm{Bl}_{\{x_+,x_-\}}X$, which is viewed as the sublinear system $|H_0(-x_+ - x_-)| \subset |H_0| \simeq \mathbb{P}^{n+1}$ on X. To be more explicit, for each $n-1$-dimensional linear subspace Λ of

$$\mathrm{H}^0(X, \mathcal{I}_{x_+}\mathcal{I}_{x_-}(H_0)) \subset \mathrm{H}^0(Q_n, \mathcal{O}(1)),$$

we associate $[C] \in W\langle x_+, x_-\rangle$, where C is the plane conic cut out of Q_n by the $n-1$ hyperplanes $\in \Lambda$ through x_+, x_-.

References

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