ON THE CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS WITH $\rho = 1$ AND $f \geq 2$

KAORI SUZUKI

Abstract. This article considers $\mathbb{Q}$-Fano 3-folds X with $\rho = 1$. The aim is to show the possible maximal Fano index f of X is 19. In case of equality, the Hilbert series of X equals that of weighted projective space $\mathbb{P}(3, 4, 5, 7)$. We also consider all the possible candidates ofthe baskets of singularities of X for $f \geq 9$.

0. Introduction

We say that X is a $\mathbb{Q}$-Fano variety if it has only terminal singularities, the anticanonical Weil divisor $-K_X$ is ample, and it satisfies $\mathbb{Q}$-factorial. X is $\mathbb{Q}$-factorial if for an arbitrary Weil divisor D on X, there exists a positive integer r such that rD is a Cartier divisor. For such X, there are two indices, the *Gorenstein index*, the smallest positive integer r for rK_X is Cartier, and the *Fano index* f, which is the largest positive integer such that $-K_X = fA$ for a Weil divisor A. In this paper, we shall prove that $\mathbb{Q}$-Fano index $f \leq 19$, and determine the case of $\mathbb{Q}$-Fano 3-folds X of Picard number $\rho(X) = 1$ over $\mathbb{C}$ with $f \geq 9$.

The Main Theorem is the following:

Theorem 0.1. *Let X be a $\mathbb{Q}$-Fano 3-fold over $\mathbb{C}$ which is $\mathbb{Q}$-factorial and satisfies $\rho(X) = 1$. Let $f(X)$ be the $\mathbb{Q}$-Fano index of X. Then*

1. $\max f(X) = 19$, *and* $f(\mathbb{P}(3, 4, 5, 7)) = 19$.
2. *If* $f(X) = 19$, *then the Hilbert series of* X *equals that of* $\mathbb{P}(3, 4, 5, 7)$, *namely* $1/(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^7)$.
3. $f \in \{1, \ldots, 10, 11, 13, 17, 19\}$

The classification of smooth Fano varieties was started by Fano in the 1930s and completed by Iskovskikh [I], Fujita ([F1], [F2], [F3]), Mori and Mukai ([MM1]–[MM2]). However, the classification of singular Fano varieties is still open. From the 1980s, by ingenously expliting vector bundles, Mukai has been investigating Gorenstein Fano 3-folds with canonical singularities, to give the classification under mild technical assumptions that Fano index = 1, that its anti canonical divisor $-K_X$ is primitive Cartier divisor and that $|-K_X|$ is "indecomposable". Here indecomposable means that there is no non-trivial decomposition $-K_X = A + B$ into effective Weil divisors A and B. In particular, this condition holds if X is $\mathbb{Q}$-factorial and the Picard number of X is 1.

Most of people who study Fano varieties by MMP assume the Fano index to be one; nevertheless it has not been completed yet despite of their efforts.

On the other hand, the classification of Fano manifolds of Fano index ≥ 2 tends to be easier by experience. T. Sano studied the classification of non Gorenstein case when the Gorenstein index is less than or equal to the Fano index ([Sa1]–[Sa2]). It is thus an interesting question to see what happens if we drop this condition. For instance, under Sano's condition there are only eight types of varieties, but we have more than 60 types if we drop the assumption [Su]. Our classification is conducted through the method of anticanonical rings and Magma (computer program).

Acknowledgements. The author would like to express her deep gratitude to Professor Miles Reid for his valuable comments and unceasing encouragement. She also thanks Professor Youichi Miyaoka, Professor Keiji Oguiso and Dr. Hiromichi Takagi for the discussion. Special thanks to Dr. Gavin Brown for teaching how to program in Magma. This paper was written during the author's stay at Univ. of Warwick, U.K., 2001.

1. Graded ring method for Fano 3-folds

Let X be a $\mathbb{Q}$-Fano 3-fold with at worst terminal singularities, and

$$\mathcal{B} := \left\{ \frac{1}{r_\alpha}(1, a_\alpha, r_\alpha - a_\alpha) \right\}$$

its basket. In apriori, there is no reason why the set of f, A^3, $1/12Ac_2(X)$ and $\mathcal{B}$, wihch determine X as invariants, are finite. However, in the sequel, we shall show that this set is finite and of reasonable order in the following. For this reason we can use Magma to get the final result.

For the pair (X, A), we consider the graded ring as follows :

$$R(X, A) := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}(nA))$$

with $X = \operatorname{Proj} R(X, A)$. Write $P_n(X) = h^0(X, nA) = \dim H^0(X, \mathcal{O}(nA))$. Then P_n is given by the singular Riemann–Roch formula of [YPG, Theorem 9.1]. Note first that, since $-K_X$ generates the local class group at each virtual point $1/r(1, a, r-a)$, it follows that locally $nA \sim iK_X$ for some $i \in [0, r-1]$. Equivalently, f is coprime to r and $i \equiv -n/f$ mod r. Then

$$\chi(\mathcal{O}_X(nA)) = \chi(\mathcal{O}_X) + \frac{n(n+f)(2n+f)}{12}A^3$$
$$- \frac{2n}{f} \sum_{\mathcal{B}} \frac{r^2-1}{12r} \frac{Ac_2(X)}{12} + \sum_{\mathcal{B}} \left\{ -i\frac{r^2-1}{12r} + \sum_{j=1}^{i-1} \frac{\overline{bj}(r - \overline{bj})}{2r} \right\},$$

where the sum runs over the basket $\mathcal{B}$ of X, and $\overline{x}$ is the smallest residue mod r.

Remark 1.1. The formula for $\chi(nA)$ holds for all n. When $n > -f$, Kodaira vanishing gives $H^i(nA) = 0$, so that $\chi(nA) = H^0(nA)$.

Note in particular that $\chi(nA) = 0$ for $-f < n < 0$. This is the starting point for obtaining combinatorial constraints on f.

Then we define the *Hilbert series* of X as the formal power series

$$P_X(t) := \sum_{n\geq 0} P_n(X)t^n.$$

From the Riemann–Roch formula and $\chi(\mathcal{O}_X) = 1$, we rewrite this Hilbert function as:

$$P_X(t) = \frac{1}{1-t} + \frac{(f^3+3f+2)t + (2f^2+8)t^2 + (f^2-3f+2)t^3}{12(1-t)^4} A^3$$
$$+ \frac{t}{(1-t)^2}\frac{Ac_2}{12} + \sum_{\mathcal{B}}\left[\frac{1}{1-t^r}\sum_{k=1}^{r-1}\left\{-i\frac{r^2-1}{12r} + \sum_{j=1}^{i-1}\frac{\overline{bj}(r-\overline{bj})}{2r}\right\}t^k\right]$$

We can calculate this Hilbert series from the baskets of singularities.

Note that from $P_{-1} = 0$, we have the formula for A^3 as :

$$A^3 = \frac{12}{(f-1)(f-2)}\left\{1 - \frac{1}{12}Ac_2(X) + \sum_{j=1}^{r-1}\frac{\overline{bj}(r-\overline{bj})}{2r}\right\}$$

for $f \geq 3$. Moreover,

$$\frac{Ac_2(X)}{12} = \frac{2}{f}\left(1 - \sum_{\mathcal{B}}\frac{r^2-1}{12r}\right).$$

(Reid, [YPG, Cor. 10.3])

Now we explain about Hilbert function method. If the ring $R(X, A)$ has a nonzero element x_1 of degree a_1 then $(1-t_1^a)P(t)$ is the Hilbert series of a graded ring built over the section $(x_1 = 0)$ of X. Successively, if $x_1, x_2 \in R(X, A)$ form a regular sequence then $(1-t^{a_1})(1-t^{a_2})P(t)$ is the Hilbert series corresponding to the codimension 2 complete intersection $x_1 = x_2 = 0$ in X.

For example, consider the case $f = 6$, where X has a quotient singularity $\frac{1}{5}(1, 2, 3)$ as its only singularity, and $A^3 = 1/5$. Then $P_1(t) = 2$, $P_2(t) = 4$, $P_3(t) = 7$ so that we have at least one degree 2 generator and one degree 5 generator.

$$P(t) = \frac{t^2-t+1}{t^8-3t^7+3t^6-t^5-t^3+3t^2-3t+1},$$

and we have by the calculation that

$$(1-t)^2(1-t^2)(1-t^3)(1-t^5)P(t) = 1-t^6$$

This tells us that a plausible model for X would be hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3,5)$.

Remark 1.2 (Altinok, [Al]). For the type of singularities $\frac{1}{r}(f,a,r-a)$, the graded ring has at least 4 generators of degrees $\equiv 0, f, a, r-a \mod r$.

2. Proof of Theorem 0.1

To show $\max f(X) = 19$, we use a program written in Magma.

In Kawamata's paper (Kawamata, [K]), we have the following theorem:

Theorem 2.1. *Let X be a $\mathbb{Q}$-Fano 3-fold which is $\mathbb{Q}$-factorial and satisfies $\rho(X) = 1$. We set $\mathcal{E}$ as the double dual of the sheaf of Kähler differentials Ω^1_X. Then we have*

1. *If $\mathcal{E}$ is semistable, $(-K_X)^3 \leq 3(-K_X)\cdot c_2(X)$*
2. *If $\mathcal{E}$ is not semistable, we have $\mathcal{F}$ as the maximal destabilizing sheaf of $\mathcal{E}$. Set $s := \operatorname{rank}\mathcal{F}$, so that $s = 1$ or 2. Define t by $c_1(\mathcal{F}) \simeq tK_X$. Then we have*
   (a) *If $s = 1$ then $(1-t)(1+3t)(-K_X)^3 \leq 4(-K_X)\cdot c_2(X)$*
   (b) *If $s = 2$ then $t(4-3t)(-K_X)^3 \leq 4(-K_X)\cdot c_2(X)$.*

Remark 2.2. From the assumption $\rho(X) = 1$, we have $0 < t < s/3$ and $t \in \mathbb{Z}/f$.

Assume that $f \geq 3$, we only need to consider the case 2(b). The minimum value of the coefficient of left hand side occurs for $t = 1/f$. Let $R = \operatorname{lcm}\{r_\alpha\}$ be the Gorenstein index of X. We have $A^3 \geq 1/R$. We can rewrite the above formula using R as

$$4f^2 - 3f \leq 4R(24 - \sum_n (r_n - 1/r_n)).$$

We have the maximal value for the possible baskets of singularities and get $4f^2 - 3f \leq 9948$. This implies $f \leq 50$.

For this $f \leq 50$, we again check all the possible baskets of terminal singularities which satisfying the following conditions :

1. $\sum_{i=1}^m (r_i - 1/r_i) < 24$, $m \leq 15$.
2. $A^3 > 0$.
3. $P_{-i} = 0$ for $i = 1 \ldots f-1$.
4. Theorem 2.1

Using a program in Magma, we get finally $f \leq 19$ and if $f = 19$ the only possibility is

$$A^3 = \frac{1}{3\cdot 4\cdot 5\cdot 7}, \quad \text{and} \quad \mathcal{B} = \left\{\frac{1}{3}(1,1,2), \frac{1}{4}(1,1,3), \frac{1}{5}(1,2,3), \frac{1}{7}(1,2,5)\right\}.$$

For this X we calculate the Hilbert function $1/(1-t^3)(1-t^4)(1-t^5)(1-t^7)$. This is also the Hilbert function of $\mathbb{P}(3,4,5,7)$. The last equality of 1., Theorem 0.1 is clear from the canonical bundle formula.

Remark 2.3. The rest of the list for each $f \leq 8$ is in preparation. A partial results for the case of $f = 2$ will appear in (Reid and Suzuki [RS]).

3. List of $\mathbb{Q}$-Fano 3-folds

Table 1 is the complete list of families of $\mathbb{Q}$-Fano 3-folds satisfying the assumption of Theorem 0.1 with Fano index ≥ 9.

f	$X/\sim$	$\mathcal{B} = \{[r,a]\}$	$-K_X^3$
9	$X_6 \subset \mathbb{P}(1,2,3,4,5)$	$\{[2,1],[4,1],[5,2]\}$	$\frac{729}{20}$
	$X_{12} \subset \mathbb{P}(2,3,4,5,7)$	$\{3 \times [2,1],[5,2],[7,2]\}$	$\frac{729}{70}$
10	codim ≥ 7	$\{[7,3],[11,3]\}$	$\frac{2000}{77}$
11	$\mathbb{P}(1,2,3,5)$	$\{[2,1],[3,1],[5,2]\}$	$\frac{1331}{30}$
	$X_{12} \subset \mathbb{P}(1,4,5,6,7)$	$\{[2,1],[5,1],[7,2]\}$	$\frac{1331}{70}$
	$X_{10} \subset \mathbb{P}(2,3,4,5,7)$	$\{[2,1],[3,1],[4,1],[7,3]\}$	$\frac{1331}{84}$
13	$\mathbb{P}(1,3,4,5)$	$\{[3,1],[4,1],[5,2]\}$	$\frac{2197}{60}$
17	$\mathbb{P}(2,3,5,7)$	$\{[2,1],[3,1],[5,1],[7,3]\}$	$\frac{4913}{210}$
19	$\mathbb{P}(3,4,5,7)$	$\{[3,1],[4,1],[5,2],[7,2]\}$	$\frac{6859}{420}$

Table 1. The list of $\mathbb{Q}$-Fano 3-folds for $f \geq 9$

Remark 3.1. $X \sim Y$ means that the Hilbert series of X is equal to that of Y.

Remark 3.2. We have no idea to write down the complete structure of codim ≥ 7 X. We also believe the non-existence for this case.

4. Appendix

Here is a part of Magma program which is used to find $f \leq 50$.
Here, BB is a list of Baskets generated automatically by computer.

```
function Kc2_is(BB)

    sumpart := &+[ Rationals() |
                    (r^2-1)/r where r is p[1] : p in BB ];
    return (24-sumpart);

end function;
```

f	$X/\sim$	$\mathcal{B} = \{[r,a]\}$	$-K_X^3$
1	$X_7 \subset \mathbb{P}(1,1,1,1,2)$	$[2,1]$	$\frac{7}{2}$
2	$X_{10} \subset \mathbb{P}(1,1,2,3,5)$	$[3,1]$	$\frac{8}{3}$
3	$X_4 \subset \mathbb{P}(1,1,1,2,2)$	$2 \times [2,1]$	27
4	$X_4 \subset \mathbb{P}(1,1,1,2,3)$	$[3,1]$	$\frac{128}{3}$
5	$\mathbb{P}(1,1,1,2)$	$[2,1]$	$\frac{125}{2}$
6	$X_6 \subset \mathbb{P}(1,1,2,3,5)$	$[5,2]$	$\frac{216}{5}$
7	$\mathbb{P}(1,1,2,3)$	$\{[2,1],[7,1]\}$	$\frac{343}{6}$
8	$X_{10} \subset \mathbb{P}(1,2,3,5,7)$	$\{[3,1],[7,2]\}$	$\frac{512}{21}$

TABLE 2. Examples of $\mathbb{Q}$-Fano 3-folds of $2 \le f \le 8$

```
 L := [ 3*LCM([ p[1] : p in BB[i] ])*Kc2_is(BB[i]) :
               i in [2..#BB]];
 M := Maximum(L);
 Bigbaskets := [ i : i in [1..#L] | L[i] eq M];
 Bigbaskets;
```

This is also a part of program for the calculation of A^3, $1/12Ac_2(X)$, $\mathrm{P_n(X)}$ and Hibert series.

```
/////////////////////////////////////////////////////
// Build the Hilbert series
/////////////////////////////////////////////////////

forward Ac2over12_is, contribution;
intrinsic FanoHilbertSeries(f::RngIntElt,B::SeqEnum)
-> RngElt
{The Hilbert series of a Fano 3-fold of Fano index f

 and basket B}
   K := RationalFunctionField(Rationals());
   t := K.1;
   I := 1/(1-t);
```

```
    II := 1/12 * A3_is(f,B) *
        ((f^2+3*f+2)*t+(-2*f^2+8)*t^2+(f^2-3*f+2)*t^3)
        /(1-t)^4;
    III := Ac2over12_is(f,B) * t/(1-t)^2;
    IV := &+[ Parent(t) |
        &+[ Parent(t) | contribution(f,r,a,n)*t^n
              : n in [1..r-1] ] / (1-t^r)
                where r is p[1]
                where a is p[2] : p in B ];
    return I + II + III + IV;
end intrinsic;

/////////////////////////////////////////////////////
// Auxilliary functions
/////////////////////////////////////////////////////

function i_is(f,r,n)
    h,u,v := XGCD(f,r);
    return (-n*u) mod r;
end function;

bar := func< m,r | m mod r >;

inv := func< a,r | i_is(a,r,1) >;

function contribution(f,r,a,n)
    i := i_is(f,r,n);
        b := inv(a,r);
    first := -i*(r^2-1)/(12*r);
    if i in {0,1} then
        extra := 0;
    else
        extra := &+[ bar(b*j,r)*(r-bar(b*j,r))/(2*r) :
                         j in [0..i-1] ];
    end if;
    return first + extra;
end function;

function Ac2over12_is(f,B)
    sumpart := &+[ Rationals() | (r^2-1)/(12*r) where r is p[1]
 : p in B ];
    return (2-sumpart)/f;
end function;

function A3_is(f,B)
```

```
       factor := 12/((f-1)*(f-2));
        c2_part := Ac2over12_is(f,B);
        periodic := &+[ Rationals() | contribution(f,r,a,-1)
                                 where a is p[2]
                                 where r is p[1] : p in B ];
        return factor * (1 - c2_part + periodic);
end function;

intrinsic FanoCoefficient(f::RngIntElt,B::SeqEnum,n::RngIntElt)
-> RngElt
{The n-th coefficient of the Hilbert series of Fano
                             with Fano index f and basket B}
    V := 1+1/6*A3_is(f,B)*n*(n+1)*(n+2)+n*Ac2over12_is(f,B)+
         &+[Rationals()| contribution(f,r,a,n)
                     where a is p[2]
                                 where r is p[1] :  p in B];
    vprintf User1: "\tP_(%o) = %o\n",n,V;
    return V;
end intrinsic;
```

References

[ABR] S. Altinok, G. Brown, M. Reid, *Fano3-folds, $K3$ surfaces and graded rings*, in Singapore International Symposium in Topology and Geometry (NUS, 2001), Ed. A. J. Berrick and others, Contemp. Math., AMS, 2002, 29pp.

[Al] S. Altinok,*Graded rings corresponding to polarized K3 surfaces and $\mathbb{Q}$-Fano 3-folds*, Univ. of Warwick Ph.D.thesis, Sep. 1998.

[BB] A. A. Borisov, L. A. Borisov, *Singular toric Fano three-folds*, in Russian Acad. Sci.Sb. Math. 75 (1993), no. 1, 277–283.

[F1] T.@Fujita, *On the structure of polarized manifolds with total deficiency one,@Part I*, J. Math. Soc. of Japan 32 (1980), 709-725.

[F2] T.@Fujita, *On the structure of polarized manifolds with total deficiency one, Part II*, J. Math. Soc. of Japan 33 (1981), 415-434.

[F3] T.@Fujita, *On the structure of polarized manifolds with total deficiency one, Part III*, J. Math. Soc. of Japan 36 (1980), 75-89.

[I] V. A. Iskovskikh, *Anticanonical models of three-dimensional algebraic varieties*, in J. Soviet. Math. 13 (1980), 745-814.

[K] Y. Kawamata, *Boundedness of $\mathbb{Q}$–Fano Threefolds*, in Contemp. Math. 131 (1992), 439–445.

[MM1] S. Mori S. Mukai, *Classification of Fano 3-folds with $\mathcal{B}_2 \geq 2$*, in Manusucripta Math. 36 (1981/82) 147–162.

[MM2] S. Mori S. Mukai, *On Fano 3-folds with $\mathcal{B}_2 \geq 2$, in Algebraic varieties and analytic varieties (Tokyo, 1981)*, in Adv Stud. Pure Math. 1, Kinokuniya and NorthHolland, 1983, 101-129

[RS] M. Reid, K. Suzuki, *Cascades of projections form log del Pezzo surfaces*, preprint.

[Sa1] T. Sano, *On classification of non-Gorenstein $\mathbb{Q}$-Fano 3-folds of Fano index* 1, in J. Math. Soc. Japan 47 (1995), 369–380.

[Sa2] T. Sano, *Classification of non-Gorenstein $\mathbb{Q}$–Fano d-folds of Fano index greater than* $d-2$, in Nagoya Math. J. 142 (1996), 133–143

[Su] K. Suzuki, *On* $\mathbb{Q}$*-Fano* 3*-folds with Fano index* ≥ 2, Ph. D. thethis, Graduate school of Math. Sc. University of Tokyo, submitted on January 2003.

[YPG] M. Reid, *Graded rings and birational geometry*, in Algebraic Geometry(1985), ed. S.Bloch, in Proc. of Sym. Pure Math. 46, A.M.S. (1987), vol1,345-414.

Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153–8914, Japan

E-mail address: `suzuki@ms.u-tokyo.ac.jp`