Let $F$ be an algebraic number field. Let $\Sigma_F = \text{Hom}(F, \mathbb{C})$ and $X_F$ the free abelian group generated by $\Sigma_F$. Let $\iota$ denote the complex conjugation on $\Sigma_F$ and $X_F$. Then one can define a map called regulator

$$\rho: K_{2m-1}(F) \rightarrow (X_F \otimes \mathbb{R}(m - 1))^{\tau = \text{id}}.$$ 

In [9], Zagier conjectured that the image of $\rho$ can be expressed in terms of the $m$-th polylogarithm function

$$\text{Li}_m(z) = \sum_{k \geq 1} \frac{z^k}{k^m}, |z| < 1.$$ 

More precisely, he conjectured that $K_{2m-1}(F)$ is isomorphic modulo torsion to a certain subquotient $B_m(F)$ of $\mathbb{Z}[F^\times - \{1\}]$ called the $m$-th Bloch group of $F$, and that the composite of the isomorphism with the regulator $\rho$ can be expressed by values of $\text{Li}_m(z)$ at algebraic numbers. The conjecture was solved affirmatively for $m \leq 3$, but it is still open when $m \geq 4$.

As for the construction of the map $B_m(F) \rightarrow K_{2m-1}(F)_{\mathbb{Q}}$, two methods are well-known. Beilinson and Deligne showed that the above map can be constructed under the existence of an appropriate category of mixed Tate motives [1]. On the other hand, de Jeu also constructed the map by using the wedge complexes developed by himself [5]. In this note, we will present an alternative approach. The main tool here is higher Bott-Chern forms developed by Burgos and Wang [4].

1. The Bloch group

First we introduce a one-valued version of the polylogarithm function:

$$P_m(z) = \Re_m \left( \sum_{k=0}^{m-1} \frac{2^k B_k}{k!} (\log |z|)^k \text{Li}_{m-k}(z) \right),$$

where $\Re_m$ is $\text{Re}$ or $\text{Im}$ according as $m$ is odd or even and $B_k$ is the $k$-th Bernoulli number. This is a one-valued real analytic function on $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. 
In this note, we are interested in $K$-theory in rational coefficients, therefore we will define Bloch group with rational coefficients. Let $F^\times_\mathbb{Q} = F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and define a map
\[
\beta_2 : \mathbb{Q}[F^\times - \{1\}] \to F^\times_\mathbb{Q} \wedge F^\times_\mathbb{Q}
\]
by $[x] \mapsto x \wedge (1 - x)$. Set $A_2(F) = \text{Ker } \beta_2$ and
\[
\mathcal{C}_2(F) = \left\{ \sum_i n_i[x_i] \in A_2(F) ; \sum_i n_i P_2(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.
\]
It turns out that $\mathcal{C}_2(F)$ is generated by
\[
[x] + [y] + [\frac{1-x}{1-xy}] + [1 - xy] + [\frac{1-y}{1-xy}]
\]
for all $x, y \in F^\times - \{1\}$ with $xy \neq 1$. The quotient group $\mathcal{B}_2(F) = A_2(F)/\mathcal{C}_2(F)$ is called Bloch group of $F$.

Suppose $m \geq 3$ and subgroups $\mathcal{C}_{m-1}(F) \subset A_{m-1}(F) \subset \mathbb{Q}[F^\times - \{1\}]$ are given. Define a map
\[
\beta_m : \mathbb{Q}[F^\times - \{1\}] \to F^\times_\mathbb{Q} \otimes (\mathbb{Q}[F^\times - \{1\}]/\mathcal{C}_{m-1}(F))
\]
by $\beta_m([x]) = x \otimes [x]$. Set $A_m(F) = \text{Ker } \beta_m$ and
\[
\mathcal{C}_m(F) = \left\{ \sum_i n_i[x_i] \in A_m(F) ; \sum_i n_i P_m(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.
\]
The quotient group $\mathcal{B}_m(F) = A_m(F)/\mathcal{C}_m(F)$ is called $m$-th Bloch group of $F$.

**Zagier Conjecture** ([9]). For $m \geq 2$, the rational algebraic $K$-theory $K_{2m-1}(F)_{\mathbb{Q}}$ is isomorphic to $\mathcal{B}_m(F)$, and the composite
\[
\mathcal{B}_m(F) \simeq K_{2m-1}(F)_{\mathbb{Q}} \xrightarrow{\rho} (X_F \otimes \mathbb{R}(m-1))^{\tau=id}
\]
is written as
\[
\sum_i n_i[x_i] \mapsto \left( (\sqrt{-1})^{\alpha_m} \sum_i n_i P_m(x_i^\sigma) \right)_{\sigma \in \Sigma_F},
\]
where $\alpha_m$ is 0 or 1 according as $m$ is odd or even.

2. **The complex of exact hermitian cubes**

Let $< -1,0,1 >$ be the ordered set consisting of three elements and $< -1,0,1 >^n$ its $n$-th power. For a small exact category $\mathfrak{A}$ with a fixed zero object 0, a functor $\mathcal{F} : < -1,0,1 >^n \to \mathfrak{A}$ is called an $n$-cube of $\mathfrak{A}$. Let $\mathcal{F}_{\alpha_1,\ldots,\alpha_n}$ denote the image of an object $(\alpha_1,\ldots,\alpha_n)$ of $< -1,0,1 >^n$. For integers $1 \leq i \leq n$ and $-1 \leq j \leq 1$, an $(n - 1)$-cube $\partial_i^j \mathcal{F}$ is defined by $(\partial_i^j \mathcal{F})_{\alpha_1,\ldots,\alpha_{n-1}} = \mathcal{F}_{\alpha_1,\ldots,\alpha_{i-1},j,\alpha_{i+1},\ldots,\alpha_n}$. It is called a face of $\mathcal{F}$. For
an object $\alpha$ of $<-1, 0, 1>^{n-1}$ and an integer $1 \leq i \leq n$, a 1-cube $\partial^0_i \mathcal{F}$ called an edge of $\mathcal{F}$ is defined by

$$\mathcal{F}_{\alpha_1, \ldots, \alpha_{i-1}, -1, \alpha_i, \ldots, \alpha_{n-1}} \rightarrow \mathcal{F}_{\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_i, \ldots, \alpha_{n-1}} \rightarrow \mathcal{F}_{\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_i, \ldots, \alpha_{n-1}}.$$ 

An $n$-cube $\mathcal{F}$ is said to be exact if all edges of $\mathcal{F}$ are short exact sequences.

Let $C_n \mathcal{A}$ denote the set of all exact $n$-cubes of $\mathcal{A}$. If $\mathcal{F}$ is an exact $n$-cube, then any face $\partial^j_i \mathcal{F}$ is also exact. Hence $\partial^j_i$ induces a map

$$\partial^j_i : C_n \mathcal{A} \rightarrow C_{n-1} \mathcal{A}.$$

Let $\mathcal{F}$ be an exact $n$-cube of $\mathcal{A}$. For an integer $1 \leq i \leq n + 1$, let $s^1_i \mathcal{F}$ be an exact $(n + 1)$-cube such that its edge $\partial^p_i (s^1_i \mathcal{F})$ is $\mathcal{F}_\alpha \xrightarrow{id} \mathcal{F}_\alpha \rightarrow 0$. Similarly, let $s^{-1}_i \mathcal{F}$ be an exact $(n + 1)$-cube such that $\partial^p_i (s^{-1}_i \mathcal{F})$ is $0 \rightarrow \mathcal{F}_\alpha \xrightarrow{id} \mathcal{F}_\alpha$. An exact cube written as $s^1_i \mathcal{F}$ is said to be degenerate.

Let $\mathcal{Q}C_n \mathcal{A}$ be the free $\mathcal{Q}$-module generated by $C_n \mathcal{A}$ and $D_n \subset \mathcal{Q}C_n \mathcal{A}$ the submodule generated by all degenerate exact $n$-cubes. Let $\mathcal{Q}C_n \mathcal{A} = \mathcal{Q}C_n \mathcal{A}/D_n$ and

$$\partial = \sum_{i=1}^{n} \sum_{j=-1}^{1} (-1)^{i+j+1} \partial^j_i : \mathcal{Q}C_n \mathcal{A} \rightarrow \mathcal{Q}C_{n-1} \mathcal{A}.$$ 

Then $\mathcal{Q}C_n \mathcal{A} = (\mathcal{Q}C_n \mathcal{A}, \partial)$ becomes a homological complex.

**Theorem 2.1** ([6]). The homology of $(\mathcal{Q}C_n \mathcal{A}, \partial)$ is isomorphic to the rational algebraic $K$-theory of $\mathcal{A}$:

$$H_n(\mathcal{Q}C_n \mathcal{A}, \partial) \simeq K_n(\mathcal{A})_{\mathcal{Q}}.$$

This isomorphism preserves products on the both sides if $\mathcal{A}$ is equipped with a strictly associative tensor product.

### 3. The higher Bott-Chern forms

Let $M$ be a compact complex algebraic manifold, namely, the analytic space consisting of all $\mathbb{C}$-valued points of a smooth proper algebraic variety over $\mathbb{C}$. Let $\mathcal{E}_p^p(M)$ be the space of real smooth differential forms of degree $p$ on $M$ and $\mathcal{E}^p(M) = \mathcal{E}_p^p(M) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{E}^{p,q}(M)$ be the space of complex differential forms of type $(p, q)$ on $M$. Set

$$\mathcal{D}^n(M, p) = \begin{cases} 
\mathcal{E}_p^{n-1}(M)(p-1) \cap \bigoplus_{p'+q'=n-1} \mathcal{E}_{p'}^{q'}(M), & n < 2p, \\
\mathcal{E}_p^{2p}(M)(p) \cap \mathcal{E}^{p, p}(M) \cap \text{Ker} d, & n = 2p, \\
0, & n > 2p 
\end{cases}$$
and define a differential $d_D : \mathcal{D}^n(M, p) \to \mathcal{D}^{n+1}(M, p)$ by
\[
d_D(\omega) = \begin{cases} 
-\pi(d\omega), & n < 2p - 1, \\
-2i\partial\bar{\partial}\omega, & n = 2p - 1, \\
0, & n > 2p - 1,
\end{cases}
\]
where $\pi : E^n(M) \to \mathcal{D}^n(M, p)$ is the canonical projection. Then it is shown in [3, Thm.2.6] that $(\mathcal{D}^*(M, p), d_D)$ becomes a complex of $\mathbb{R}$-vector spaces computing the real Deligne cohomology, that is, for $n \leq 2p$ we have
\[
H^n(\mathcal{D}^*(M, p), d_D) \simeq H^n_\mathcal{D}(M, \mathbb{R}(p)).
\]

By a hermitian vector bundle $E = (E, h)$ on $M$ we mean an algebraic vector bundle $E$ on $M$ with a smooth hermitian metric $h$. Let $K_E$ denote the curvature form of the unique connection on $E$ that is compatible with both the metric and the complex structure. The Chern form of $E$ is defined as
\[
\text{ch}_0(E) = \text{Tr}(\exp(-K_E)) \in \oplus_p \mathcal{D}^{2p}(M, p).
\]

An exact hermitian $n$-cube on $M$ is an exact $n$-cube made of hermitian vector bundles on $M$. Let $\mathcal{F} = \{E_\alpha\}$ be an exact hermitian $n$-cube on $M$. We call $\mathcal{F}$ an emi-$n$-cube if the metric on any $E_\alpha$ with $\alpha_1 = 1$ coincides with the metric induced from $E_{\alpha_1, \ldots, \alpha_i=1,0,\alpha_{i+1},\ldots,\alpha_n}$ by the surjection $E_{\alpha_1,\ldots,\alpha_i,0,\alpha_{i+1},\ldots,\alpha_n} \to E_\alpha$.

For an emi-1-cube $E : \overline{E}_{-1} \to \overline{E}_0 \to \overline{E}_1$, a canonical way of constructing a hermitian vector bundle $\text{tr}_1 E$ on $M \times \mathbb{P}^1$ connecting $\overline{E}_0$ with $\overline{E}_{-1} \oplus \overline{E}_1$ is given in [4]. If $(x : y)$ denotes the homogeneous coordinate of $\mathbb{P}^1$ and $z = x/y$, then $\text{tr}_1 E$ fulfills the conditions $(\text{tr}_1 E)|_{z=0} \simeq \overline{E}_0$ and $(\text{tr}_1 E)|_{z=\infty} \simeq \overline{E}_{-1} \oplus \overline{E}_1$. For an emi-$n$-cube $\mathcal{F}$, let $\text{tr}_1(\mathcal{F})$ be an emi-$(n-1)$-cube on $M \times \mathbb{P}^1$ defined by $\text{tr}_1(\mathcal{F})_\alpha = \text{tr}_1(\partial_1^{\alpha}(\mathcal{F}))$ for any object $\alpha$ of $< -1, 0, 1 >^{n-1}$, and $\text{tr}_n(\mathcal{F})$ a hermitian vector bundle on $M \times (\mathbb{P}^1)^n$ given by taking $\text{tr}_1$ $n$ times.

Let $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ be the $i$-th projection and $z_i = \pi_i^* z$. Let $\mathfrak{S}_n$ be the symmetric group of $n$-letters. For an integer $1 \leq i \leq n$, a differential form with logarithmic poles $S_n^i$ on $(\mathbb{P}^1)^n$ is defined as
\[
S_n^i = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log |z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}},
\]
and $T_n$ is defined as
\[
T_n = \frac{(-1)^n}{2n!} \sum_{i=1}^n (-1)^i S_n^i.
\]

Let us define the Bott-Chern form of an emi-$n$-cube $\mathcal{F}$ as
\[
\text{ch}_n(\mathcal{F}) = \frac{1}{(2\pi\sqrt{-1})^n} \int_{(\mathbb{P}^1)^n} \text{ch}_0(\text{tr}_n(\mathcal{F})) \wedge T_n \in \oplus_p \mathcal{D}^{2p-n}(M, p).
\]
A process to make an emi-$n$-cube $\lambda F$ from an arbitrary exact hermitian $n$-cube $F$ has been given in [4]. By virtue of this process, we can extend the definition of the Bott-Chern form to an arbitrary exact hermitian $n$-cube.

**Definition 3.1.** The Bott-Chern form of an exact hermitian $n$-cube $F$ is an element of $\bigoplus_p \mathcal{D}^{2p-n}(M, p)$ defined as

$$
\text{ch}_n(F) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{(p_1)^n} \text{ch}_0(\text{tr}_n(\lambda F)) \wedge T_n.
$$

**Theorem 3.2 ([4]).** Let $\hat{P}(M)$ denote the category of hermitian vector bundles on $M$ and let $\tilde{Q} \hat{C}_s(M) = \hat{Q}C_s \hat{P}(M)$. Then the higher Bott-Chern forms induce a homomorphism of complexes

$$
\text{ch} : \tilde{Q} \hat{C}_s(M) \rightarrow \bigoplus_p \mathcal{D}^*(M, p)[2p].
$$

Moreover, the following map

$$
K_n(M)_Q \simeq H_n(\tilde{Q} \hat{C}_s(M)) \xrightarrow{\text{ch}} \bigoplus_p H^{2p-n}_D(M, \mathbb{R}(p))
$$

agrees with the higher Chern character with values in the real Deligne cohomology.

Let $X$ be a smooth proper variety defined over $\mathbb{Q}$. By a hermitian vector bundle $E = (E, h)$ on $X$, we mean a vector bundle $E$ on $X$ with an $i$-invariant smooth hermitian metric $h$ on the holomorphic vector bundle $E(\mathbb{C})$. In the same way as above, one can consider an exact hermitian $n$-cube $F$ on $X$ and define its Bott-Chern form $\text{ch}_n(F)$. Let $\tilde{Q} \hat{C}_s(X)$ denote the complex of exact hermitian cubes on $X$. Then we have an isomorphism preserving products

$$
K_s(X)_Q \simeq H_s(\tilde{Q} \hat{C}_s(X))
$$

and the Bott-Chern forms leads to the regulator map for $X$:

$$
K_n(X)_Q \xrightarrow{\text{ch}} \bigoplus_p H^{2p-n}_D(X, \mathbb{R}(p)) := \bigoplus_p H^{2p-n}_D(X(\mathbb{C}), \mathbb{R}(p))^{\tau=\text{id}}
$$

4. **The main theorem**

We introduce a new real analytic function on $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ coming from the polylogarithm function. Let

$$
I_m(z) = \sum_{j=0}^{m-1} \frac{(- \log |z|)^j}{j!} \text{Li}_{m-j}(z),
$$

where $\text{Li}_s(z)$ is the polylogarithm function.

and
\[ L_m(z) = \mathcal{R}_m \left( \sum_{0 \leq 2r < m} \frac{(-1)^r}{2^r r!} (\log |z|)^{2r} (2m-3)(2m-5) \cdots (2m-2r-1) I_{m-2r}(z) \right). \]

When \( m \leq 3 \), \( L_m(z) \) is equal to \( P_m(z) \), but is not so when \( m \geq 4 \). However, for \( \sum_i n_i [x_i] \in A_m(F) \), we have
\[ \sum_i n_i L_m(x_i^p) = \sum_i n_i P_m(x_i^p). \]

The function \( L_m(z) \) satisfies the following differential equation, which is obtained by a direct calculation.

**Theorem 4.1.** [8, Thm.5.4] If \( m \geq 2 \), then
\[ (-1)^m dL_m(z) = Im \left( \frac{dz}{z} \right) L_{m-1}(z) - \frac{\sqrt{-1}}{2m-3} \log |z| (\partial L_{m-1}(z) - \partial L_{m-1}(z)). \]

Let \( X = \mathbb{P}^1 - \{0, 1, \infty\} \) over \( \mathbb{Q} \) and let \( z \) be the absolute coordinate of \( X \). Hence we can write \( X = \text{Spec} \mathbb{Q}[z, 1/z, 1/(1-z)] \). We want to apply the theory of Bott-Chern forms to \( X \). But since \( X \) is not proper over \( \mathbb{Q} \), we can not apply directly the results mentioned in the preceding section.

For an exact hermitian \( n \)-cube \( F \) on \( X \), one can define \( \text{ch}_n(F) \) as a differential form on \( X(\mathbb{C}) \) by the same integral expression. Moreover, for \( n \geq 2 \), we have \( d \text{ch}_n(F) = -\text{ch}_{n-1}(\partial F) \). Hence when \( n \geq 2 \), \( \text{ch}_n(F) \) induces a map from the rational \( K \)-theory of \( X \) to the de Rham cohomology of \( X(\mathbb{C}) \).

For \( f \in \mathcal{O}_X \), let \( \langle f \rangle \) be an exact hermitian 1-cube on \( X \) given as
\[ 0 \to \mathcal{O}_X \xrightarrow{f} \mathcal{O}_X. \]

**Proposition 4.2.** [8, Prop.6.1] There exists an element \( h_n(z) \in \tilde{\mathcal{Q}}C_{2n-1}(X) \) for each \( n \geq 1 \) satisfying the following conditions:
1. \( h_1(z) = \langle z \rangle \).
2. \( \partial h_n(z) = \sum_{i=1}^{n-1} h_i(z) \otimes h_{n-i}(z) \).
3. \( \text{ch}_{2n-1}(h_n(z)) = 0 \) for \( n \geq 2 \).

**Theorem 4.3.** [8, Thm.6.2] For each \( m \geq 1 \), there exists \( \mathcal{L}_m(z) \in \tilde{\mathcal{Q}}C_{2m-1}(X) \) satisfying the following conditions:
1. \( \mathcal{L}_1(z) = -2 \langle 1-z \rangle \).
2. \( \partial \mathcal{L}_m(z) = \sum_{i=1}^{m-1} 2^i h_i(z) \otimes \mathcal{L}_{m-i}(z) \) for \( m \geq 2 \).
3. If \( \text{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} \) denotes the part of degree 0 of \( \text{ch}_{2m-1}(\mathcal{L}_m(z)) \), then
\[ \text{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = (\sqrt{-1})^{m} L_m(z), \]
where $\alpha_m$ is 0 or 1 according as $m$ is odd or even.

Outline of the proof: We will prove the theorem by induction on $m$. Assume that $\mathcal{L}_1(z), \cdots, \mathcal{L}_{m-1}(z)$ exist. By the product formula for Bott-Chern forms [7, Prop.4.2],

$$
\text{ch}_{2m-2} \left( \sum_{i=1}^{m-1} 2^i h_i(z) \otimes \mathcal{L}_{m-i}(z) \right) = (-\sqrt{-1})^{\alpha_m} \left( Im \left( \frac{dz}{z} \right) L_{m-1}(z) - \frac{\sqrt{-1}}{2m-3} \log |z|(\partial L_{m-1}(z) - \partial L_{m-1}(z)) \right)
$$

$$
= -(\sqrt{-1})^{\alpha_m} dL_m(z).
$$

It can be shown that the map

$$
\text{ch}_{2m-2} : K_{2m-2}(X)_\mathbb{Q} \to H^1_{dR}(X(\mathbb{C}), \mathbb{R}(m-1))^{\tau = \text{id}}
$$

is injective. Hence there exists $\mathcal{L}_m(z) \in \widetilde{\mathbb{Q}}\hat{C}_{2m-1}(X)$ satisfying the equation (2). Then

$$
d\text{ch}_{m-2}(\mathcal{L}_m(z) \otimes 0) = -\text{ch}_{2m-2}(\partial \mathcal{L}_m(z)) = (\sqrt{-1})^{\alpha_m} dL_m(z),
$$

therefore

$$
\text{ch}_{2m-1}(\mathcal{L}_m(z) \otimes 0) = (\sqrt{-1})^{\alpha_m} L_m(z) + a_m
$$

for some constant $a_m$. We can eliminate this constant term by using the Borel’s theorem for the regulator map of $\mathbb{Q}$ [2].

\hfill $\square$

**Theorem 4.4.** [8, Thm.7.2, Thm.7.5] There exists a homomorphism

$$
\mathcal{P}_m : \mathcal{A}_m(F) \to \widetilde{\mathbb{Q}}\hat{C}_{2m-1}(F)
$$

satisfying the following conditions:

1. $\text{ch}_{2m-1}(\mathcal{P}_m(\xi)) = 0$ for any $\xi \in \mathcal{A}_m(F)$.
2. $\partial(\mathcal{L}_m(\xi) + \mathcal{P}_m(\xi)) = 0$ for any $\xi \in \mathcal{A}_m(F)$.

By virtue of the above theorems, one can define a map

$$
\mathcal{B}_m(F) = \mathcal{A}_m(F)/C_m(F) \to K_{2m-1}(F)_\mathbb{Q}
$$

by $[\xi] \mapsto \mathcal{L}_m(\xi) + \mathcal{P}_m(\xi)$. It is easy to see that the composite of this map with the regulator satisfies the condition of the Zagier conjecture.

**References**


Graduate School of Mathematics, Kyushu University 33, Fukuoka, 812-8581, Japan

E-mail address: yutakeda@math.kyushu-u.ac.jp