Let $F$ be an algebraic number field. Let $\Sigma_F = \text{Hom}(F, \mathbb{C})$ and $X_F$ the free abelian group generated by $\Sigma_F$. Let $\iota$ denote the complex conjugation on $\Sigma_F$ and $X_F$. Then one can define a map called regulator

$$\rho : K_{2m-1}(F) \to (X_F \otimes \mathbb{R}(m - 1))^{\iota = \text{id}}.$$ 

In [9], Zagier conjectured that the image of $\rho$ can be expressed in terms of the $m$-th polylogarithm function

$$Li_m(z) = \sum_{k \geq 1} \frac{z^k}{k^m}, |z| < 1.$$ 

More precisely, he conjectured that $K_{2m-1}(F)$ is isomorphic modulo torsion to a certain subquotient $\mathcal{B}_m(F)$ of $\mathbb{Z}[F^\times - \{1\}]$ called the $m$-th Bloch group of $F$, and that the composite of the isomorphism with the regulator $\rho$ can be expressed by values of $Li_m(z)$ at algebraic numbers. The conjecture was solved affirmatively for $m \leq 3$, but it is still open when $m \geq 4$.

As for the construction of the map $\mathcal{B}_m(F) \to K_{2m-1}(F)_{\mathbb{Q}}$, two methods are well-known. Beilinson and Deligne showed that the above map can be constructed under the existence of an appropriate category of mixed Tate motives [1]. On the other hand, de Jeu also constructed the map by using the wedge complexes developed by himself [5]. In this note, we will present an alternative approach. The main tool here is higher Bott-Chern forms developed by Burgos and Wang [4].

1. The Bloch group

First we introduce a one-valued version of the polylogarithm function:

$$P_m(z) = \Re_m \left( \sum_{k=0}^{m-1} \frac{2^k B_k}{k!} (\log |z|)^k Li_{m-k}(z) \right),$$

where $\Re_m$ is $\Re$ or $\Im$ according as $m$ is odd or even and $B_k$ is the $k$-th Bernoulli number. This is a one-valued real analytic function on $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. 
In this note, we are interested in $K$-theory in rational coefficients, therefore we will define Bloch group with rational coefficients. Let $F^\times_Q = F^\times \otimes_\mathbb{Z} \mathbb{Q}$ and define a map

$$\beta_2 : \mathbb{Q}[F^\times - \{1\}] \rightarrow F^\times_Q \wedge F^\times_Q$$

by $[x] \mapsto x \wedge (1 - x)$. Set $A_2(F) = \text{Ker} \; \beta_2$ and

$$C_2(F) = \left\{ \sum_i n_i [x_i] \in A_2(F); \sum_i n_i P_2(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.$$ 

It turns out that $C_2(F)$ is generated by

$$[x] + [y] + \left[ \frac{1 - x}{1 - xy} \right] + \left[ 1 - xy \right] + \left[ \frac{1 - y}{1 - xy} \right]$$

for all $x, y \in F^\times - \{1\}$ with $xy \neq 1$. The quotient group $B_2(F) = A_2(F)/C_2(F)$ is called the Bloch group of $F$.

Suppose $m \geq 3$ and subgroups $C_{m-1}(F) \subset A_{m-1}(F) \subset \mathbb{Q}[F^\times - \{1\}]$ are given. Define a map

$$\beta_m : \mathbb{Q}[F^\times - \{1\}] \rightarrow F^\times_Q \otimes (\mathbb{Q}[F^\times - \{1\}]/C_{m-1}(F))$$

by $\beta_m([x]) = x \otimes [x]$. Set $A_m(F) = \text{Ker} \; \beta_m$ and

$$C_m(F) = \left\{ \sum_i n_i [x_i] \in A_m(F); \sum_i n_i P_m(x_i^\sigma) = 0 \text{ for any } \sigma \in \Sigma_F \right\}.$$ 

The quotient group $B_m(F) = A_m(F)/C_m(F)$ is called $m$-th Bloch group of $F$.

**Zagier Conjecture** ([9]). For $m \geq 2$, the rational algebraic $K$-theory $K_{2m-1}(F)_\mathbb{Q}$ is isomorphic to $B_m(F)$, and the composite

$$B_m(F) \simeq K_{2m-1}(F)_\mathbb{Q} \xrightarrow{\rho} (X_F \otimes \mathbb{R}(m-1))^{\tau = \text{id}}$$

is written as

$$\sum_i n_i [x_i] \mapsto \left( \sqrt{-1} \alpha_m \sum_i n_i P_m(x_i^\sigma) \right)_{\sigma \in \Sigma_F},$$

where $\alpha_m$ is 0 or 1 according as $m$ is odd or even.

2. THE COMPLEX OF EXACT HERMITIAN CUBES

Let $< -1, 0, 1 >^n$ be the ordered set consisting of three elements and $< -1, 0, 1 >^n$ its $n$-th power. For a small exact category $\mathcal{A}$ with a fixed zero object 0, a functor $\mathcal{F} : < -1, 0, 1 >^n \rightarrow \mathcal{A}$ is called an $n$-cube of $\mathcal{A}$. Let $\mathcal{F}_{a_1, \ldots, a_n}$ denote the image of an object $(a_1, \ldots, a_n)$ of $< -1, 0, 1 >^n$. For integers $1 \leq i \leq n$ and $-1 \leq j \leq 1$, an $(n-1)$-cube $\partial_i \mathcal{F}$ is defined by $(\partial_i \mathcal{F})_{a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots, a_n} = \mathcal{F}_{a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots, a_n}$. It is called a face of $\mathcal{F}$. For
an object $\alpha$ of $<-1,0,1>_n$ and an integer $1 \leq i \leq n$, a 1-cube $\partial_i^p F$ called an edge of $F$ is defined by

$$F_{\alpha_1, \cdots, \alpha_{i-1}, -1, \alpha_i, \cdots, \alpha_{n-1}} \rightarrow F_{\alpha_1, \cdots, 0, \alpha_i, \cdots, \alpha_{n-1}} ightarrow F_{\alpha_1, \cdots, \alpha_{i-1}, 1, \alpha_i, \cdots, \alpha_{n-1}}.$$    

An $n$-cube $F$ is said to be exact if all edges of $F$ are short exact sequences.

Let $C_n A$ denote the set of all exact $n$-cubes of $A$. If $F$ is an exact $n$-cube, then any face $\partial_i^j F$ is also exact. Hence $\partial_i^j$ induces a map

$$\partial_i^j : C_n A \rightarrow C_{n-1} A.$$    

Let $F$ be an exact $n$-cube of $A$. For an integer $1 \leq i \leq n + 1$, let $s_i^1 F$ be an exact $(n + 1)$-cube such that its edge $\partial_i^p (s_i^1 F)$ is $F_{\alpha} \overset{id}{\rightarrow} F_{\alpha}$ to 0. Similarly, let $s_i^{-1} F$ be an exact $(n + 1)$-cube such that $\partial_i^p (s_i^{-1} F)$ is 0 $\rightarrow F_{\alpha} \overset{id}{\rightarrow} F_{\alpha}$. An exact cube written as $s_i^j F$ is said to be degenerate.

Let $\tilde{Q}C_n A$ be the free $\tilde{Q}$-module generated by $C_n A$ and $D_n \subset \tilde{Q}C_n A$ the submodule generated by all degenerate exact $n$-cubes. Let $\tilde{Q}C_n A = \tilde{Q}C_n A / D_n$ and

$$\partial = \sum_{i=1}^{n} \sum_{j=-1}^{1} (-1)^{i+j+1} \partial_i^j : \tilde{Q}C_n A \rightarrow \tilde{Q}C_{n-1} A.$$ Then $\tilde{Q}C_n A = (\tilde{Q}C_n A, \partial)$ becomes a homological complex.

**Theorem 2.1 ([6]).** The homology of $(\tilde{Q}C_n A, \partial)$ is isomorphic to the rational algebraic $K$-theory of $A$:

$$H_n(\tilde{Q}C_n A, \partial) \simeq K_n(A)_{\tilde{Q}}.$$ This isomorphism preserves products on the both sides if $A$ is equipped with a strictly associative tensor product.

### 3. The higher Bott-Chern forms

Let $M$ be a compact complex algebraic manifold, namely, the analytic space consisting of all $\mathbb{C}$-valued points of a smooth proper algebraic variety over $\mathbb{C}$. Let $\mathcal{E}_R^p(M)$ be the space of real smooth differential forms of degree $p$ on $M$ and $\mathcal{E}^p(M) = \mathcal{E}_R^p(M) \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{E}_{p,q}(M)$ be the space of complex differential forms of type $(p,q)$ on $M$. Set

$$\mathcal{D}^n(M, p) = \begin{cases} \mathcal{E}_{R}^{n-1}(M)(p-1) \cap \mathcal{E}^{p',q}(M), & n < 2p, \\
\mathcal{E}_{R}^{2p}(M)(p) \cap \mathcal{E}^{p,p}(M) \cap \text{Ker} d, & n = 2p, \\
0, & n > 2p\end{cases}$$
and define a differential \( d_D : \mathcal{D}^n(M, p) \rightarrow \mathcal{D}^{n+1}(M, p) \) by

\[
d_D(\omega) = \begin{cases} 
-\pi(d\omega), & n < 2p - 1, \\
-2\partial\bar{\partial}\omega, & n = 2p - 1, \\
0, & n > 2p - 1,
\end{cases}
\]

where \( \pi : \mathcal{E}^n(M) \rightarrow \mathcal{D}^n(M, p) \) is the canonical projection. Then it is shown in [3, Thm.2.6] that \((\mathcal{D}^*(M, p), d_D)\) becomes a complex of \(\mathbb{R}\)-vector spaces computing the real Deligne cohomology, that is, for \( n \leq 2p \) we have

\[
H^n(\mathcal{D}^*(M, p), d_D) \cong H^n_{DR}(M, \mathbb{R}(p)).
\]

By a hermitian vector bundle \( \mathcal{E} = (E, h) \) on \( M \) we mean an algebraic vector bundle \( E \) on \( M \) with a smooth hermitian metric \( h \). Let \( K_{\mathcal{E}} \) denote the curvature form of the unique connection on \( \mathcal{E} \) that is compatible with both the metric and the complex structure. The Chern form of \( \mathcal{E} \) is defined as

\[
\text{ch}_0(\mathcal{E}) = \text{Tr}(\exp(-K_{\mathcal{E}})) \in \oplus_p \mathcal{D}^{2p}(M, p).
\]

An exact hermitian \( n \)-cube on \( M \) is an exact \( n \)-cube made of hermitian vector bundles on \( M \). Let \( \mathcal{F} = \{\mathcal{E}_\alpha\} \) be an exact hermitian \( n \)-cube on \( M \). We call \( \mathcal{F} \) an emi-\( n \)-cube if the metric on any \( \mathcal{E}_\alpha \) with \( \alpha_i = 1 \) coincides with the metric induced from \( \mathcal{E}_{\alpha_1,\ldots,\alpha_i,0,\alpha_{i+1},\ldots,\alpha_n} \) by the surjection \( \mathcal{E}_{\alpha_1,\ldots,\alpha_i,0,\alpha_{i+1},\ldots,\alpha_n} \rightarrow \mathcal{E}_\alpha \).

For an emi-1-cube \( \mathcal{E} : \mathcal{E}_{-1} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \), a canonical way of constructing a hermitian vector bundle \( \text{tr}_1 \mathcal{E} \) on \( M \times \mathbb{P}^1 \) connecting \( \mathcal{E}_0 \) with \( \mathcal{E}_{-1} \oplus \mathcal{E}_1 \) is given in [4]. If \((x : y)\) denotes the homogeneous coordinate of \( \mathbb{P}^1 \) and \( z = x/y \), then \( \text{tr}_1 \mathcal{E} \) fulfills the conditions \( (\text{tr}_1 \mathcal{E})|_{z=0} \cong \mathcal{E}_0 \) and \( (\text{tr}_1 \mathcal{E})|_{z=\infty} \cong \mathcal{E}_{-1} \oplus \mathcal{E}_1 \). For an emi-\( n \)-cube \( \mathcal{F} \), let \( \text{tr}_1(\mathcal{F}) \) be an emi-(\( n-1 \))-cube on \( M \times \mathbb{P}^1 \) defined by \( \text{tr}_1(\mathcal{F})_\alpha = \text{tr}_1(\partial_{\mathcal{E}_\alpha}(\mathcal{F})) \) for any object \( \alpha \) of \( < -1, 0, 1 >^{n-1} \), and \( \text{tr}_n(\mathcal{F}) \) a hermitian vector bundle on \( M \times (\mathbb{P}^1)^n \) given by taking \( \text{tr}_1 \) \( n \) times.

Let \( \pi_i : (\mathbb{P}^1)^n \rightarrow \mathbb{P}^1 \) be the \( i \)-th projection and \( z_i = \pi_i^*z \). Let \( S_n \) be the symmetric group of \( n \)-letters. For an integer \( 1 \leq i \leq n \), a differential form with logarithmic poles \( S_n^i \) on \( (\mathbb{P}^1)^n \) is defined as

\[
S_n^i = \sum_{\sigma \in S_n} (-1)^\sigma \log |z_\sigma(1)|^2 \frac{dz_\sigma(2)}{z_\sigma(2)} \wedge \cdots \wedge \frac{dz_\sigma(i)}{z_\sigma(i)} \wedge \frac{d\bar{z}_\sigma(i+1)}{\bar{z}_\sigma(i+1)} \wedge \cdots \wedge \frac{d\bar{z}_\sigma(n)}{\bar{z}_\sigma(n)},
\]

and \( T_n \) is defined as

\[
T_n = \frac{(-1)^n}{2n!} \sum_{i=1}^n (-1)^i S_n^i.
\]

Let us define the Bott-Chern form of an emi-\( n \)-cube \( \mathcal{F} \) as

\[
\text{ch}_n(\mathcal{F}) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{(\mathbb{P}^1)^n} \text{ch}_0(\text{tr}_n(\mathcal{F})) \wedge T_n \in \oplus_p \mathcal{D}^{2p-n}(M, p).
\]
A process to make an emi-$n$-cube $\lambda F$ from an arbitrary exact hermitian $n$-cube $F$ has been given in [4]. By virtue of this process, we can extend the definition of the Bott-Chern form to an arbitrary exact hermitian $n$-cube.

**Definition 3.1.** The Bott-Chern form of an exact hermitian $n$-cube $F$ is an element of $\oplus_p D^{2p-n}(M, p)$ defined as

$$\text{ch}_n(F) = \frac{1}{(2\pi \sqrt{-1})^n} \int_{(p_1)^n} \text{ch}_0(\text{tr}_n(\lambda F)) \wedge T_n.$$  

**Theorem 3.2 ([4]).** Let $\mathcal{P}(M)$ denote the category of hermitian vector bundles on $M$ and let $\mathcal{Q}\mathcal{C}_*(M) = \mathcal{Q}C_*\mathcal{P}(M)$. Then the higher Bott-Chern forms induce a homomorphism of complexes

$$\text{ch} : \mathcal{Q}\mathcal{C}_*(M) \to \oplus_p D^*(M, p)[2p].$$

Moreover, the following map

$$K_\pi(M)_\mathbb{Q} \simeq H_n(\mathcal{Q}\mathcal{C}_*(M)) \text{ ch} \oplus H_{2p-n}(M, \mathbb{R}(p))$$

agrees with the higher Chern character with values in the real Deligne cohomology.

Let $X$ be a smooth proper variety defined over $\mathbb{Q}$. By a hermitian vector bundle $E = (E, h)$ on $X$, we mean a vector bundle $E$ on $X$ with an $i$-invariant smooth hermitian metric $h$ on the holomorphic vector bundle $E(\mathbb{C})$. In the same way as above, one can consider an exact hermitian $n$-cube $F$ on $X$ and define its Bott-Chern form $\text{ch}_n(F)$. Let $\mathcal{Q}\mathcal{C}_*(X)$ denote the complex of exact hermitian cubes on $X$. Then we have an isomorphism preserving products

$$K_\pi(X)_\mathbb{Q} \simeq H_*\mathbb{C}_*(X)$$

and the Bott-Chern forms leads to the regulator map for $X$:

$$K_\pi(X)_\mathbb{Q} \text{ ch} \oplus H_{2p-n}(X, \mathbb{R}(p)) := \oplus_p H_{2p-n}(X(\mathbb{C}), \mathbb{R}(p))^{\tau = \text{id}}$$

4. The Main Theorem

We introduce a new real analytic function on $\mathbb{P}^1_\mathbb{C} - \{0, 1, \infty\}$ coming from the polylogarithm function. Let

$$I_m(z) = \sum_{j=0}^{m-1} \frac{(- \log |z|)^j}{j!} Li_{m-j}(z),$$

where $Li_j(z)$ is the polylogarithm function.
and
\[ L_m(z) = R_m \left( \sum_{0 \leq 2r < m} \frac{(-1)^r}{2^r r!} \frac{\log |z|^{2r}}{(2m-3)(2m-5) \cdots (2m-2r-1)} I_{m-2r}(z) \right). \]

When \( m \leq 3 \), \( L_m(z) \) is equal to \( P_m(z) \), but is not so when \( m \geq 4 \). However, for \( \sum_i n_i [x_i] \in A_m(F) \), we have
\[ \sum_i n_i L_m(x_i^\sigma) = \sum_i n_i P_m(x_i^\sigma). \]

The function \( L_m(z) \) satisfies the following differential equation, which is obtained by a direct calculation.

**Theorem 4.1.** [8, Thm.5.4] If \( m \geq 2 \), then
\[ (-1)^m dL_m(z) = \text{Im} \left( \frac{dz}{z} \right) L_{m-1}(z) - \frac{\sqrt{-1}}{2m-3} \log |z| (\overline{\partial L_{m-1}(z)} - \partial L_{m-1}(z)). \]

Let \( X = \mathbb{P}^1 - \{0, 1, \infty\} \) over \( \mathbb{Q} \) and let \( z \) be the absolute coordinate of \( X \). Hence we can write \( X = \text{Spec} \mathbb{Q}[z, 1/z, 1/(1-z)] \). We want to apply the theory of Bott-Chern forms to \( X \). But since \( X \) is not proper over \( \mathbb{Q} \), we can not apply directly the results mentioned in the preceding section.

For an exact hermitian \( n \)-cube \( \mathcal{F} \) on \( X \), one can define \( \text{ch}_n(\mathcal{F}) \) as a differential form on \( X(\mathbb{C}) \) by the same integral expression. Moreover, for \( n \geq 2 \), we have \( d \text{ch}_n(\mathcal{F}) = -\text{ch}_{n-1}(\partial \mathcal{F}) \). Hence when \( n \geq 2 \), \( \text{ch}_n(\mathcal{F}) \) induces a map from the rational \( K \)-theory of \( X \) to the de Rham cohomology of \( X(\mathbb{C}) \).

For \( f \in \mathcal{O}_X^\infty \), let \( \langle f \rangle \) be an exact hermitian 1-cube on \( X \) given as
\[ 0 \to \overline{\mathcal{O}_X} \overset{f}{\to} \mathcal{O}_X. \]

**Proposition 4.2.** [8, Prop.6.1] There exists an element \( h_n(z) \in \overline{Q} \hat{C}_{2n-1}(X) \) for each \( n \geq 1 \) satisfying the following conditions:
1. \( h_1(z) = \langle z \rangle \).
2. \( \partial h_n(z) = \sum_{i=1}^{n-1} h_i(z) \otimes h_{n-i}(z) \).
3. \( \text{ch}_{2n-1}(h_n(z)) = 0 \) for \( n \geq 2 \).

**Theorem 4.3.** [8, Thm.6.2] For each \( m \geq 1 \), there exists \( L_m(z) \in \overline{Q} \hat{C}_{2m-1}(X) \) satisfying the following conditions:
1. \( L_1(z) = -2 \langle 1 - z \rangle \).
2. \( \partial L_m(z) = \sum_{i=1}^{m-1} 2^i h_i(z) \otimes L_{m-i}(z) \) for \( m \geq 2 \).
3. If \( \text{ch}_{2m-1}(L_m(z))^{(0)} \) denotes the part of degree 0 of \( \text{ch}_{2m-1}(L_m(z)) \), then
\[ \text{ch}_{2m-1}(L_m(z))^{(0)} = (\sqrt{-1})^{m} L_m(z), \]
where $\alpha_m$ is 0 or 1 according as $m$ is odd or even.

Outline of the proof: We will prove the theorem by induction on $m$. Assume that $\mathcal{L}_1(z), \cdots, \mathcal{L}_{m-1}(z)$ exist. By the product formula for Bott-Chern forms [7, Prop.4.2],

$$
\text{ch}_{2m-2} \left( \sum_{i=1}^{m-1} 2^i h_i(z) \otimes \mathcal{L}_{m-i}(z) \right) = (-\sqrt{-1})^{\alpha_m} \left( \text{Im} \left( \frac{dz}{z} \right) \mathcal{L}_{m-1}(z) - \frac{\sqrt{-1}}{2m-3} \log |z| (\partial \mathcal{L}_{m-1}(z) - \partial \mathcal{L}_{m-1}(z)) \right)
$$

$$
= - (\sqrt{-1})^{\alpha_m} d\mathcal{L}_m(z).
$$

It can be shown that the map

$$
\text{ch}_{2m-2} : K_{2m-2}(X)_\mathbb{Q} \to H^1_{dR}(X(\mathbb{C}), \mathbb{R}(m-1))^{\zeta = \text{id}}
$$

is injective. Hence there exists $\mathcal{L}_m(z) \in \tilde{\mathbb{Q}}\hat{C}_{2m-1}(X)$ satisfying the equation (2). Then

$$
d \text{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = - \text{ch}_{2m-2}(\partial \mathcal{L}_m(z)) = (\sqrt{-1})^{\alpha_m} d\mathcal{L}_m(z),
$$

therefore

$$
\text{ch}_{2m-1}(\mathcal{L}_m(z))^{(0)} = (\sqrt{-1})^{\alpha_m} \mathcal{L}_m(z) + a_m
$$

for some constant $a_m$. We can eliminate this constant term by using the Borel’s theorem for the regulator map of $\mathbb{Q}$ [2].

\begin{flushright}
$\Box$
\end{flushright}

**Theorem 4.4.** [8, Thm.7.2, Thm.7.5] There exists a homomorphism

$$
P_m : A_m(F) \to \tilde{\mathbb{Q}}\hat{C}_{2m-1}(F)
$$

satisfying the following conditions:

1. $\text{ch}_{2m-1}(P_m(\xi)) = 0$ for any $\xi \in A_m(F)$.
2. $\partial(\mathcal{L}_m(\xi) + P_m(\xi)) = 0$ for any $\xi \in A_m(F)$.

By virtue of the above theorems, one can define a map

$$
\mathcal{B}_m(F) = A_m(F)/C_m(F) \to K_{2m-1}(F)_\mathbb{Q}
$$

by $[\xi] \mapsto \mathcal{L}_m(\xi) + P_m(\xi)$. It is easy to see that the composite of this map with the regulator satisfies the condition of the Zagier conjecture.

**References**


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