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Note	Hodge Theory and Algebraic Geometry 2002/10/7-11 Department of Mathematics, Hokkaido University 石井志保子（東工大）Nash problem on arc families for singularities 内藤広嗣　（名大多元） 村上雅亮（京大理）Surfaces with c^2_1= 3 and \kai(O) = 2, which have non-trivial 3-torsion divisors 大野浩二（大阪大）On certain boundedness of fibred Calabi-Yau s threefolds 阿部健（京大理） 春井岳（大阪大）The gonarlity of curves on an elliptic ruled surface 山下剛（東大数理）開多様体のp進etale cohomology と crystalline cohomology 中島幸喜（東京電機大）Theorie de Hodge III pour cohomologies p-adiques 皆川龍博　（東工大）On classification of weakened Fano 3-folds 斉藤夏男（東大数理）Fano threefold in positive characteristic 石井亮（京大工）Variation of the representation moduli of the McKay quiver 前野俊昭（京大理）群のコホモロジーと量子変形 宮岡洋一（東大数理）　次数が低い有理曲線とファノ多様体 池田京司（大阪大）Subvarieties of generic hypersurfaces in a projective toric variety 竹田雄一郎（九大数理）Complexes of hermitian cubes and the Zagier conjecture 臼井三平（大阪大）SL(2)-orbit theorem and log Hodge structures (Joint work with Kazuya Kato) 鈴木香織（東大数理) \rho(X) = 1, f \le 2 のQ-Fano 3-fold Fanoの分類
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On certain boundedness of fibred Calabi-Yau's threefolds *

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Abstract

In this report, the Euler characteristic formula for projective logarithmic minimal degenerations of surfaces with Kodaira dimension zero over a 1-dimensional complex disk is given under a reasonable assumption and as its application, the singularity of logarithmic minimal degenerations are determined in the abelian or hyperelliptic case. By globalizing this local analysis of singular fibres via generalized canonical bundle formulae due to Fujino-Mori, we bound the number of singular fibres of abelian fibred Calabi-Yau threefolds from above, which was previously done by Oguiso in the potentially good reduction case.

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1 Introduction

Based on the 2-dimensional minimal model theory, Kodaira classified the singular fibres of degenerations of elliptic curves ([19], Theorem 6.2). It is quite natural that many people have been interested in the degenerations of surfaces with Kodaira dimension zero as a next problem. The first effort began by his student Iitaka and Ueno who studied the first kind degeneration (i.e., degeneration with the finite monodromy) of abelian surfaces with a principal polarization ([42] and [43]) while in that time, 3-dimensional minimal model theory had not been known. After Kulikov succeeded to construct the minimal models of degenerations of algebraic K3 surfaces in the analytic category from semistable degenerations and to classify their singular fibres ([22]), extension to the case of the other surfaces with Kodaira dimension zero has been done (see for example, [33], [26], [38]). As for the

*This is a resume of [32]

non-semistable case, there are works due to Crauder and Morrison who classified triple point free degeneration ([3], [4]). 3-dimensional minimal model theory in the projective category was established by Mori ([28]) but we can not start studying the degenerations from minimal models because of their complexity while it has been known that log minimal models of degenerations of elliptic curves behaves nicely (see [35], (8.9) Added in Proof.). After the establishment of 3-dimensional log minimal model theory, we introduced the notion of a *logarithmic minimal degeneration* in [31] as a good intermediate model to a minimal model which acts like a "quotient" of minimal semistable degeneration by the transformation group induced from a semistable reduction. Of course, because of the non-uniqueness of minimal models, the transformation group does not act holomorphically on the total space in general.

Definition 1.1 Let $f : X \to B$ be a proper connected morphism from a normal $\mathbf{Q}$-factorial variety defined over the complex number field $\boldsymbol{C}$ (resp. a normal $\mathbf{Q}$-factorial complex analytic space) X onto a smooth projective curve (resp. a unit disk $\mathcal{D} := \{z \in \boldsymbol{C}; |z| < 1\}$) B such that a general fibre $f^*(p)$ (resp. any fibre $f^*(p)$ where p is not the origin) is a normal algebraic variety with only terminal singularity. Let Σ be a set of points in B (resp. the origin 0) such that the fibre $f^*(p)$ is not a normal algebraic variety with only terminal singularity. Put $\Theta_p := f^*(p)_{\mathrm{red}}$ and $\Theta := \sum_{p\in\Sigma} \Theta_p$.

(1) $f : X \to B$ is called a *minimal fibration* (*resp. degeneration*) if X has only terminal singularity and K_X is f-nef (i.e., The intersection number of K_X and any complete curve contained in a fibre of f is non-negative).

(2) $f : X \to B$ is called a *logarithmic minimal* (*or abbreviated, log minimal*) *fibration* (*resp. degeneration*) if (X, Θ) is divisorially log terminal and $K_X + \Theta$ is f-nef.

(3) $f : X \to B$ is called a *strictly logarithmic minimal* (*or abbreviated, strictly log minimal*) *fibration* (*resp. degeneration*) if (X, Θ) is log canonical with K_X, Θ being both f-nef.

Remark 1.1 We note that any fibrations (resp. degenerations) of algebraic surfaces with Kodaira dimension zero over a smooth projective curve (resp. 1-dimensional unit disk) B are birationally (resp. bimeromorphically) equivalent over B to a projective log minimal fibration (resp. degeneration) and also to a projective strictly log minimal fibration (resp. degeneration). In fact, firstly we can take a birational (resp. bimeromorphic) model $g : Y \to B$, where g is a relatively projective connected morphism from a smooth variety (resp. complex analytic space) Y such that for any singular fibre of g, its support has only simple normal crossing singularity with each component smooth by the Hironaka's theorem ([12]). Let $\Sigma \subset B$ be a set of all the points such that g is not smooth over $p \in B$ (resp. the origin 0). By the existence theorem of log minimal models established in [37], [16], Theorem 2, [21], Theorem 1.4, we can run the log minimal program with respect to $K_Y + \sum_{p\in\Sigma} g^*(p)_{\mathrm{red}}$ starting from Y to get a log minimal model $f : X \to B$ which is a log minimal fibration (resp. degenerations) in the sense of Definition 1.1. Here we note that by the Base Point Free Theorem in [29], we infer that $K_X + \Theta \sim_{\boldsymbol{Q}} f^*D$ for some $\mathbf{Q}$-divisor D on B. By applying the log minimal program with respect to K_X starting from X, we obtain a model $f^s : X^s \to B$ which obviously turns out to be a strictly log minimal fibration (resp. degenerations).

Definition 1.2 Let G be a finite group and $\rho : G \to \mathrm{GL}(3, \boldsymbol{C})$ be a faithful representation. Let $\boldsymbol{C}^3/(G, \rho)$ denote the quotient of $\boldsymbol{C}^3$ by the action of G defined by ρ. We assume that the quotient map $\boldsymbol{C}^3 \to \boldsymbol{C}^3/(G, \rho)$ is étale in codimension one. A pair (X, D) which consists of a normal complex analytic space X and a reduced divisor D on X is said to *have singularity of type* $V_1(G, \rho)$ (*resp.* $V_2(G, \rho)$) *at* $p \in X$ if there exists an analytic isomorphism $\varphi : (X, p) \to (\boldsymbol{C}^3/(G, \rho), 0)$ between germs and a hypersurface H in $\boldsymbol{C}^3$ defined by the equation $z = 0$ (resp. $xy = 0$), where x, y and z

are semi-invariant coordinates of $\boldsymbol{C}^3$ at 0 such that $D = \varphi^*(H/(G,\rho))$. In particular, if G is cyclic with a generator $\sigma \in G$ and $(\rho(\sigma)^*x, \rho(\sigma)^*y, \rho(\sigma)^*z) = (\zeta^a x, \zeta^b y, \zeta^c z)$, where a, b, $c \in \boldsymbol{Z}$ and ζ is a primitive r-th root of unity for some coordinate x, y and z of $\boldsymbol{C}^3$ at 0, we shall use the notation $V_1(r; a, b, c)$ (resp. $V_2(r; a, b, c)$) instead of $V_1(G, \rho)$ (resp. $V_2(G, \rho)$).

Remark 1.2 We note that if (X, D) has singularity of type $V_i(G, \rho)$ at p, the local fundamental group at p of the singularity of X is isomorphic to G by its definition.

Let $f : X \to \mathcal{D}$ be a log minimal degeneration and let $\Theta = \sum_i \Theta_i$ be the irreducible decomposition and put $\Delta_i := \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i)$ for any i. For $p \in X$, let $d(p)$ be the number of irreducible components of Θ passing through $p \in X$. Then the followings hold.

- **(a)** For any i, Θ_i is normal, Δ_i is a standard boundary and (Θ_i, Δ_i) is log terminal (see [37], Lemma 3.6, (3.2.3) and Corollary 3.10).
- **(b)** $d(p) \leq 3$.
- **(c)** If $d(p) = 2$, (X, Θ) has singularity of type $V_2(r; a, b, 1)$ at p, where $r \in \boldsymbol{N}$, $a, b \in \boldsymbol{Z}$ and $(r, a, b) = 1$ (see [2], Theorem 16.15.2).
- **(d)** If $d(p) = 3$, $p \in \Theta \subset X$ is analytically isomorphic to the germ of the origin $0 \in \{(x, y, z); xyz = 0\} \subset \boldsymbol{C}^3$ (see [2], Theorem 16.15.1).
- **(e)** For any i and $p \in \Theta_i \setminus \mathrm{Supp}\ \Delta_i$, if Θ_i is smooth at p, then X is smooth at p (see [37], Corollary 3.7).

One of the aims of this paper is to give the following Euler characteristic formula for log minimal degenerations with $K_X + \Theta$ being Cartier. We note here that the study of log minimal degenerations of surfaces with Kodaira dimension zero reduces to this case by taking the log canonical cover with respect to $K_X + \Theta$ globally (see §6).

Theorem 1.1 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of surfaces with Kodaira dimension zero such that $K_X + \Theta$ is Cartier. Let $f^*(0) = \sum_i m_i \Theta_i$ be the irreducible decomposition. Then for $t \in \mathcal{D}^* := \mathcal{D} \setminus \{0\}$, the following formula holds.*

$$e_{\mathrm{top}}(X_t) = \sum m_i (e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i) + \sum_{p \in \Theta_i \setminus \Delta_i} \delta_p(X, \Theta_i)),$$

where $X_t := f^*(t)$, $e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i)$ is the orbifold Euler number of $\Theta_i \setminus \Delta_i$ and $\delta_p(X, \Theta_i)$ is the invariant of the singularity of the pair (X, Θ_i) at $p \in \Theta_i \setminus \Delta_i$ which is well defined and can be calculated explicitly as explained in the next section.

The above formula turns out to be quite useful for further study of degenerations. In fact, we apply the following corollary to the study on non-semistable degenerations of abelian or hyperelliptic surfaces.

Corollary 1.1 *Let notation and assumptions be as in Theorem 1.1. Assume that $e_{\mathrm{top}}(X_t) = 0$ for $t \in \mathcal{D}^*$. Then, for any i, we have $e_{\mathrm{orb}}(\Theta_i \setminus \Delta_i) = 0$ and for any $p \in \Theta_i \setminus \Delta_i$, (X, Θ) has only singularity of type $V_1(r; a, -a, 1)$ at p, where $(r, a) = 1$.*

Based on the result of Corollary 1.1, we get the following theorem.

Theorem 1.2 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of abelian or hyperelliptic surfaces, not neccesarily assuming that $K_X + \Theta$ is Cartier. Then the possible singularities of (X, Θ) at $p \in X$ are the following three types :*

(0) *X is smooth at $p \in X$ and Θ has only normal crossing singularity at p,*

(1) *(X, Θ) has singularity of type $V_2(r; a, b, 1)$ at p, where $r \in \boldsymbol{N}$, $a, b \in \boldsymbol{Z}$ and $(r, a, b) = 1$.*

(2) *(X, Θ) has singularity of type $V_1(G, \rho)$ at p.*

More precisely, if f is of type II, *we have $r = 2, 3, 4$ or 6 in (1), and $G \simeq \boldsymbol{Z}/n\boldsymbol{Z}$ or $\boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/n\boldsymbol{Z}$, where $n = 2, 3, 4$ or 6 in (2). The dual graph of Θ is a linear chain or a cycle. Moreover, there exists a projective bimeromorphic morphism $\psi : X \to X^s$ over $\mathcal{D}$ such that for the induced projective degeneration $f^s : X^s \to \mathcal{D}$, we have $K_{X^s} \sim_{\boldsymbol{Q}} 0$ and $f^{s*}(0) = m\Theta^s$ for some $m \in \boldsymbol{N}$, where $\Theta^s := \psi_*\Theta$. The possible types of singularity of (X^s, Θ^s) and the dual graph of the support of the singular fibre are the same as ones of (X, Θ) (but the components of the singular fiber may become non-normal). If f is of type* III, *we have $r = 2$ in (1), and (2) is reduced to the following three types.*

(III-2.1) *(X, Θ) has singularity of type $V_1(r; a, -a, 1)$ at p, where $r = 2, 3, 4$ or 6, $a \in \boldsymbol{Z}$ and $(r, a) = 1$,*

(III-2.2) *(X, Θ) has singularity of type $V_1(2; 1, 0, 1)$ at p,*

(III-2.3) *(X, Θ) has singularity of type $V_1(G, \rho)$ at p, where $G \simeq \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z}$ and letting $\{\sigma, \tau\}$ denote a set of generators,*

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, if f is of type III, *then X has only canonical quotient singularity.*

For the definition of types I, II and III, see Definition 3.3.

Problem 1.1 Let $f : X \to \mathcal{D}$ a projective log minimal degeneration of abelian or hyperelliptic surfaces of type III. Applying the log minimal program to f with respect to K_X, we see that there exists a projective bimeromorphic map $\psi : X - \to X^s$ over $\mathcal{D}$ such that for the induced projective degeneration $f^s : X^s \to \mathcal{D}$, we have $K_{X^s} \sim_{\boldsymbol{Q}} 0$ and $f^{s*}(0) = m\Theta^s$ for some $m \in \boldsymbol{N}$, where $\Theta^s := \psi_*\Theta$ and that X^s has only canonical singularity but the possible types of singularity of (X^s, Θ^s) may differ from the ones of (X, Θ). So determination of the types of singularity of (X^s, Θ^s) remains to be done.

Refining a canonical bundle formula in [8] by the Log Minimal Model Program, we obtain the following theorem as an application of Theorem 1.2. [8].

Theorem 1.3 (Theorem 5.1) *The number of singular fibres of abelian fibred Calabi Yau threefolds over a smooth rational curves are bounded from above not depending on relative polarizations.*

Remark 1.3 (1) The number of singular fibres and the bounding problem of Euler numbers are closely related to each other. [13] asserts the boundedness of Euler numbers of fibred Calabi-Yau threefolds, but unfortunately, [13] contains a several crucial gaps. In fact, one of the aims of this paper is to remedy the results in [13]. For example, the crucial Lemma 4 in [13] has a counter-example as follows. Let $f : X \to \mathcal{D}$ be a projective connected morphism from a complex manifold X onto a 1-dimensional complex disk $\mathcal{D}$. Assume that $f_\circ := f|_{f^{-1}(\mathcal{D}^*)}$ is a smooth family of abelian varieties, where $\mathcal{D}^* := \mathcal{D} \setminus \{0\}$. Assume moreover f does not admit any sections. According to [44], there exists a projective morphism $f^b : X^b \to \mathcal{D}$ from a complex manifold X^b onto $\mathcal{D}$ such that $f^b_\circ := f^b|_{f^{b-1}(\mathcal{D}^*)}$ is a basic polarized bundle associated with $f_\circ$, that is, the pairs of period maps and the monodromies associated with $f_\circ$ and $f^b_\circ$ are equivalent. But f and f^b are not bimeromorphically equivalent because f^b admits a section while f does not. By the same reason, the assertion in §3, Step 1 in [13] saying that $\pi' : X' \to Y'$ is birational to the pull back of $g' : \mathcal{F}_{\Gamma'} \to \Gamma' \setminus \mathcal{D}^\wedge$ is incorrect.

(2) Moreover, fixing the degrees of direct image sheaves of relative dualizing sheaves as in [13] gives no condition on the number of singular fibres. For example, for any given integer g, there exists a elliptic surface $f_g : X_g \to B$ over a projective line B such that $\deg f_{g*}\mathcal{O}(K_{X_g}/B) = 0$ and f_g has $2g+2$ singular fibres. These can be constructed by taking quotients of products of elliptic curves and hyperelliptic curves by involutions which are products of translations by torsion point of order two and canonical involutions of hyperelliptic curves. These are not the counter-example of boundedness of Euler numbers of fibred Calabi-Yau threefolds, but at least one has to care about the number of singular fibres which contribute to the Euler numbers. It seems that there is no argument like that in [13].

(3) As for independence of boudedness on the relative polarizations, [13] seems to be using the fact that abelian varieties defined over an algebraically closed field are isogeneous to a principally polarized abelian varieties. The problem is to bound the degree of the neccesay base change but the argument [13], pp.150, Corollary does not seem to be successful (It seems that p_Γ in the proof is just the isomorphism). Instead, we used the Zarhin's trick in our argument.

Notation and Conventions

Let X be a normal variety defined over an algebraically closed field k (if the characteristic of k is not zero, we assume the existence of a embedded resolution). An elements of Weil X $\otimes$ $\boldsymbol{Q}$ is called a $\mathbf{Q}$-*divisor.* $\mathbf{Q}$-divisor D has the unique irreducible decomposition $D = \sum_\Gamma (\mathrm{mult}_\Gamma D)\Gamma$, where $\mathrm{mult}_\Gamma D \in \boldsymbol{Q}$ and the summation is taken over all the prime divisors Γ on X. $\mathbf{Q}$-divisor Δ is called a $\mathbf{Q}$-*boundary* if $\mathrm{mult}_\Gamma \Delta \in [0,1] \cap \boldsymbol{Q}$ for any prime divisor Γ. $\mathbf{Q}$-divisor D is said to be $\mathbf{Q}$-*Cartier* if $rD \in \mathrm{Div}\ X$ for some $r \in \boldsymbol{Q}$. X is said to be $\mathbf{Q}$-Gorenstein if a canonical divisor K_X is $\mathbf{Q}$-Cartier. X is said to be $\mathbf{Q}$-*factorial* if any Weil divisor on X is $\mathbf{Q}$-Cartier. A pair (X, Δ) which consists of a normal variety X and $\mathbf{Q}$-boundary Δ on X is called a normal *log variety.* For a normal log variety (X, Δ), a resolution $\mu : Y \to X$ is called a *log resolution* of (X, Δ) if each component of the support of $\mu_*^{-1}\Delta + \sum_{i\in I} E_i$ are smooth and cross normally, where $\{E_i\}_{i\in I}$ is a set of all the exceptional divisors of μ. Assume that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. The *log discrepancy* $a_l(E_i; X, \Delta) \in \boldsymbol{Q}$ of E_i with respect to (X, Δ) is defined by

$$a_l(E_i; X, \Delta) := \mathrm{mult}_{E_i}(K_Y + \mu_*^{-1}\Delta + \sum_{i\in I} E_i - \mu^*(K_X + \Delta)) \in \boldsymbol{Q}$$

and the *discrepancy* $a(E_i; X, \Delta) \in \boldsymbol{Q}$ is defined by $a(E_i; X, \Delta) := a_l(E_i; X, \Delta) + 1$. The closure of $\mu(E_i) \subset X$ is called a *center* of E_i at X which is denoted by $\mathrm{Center}_X(E_i)$. The above definitions of discrepancies, a log discrepancies and centers are known to be well-defined and depend only on the rank one discrete valuation of the function field of X associated with E_i's. E_i's are called a *exceptional divisors of the function field of* X and it has its meaning saying discrepancies, a log discrepancies and centers of exceptional divisors of the function field of X. A normal log variety (X, Δ) is said to be *terminal* (resp. *canonical*, resp. *purely log terminal*) if $a(E_i; X, \Delta) > 0$ (resp. $a(E_i; X, \Delta) \geq 0$, resp. $a_l(E_i; X, \Delta) > 0$) for any log resolution μ and any $i \in I$. For some log resolution μ, if $a_l(E_i; X, \Delta) > 0$ for any $i \in I$, (X, Δ) is said to be *log terminal*, moreover if the exceptional loci of μ is purely one codimensional, (X,Δ) is said to be *divisorially log terminal.* We shall say that X has only terminal (resp. canonical, resp. log terminal) singularity if $(X,0)$ is terminal (resp. canonical, resp. log terminal) as usual (see [21] or [37] and see also [14], §1 for the treatment in the complex analytic case).

In this paper, we shall use the following notation:

$\nu : X^\nu \to X$: The normalization of a scheme X.

$\mathrm{Diff}_{\Gamma^\nu}(\Delta)$: $\boldsymbol{Q}$-divisor which is called Shokurov's different satisfying

$$\nu^*(K_X+\Gamma+\Delta) = K_{\Gamma^\nu} + \mathrm{Diff}_{\Gamma^\nu}(elta),$$

where Γ is a reduced divisor on a normal variety X and $\Gamma+\Delta$ is a $\boldsymbol{Q}$-boundary on X such that $K_X+\Gamma+\Delta$ is $\boldsymbol{Q}$-Cartier. (see [37], §3, [2], §16).

Δ^Y : $\boldsymbol{Q}$-divisor on Y satisfying $K_Y+\Delta^Y = f^*(K_X+\Delta)$, where $f : Y \to X$ is a birational morphism between normal varieties and Δ is a $\boldsymbol{Q}$-boundary on X such that $K_X+\Delta$ is $\boldsymbol{Q}$-Cartier.

$\mathrm{ind}_p(D)$: The smallest positive integer r such that rD is Cartier on the germ of X at p, where D is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor on a normal variety or a normal complex analytic space X.

$\mathrm{Ind}(D)$: The smallest positive integer r such that $rD \sim 0$, where D is a $\boldsymbol{Q}$-Cartier $\boldsymbol{Q}$-divisor on a normal variety X such that $D \sim_{\boldsymbol{Q}} 0$.

$\mathrm{Exc} f$: Exceptional loci of a birational morphism $f : X \to Y$ between varieties X and Y, that is, loci of points in X in a neighbourhood of which f is not isomorphic.

$\sim$: Linear equivalence.

$\sim_{\boldsymbol{Q}}$: $\boldsymbol{Q}$-linear equivalence.

$\lceil\Delta\rceil$: Round up of a $\boldsymbol{Q}$-divisor Δ.

$\lfloor\Delta\rfloor$: Round down of a $\boldsymbol{Q}$-divisor Δ.

$\{\Delta\}$: Fractional part of the boundary Δ.

e_{top} : Topological Euler characteristic.

Card $\mathcal{S}$: Cardinality of a set $\mathcal{S}$.

For a normal complete surface S with at worst Du Val singularities, we shall write

$$\mathrm{Sing\ S} = \sum_{\mathcal{T}} \nu(\mathcal{T})\mathcal{T},$$

where $\nu(\mathcal{T})$ denotes the number of singular points on S of type $\mathcal{T}$. For a quasi projective complex surface S with only quotient singularity, recall that *the orbifold euler number* $e_{\mathrm{orb}}(S) \in \boldsymbol{Q}$ of S is defined by

$$e_{\mathrm{orb}}(S) := e_{\mathrm{top}}(S) - \sum_{p\in S}(1 - \frac{1}{\mathrm{Card}\pi_{S,p}}),$$

where $\pi_{S,p}$ denotes the local fundamental group of S at $p \in S$ (see [15], page 233 or [23], Definition 10.7).

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2 The Euler characteristic formula

Firstly, let us recall the following result due to Crauder and Morrison.

Proposition 2.1 ([3], Proposition (A.1)) *Let X be a smooth 3-fold and let D be a complete effective divisor on X whose support has only simple normal crossing singularities. Then the following holds.*

$$\chi(\mathcal{O}_D) = \sum_i m_i \chi(\mathcal{O}_{D_i}) + \frac{1}{6}(D^3 - \sum_i m_i D_i^3) + \frac{1}{4}(D^2 - \sum_i m_i D_i^2)K_X,$$

where $D = \sum_i m_i D_i$ is the irreducible decomposition.

Let (X, p) be a germ of 3-dimensional terminal singularity at p whose index r is equal to or greater than 2. Take a Du Val element $S \in |-K_X|$ passing through p, where we say that $S \in |-K_X|$ is a Du Val element, if S is a reduced normal $\mathbf{Q}$-Cartier divisor on X passing through p such that S has a Du Val singularity at p. The canonical cover $\pi : \tilde{X} \to X$ with respect to K_X induces a covering of Du Val singularities $\pi : \tilde{S} := \pi^{-1}(S) \to S$. There is a coordinate system x, y and z of $\boldsymbol{C}^3$ which are semi-invariant under the action of the Galois group Gal $(\tilde{S}/S)$ such that $\tilde{p} := \pi^{-1}(p) \in \tilde{S}$ is analytically isomorphic to the germ of the origin of the hypersurface defined by a equation $f(x, y, z) = 0$. Let σ be a generator of Gal $(\tilde{S}/S)$ and let ζ be a primitive r-th root of unity. The actions of σ are completely classified into the following 6 types (see [36]).

(1) $\tilde{p} \in \tilde{S}$ is of type A_{n-1} and $p \in S$ is of type A_{rn-1} $(n \geq 1)$. $f = xy + z^n$,$\sigma^* x = \zeta^a x$, $\sigma^* y = \zeta^{-a} y$ and $\sigma^* z = z$, where $(r, a) = 1$.

(2) $\tilde{p} \in \tilde{S}$ is of type A_{2n-2} and $p \in S$ is of type D_{2n+1} $(n \geq 2)$. $r = 4$, $f = x^2 + y^2 + z^{2n-1}$, $\sigma^* x = \zeta x$, $\sigma^* y = \zeta^3 y$ and $\sigma^* z = \zeta^2 z$.

(3) $\tilde{p} \in \tilde{S}$ is of type A_{2n-1} and $p \in S$ is of type D_{n+2} $(n \geq 2)$. $r = 2$, $f = x^2 + y^2 + z^{2n}$, $\sigma^* x = x$, $\sigma^* y = -y$ and $\sigma^* z = -z$.

(4) $\tilde{p} \in \tilde{S}$ is of type D_4 and $p \in S$ is of type E_6. $r = 3$, $f = x^2 + y^3 + z^3$, $\sigma^* x = x$, $\sigma^* y = \zeta y$ and $\sigma^* z = \zeta^2 z$.

(5) $\tilde{p} \in \tilde{S}$ is of type D_{n+1} and $p \in S$ is of type D_{2n}. $r = 2$, $f = x^2 + y^2 z + z^n$, $\sigma^* x = -x$, $\sigma^* y = -y$ and $\sigma^* z = z$.

(6) $\tilde{p} \in \tilde{S}$ is of type E_6 and $p \in S$ is of type E_7. $r = 2$, $f = x^2 + y^3 + z^4$, $\sigma^* x = -x$, $\sigma^* y = y$ and $\sigma^* z = -z$.

Definition 2.1 For $p \in S \subset X$ as above, we define the rational number $c_p(X, S) \in \boldsymbol{Q}$ as follows:

$$c_p(X,S) := \begin{cases} 0 & \text{Case } p \in X \text{ Gorenstein}, \\ n\{r - (1/r)\} & \text{Case (1)}, \\ 3(2n+3)/4 & \text{Case (2)}, \\ 3 & \text{Case (3)}, \\ 16/3 & \text{Case (4)}, \\ 3n/2 & \text{Case (5)}, \\ 9/2 & \text{Case (6)}. \end{cases}$$

Definition 2.2 Let $p \in S \subset X$ be as above. we define the rational number $\delta_p(X, S) \in \boldsymbol{Q}$ as follows:

$$\delta_p(X,S) := e_p(S) - \frac{1}{o_p(S)} - c_p(X,S) \in \boldsymbol{Q},$$

where $e_p(S)$ is the Euler number of the inverse image of p by the morphism induced by the minimal resolution and $o_p(S)$ is the order of the local fundamental group of S at p.

If the index of X at p is equal to or greater than 2, we obtain the following table.

Table I

	$e_p(S)$	$o_p(S)$	$c_p(X,S)$	$\delta_p(X,S)$
(1)	rn	rn	$n\{r-(1/r)\}$	$(n^2-1)/rn$
(2)	$2n+2$	$8n-4$	$3(2n+3)/4$	$n(n-1)/(2n-1)$
(3)	$n+3$	$4n$	3	$(4n^2-1)/4n$
(4)	7	24	$16/3$	$13/8$
(5)	$2n+1$	$8(n-1)$	$3n/2$	$(4n^2+4n-9)/8(n-1)$
(6)	8	48	$9/2$	$167/48$

Proposition 2.2 *$\delta_p(X,S)\geq 0$. $\delta_p(X,S)=0$ if and only if (X,S) has only singularity of type $V_1(r;a,-a,1)$ at p, where $(r,a)=1$.*

Recall the following:

Theorem 2.1 ([36], Theorem 9.1 (I)) *Let X be a projective surface with at worst Du Val singularities and let D be a Weil divisor on X. Then*

$$\chi(\mathcal{O}_X(D))=\chi(\mathcal{O}_X)+\frac{1}{2}D(D-K_X)+\sum_{p\in X}c_p(D),$$

where $c_p(D)$ is the rational number which depends only on the local analytic type of $p\in X$ and $\mathcal{O}_X(D)$.

Using Theorem 2.1, we get the following:

Proposition 2.3 *Let X be a normal $\mathbf{Q}$-Gorenstein 3-fold and D be an effective complete Cartier divisor on X such that the log 3-fold $(X,D_{\rm red})$ is divisorially log terminal and that X is smooth outside the support of D. Assume that $K_X+D_{\rm red}$ is Cartier and each irreducible component of D is algebraic and $\mathbf{Q}$-Cartier. Then the following formula holds :*

$$\begin{aligned}\chi(\mathcal{O}_D) &= \sum_i m_i\chi(\mathcal{O}_{D_i})+\frac{1}{6}(D^3-\sum_i m_iD_i^3)+\frac{1}{4}(D^2-\sum_i m_iD_i^2)K_X\\ &\quad -\frac{1}{12}\sum_i m_i\sum_{p\in D_i^o}c_p(X,D_i),\end{aligned}$$

where $D=\sum_i m_iD_i$ is the irreducible decomposition and $D_i^o:=D_i\setminus\cup_{j\neq i}D_j$.

Theorem 1.1 can be proved using Proposition 2.3.

3 Types of degenerations of algebraic surfaces with Kodaira dimension zero

Definition 3.1 (Minimal Semistable Degeneration) A minimal model $X\to\mathcal{D}$ obtained from a projective semistable degeneration of surfaces with non-negative Kodaira dimension $g:Y\to\mathcal{D}$ by applying the Minimal Model Program is called a projective minimal semistable degeneration of surfaces.

A projective log minimal degeneration of Kodaira dimension zero is related to a minimal semistable degeneration as in the following way.

Lemma 3.1 *Let $f : X \to \mathcal{D}$ be a projective log minimal degeneration of surfaces with non-negative Kodaira dimension. Then there exists a finite covering $\tau : \mathcal{D}^\sigma \to \mathcal{D}$, a projective minimal semistable degeneration $f^\sigma : X^\sigma \to \mathcal{D}^\sigma$ which is bimeromorphically equivalent to $X \times_{\mathcal{D}} \mathcal{D}^\sigma$ over $\mathcal{D}^\sigma$ and a generically finite morphism $\pi : X^\sigma \to X$ such that $f \circ \pi = \tau \circ f^\sigma$ and $K_{X^\sigma} + \Theta^\sigma = \pi^*(K_X + \Theta)$, where $\Theta^\sigma := f^{\sigma*}(0)$.*

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\pi} & X \\ {\scriptstyle f^\sigma}\downarrow & & {\scriptstyle f}\downarrow \\ \mathcal{D}^\sigma & \xrightarrow{\tau} & \mathcal{D} \end{array}$$

Let (S, Δ) be a projective log surface with a standard boundary defined over an algebraically closed field k. Assume that (S, Δ) is log terminal and $K_S + \Delta$ is numerically trivial. (S, Δ) can be roughly classified into the following three types.

I : $\lfloor \Delta \rfloor = 0$,

II : $\lfloor \Delta \rfloor \neq 0$ and $\lfloor \mathrm{Diff}_{\lfloor \Delta \rfloor^\nu}(\Delta - \lfloor \Delta \rfloor) \rfloor = 0$,

III : $\lfloor \Delta \rfloor \neq 0$ and $\lfloor \mathrm{Diff}_{\lfloor \Delta \rfloor^\nu}(\Delta - \lfloor \Delta \rfloor) \rfloor \neq 0$,

where ν denotes the normalization map $\nu : \lfloor \Delta \rfloor^\nu \to \lfloor \Delta \rfloor$.

Definition 3.2 Log surfaces (S, Δ) with the above conditions are said to be *of type* I, II *and* III respectively.

Definition 3.3 A log minimal degeneration $f : X \to \mathcal{D}$ of surfaces of Kodaira dimension zero is said to be *of type* I (resp. *of type* II, resp. *of type* III), if there exists an irreducible component Θ_i of Θ such that $(\Theta_i, \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i))$ is of type I (resp. of type II, resp. of type III).

For a projective log minimal degenerations of surfaces of Kodaira dimension zero $f : X \to \mathcal{D}$, take a projective minimal semistable degeneration $f^\sigma : X^\sigma \to \mathcal{D}^\sigma$ obtained from f as in Lemma 3.1. Then the following holds.

Proposition 3.1 *f is of type* I (*resp. of type* II, *resp. of type* III) *if and only if f^σ is of type* I (*resp. of type* II, *resp. of type* III). *Moreover two projective log minimal degenerations $f_j : X_j \to \mathcal{D}$ $(j = 1, 2)$ which are bimeromorphically equivalent to each other over $\mathcal{D}$ have exactly the same types as each other, i.e., types* I, II *and* III *are bimeromorphic notion which are independent from the choice of log minimal models.*

Remark 3.1 From Proposition 3.1, for a projective log minimal degenerations of surfaces of Kodaira dimension zero $f : X \to \mathcal{D}$, we can see that if f is of type I (resp. of type II, resp. of type III), then for any irreducible component Θ_i of Θ, $(\Theta_i, \mathrm{Diff}_{\Theta_i}(\Theta - \Theta_i))$ is of type I (resp. of type II, resp. of type III).

4 Canonical bundle formulae

4.1 Review on Fujino-Mori's canonical bundle formula

Let $f : X \to B$ be a morphism from a normal projective variety X with $\dim X = m + d$ onto a smooth projective variety B of dimension d defined over the complex number field. Assume that X has only canonical singularities and that a general fibre F of f is an irreducible variety with Kodaira dimension zero. Let b be the smallest positive integer such that the b-th plurigenera of F is not zero. According to [8], there exists a $\mathbf{Q}$-divisor $L_{X/B}$ on B such that there exists a $\mathcal{O}_B$-algebra isomorphism

$$\oplus_{i\geq 0}\mathcal{O}_B(\lfloor iL_{X/B}\rfloor) \simeq \oplus_{i\geq 0} f_*\mathcal{O}_X(ibK_X)^{**},$$

where $**$ denotes the double dual. $L_{X/B}$ is well defined in the sense that $L_{X/B}$ is unique modulo linear equivalence[1] . Mori also defined a $\mathbf{Q}$-divisor $L^{ss}_{X/B}$ on B as a "moduli contribution" to a canonical bundle formula as follows.

Proposition 4.1 ([8], Corollary 2.5) *There exists a* $\mathbf{Q}$*-divisor* $L^{ss}_{X/B}(\leq L_{X/B})$ *such that*

(i) $\tau^* L^{ss}_{X/B} \leq L_{X'/B'}$ *for any finite surjective morphism* $\tau : B' \to B$ *from an irreducible smooth projective variety* B'*, and that*

(ii) $\tau^* L^{ss}_{X/B} = L_{X'/B'}$ *at* $p' \in B'$ *if* $X \times_B B'$ *has a semistable resolution over a neighbourhood of* $p' \in B'$ *or* $f'^*(p')$ *has only canonical singularities,*

where $f' : X' \to B'$ *is a fibration by taking a non-singular model of the second projection* $X \times_B B' \to B'$.

Remark 4.1 Since $L_{X/B}$ and $L^{ss}_{X/B}(\leq L_{X/B})$ depend only on the birational equivalence class of X over B, we can define these $\mathbf{Q}$-divisors even if the singularity of X is worse than canonical by passing to a non-singular model.

Let $\pi : \tilde{F} \to F$ be a proper surjective morphism from a smooth variety $\tilde{F}$ obtained by taking b-th root of the unique element of $|bK_F|$ and desingularization. Put $N(x) := \text{L.C.M.}\{n \in \boldsymbol{N} | \varphi(n) \leq x\}$, where φ denotes the Euler's function. Let B_m be the m-th Betti number of $\tilde{F}$. The following theorem says, coefficients appearing in canonical bundle formulae can be well controlled. From what follows, $B^{(1)}$ denotes the set of all the codimension one point of B.

Theorem 4.1 ([8], Proposition 2.8 and Theorem 3.1) **(1)** $N(B_m)L^{ss}_{X/B}$ *is a Weil divisor.*

(2) *Assume that* $NL^{ss}_{X/B}$ *is a Weil divisor. Then we have* $bK_X = f^*(bK_B + L^{ss}_{X/B} + \sum_{p\in B^{(1)}} s_p p) + E$, *where* $s_p \in \boldsymbol{Q}$ *and* E *is an effective* $\mathbf{Q}$*-divisor such that*

(i) *for each point* $p \in B^{(1)}$*, there exist positive integers* u_p*,* v_p *such that* $0 < v_p \leq bN$ *and*

$$s_p = \frac{bNu_p - v_p}{Nu_p},$$

(ii) $s_p = 0$ *if the geometric fibre of* f *over* p *has only canonical singularity or if* $f : X \to B$ *has a semistable resolution in a neighbourhood of* p*, and*

(iii) $f_*\mathcal{O}_X(\lfloor nE\rfloor) = \mathcal{O}_B$ *for any* $n \in \boldsymbol{N}$.

[1] Any two $\mathbf{Q}$-divisors D_1 and D_2 on a variety are said to be linearly equivalent to each other if $D_1 - D_2$ is a principal divisor.

4.2 Canonical Bundle Formulae and Log Minimal Models

Lemma 4.1 *Let $f : X \to B$ be a proper morphism from a normal variety X onto a smooth irreducible projective curve B defined over the complex number field whose general fibre F of f is a normal variety with only canonical singularity whose Kodaira dimension is zero. Let b be the smallest positive integer such that the b-th plurigenera of F is not zero as in the previous section. Let $\Sigma \subset B$ be a finite set of points which consists of all the points $p \in B$, such that $f^*(p)$ is not a normal variety with only canonical singularity. Assume that (X, Θ) is log canonical where $\Theta := (f^*\Sigma)_{\mathrm{red}}$. Then there exists $d \in \boldsymbol{N}$, such that*

$$f_*\mathcal{O}_X(n(K_X + \Theta)) = \mathcal{O}_B(n(K_B + (1/b)L^{ss}_{X/B} + \Sigma))$$

for any $n \in d\boldsymbol{N}$.

Remark 4.2 Let $f^s : X^s \to B$ be a strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B as in Definition 1.1. Since K_{X^s} is numerically trivial over B, there exists a positive integer $\ell_p \in \boldsymbol{N}$ such that $f^{s*}(p) = \ell_p \Theta^s_p$ for any $p \in B$, where $\Theta^s_p := f^{s*}(p)_{\mathrm{red}}$. Let $\mu : Y \to X^s$ be a minimal model over X^s, that is, μ is a projective birational morphism from a normal $\mathbf{Q}$-factorial Y with only terminal singularity to X^s such that K_Y is μ-nef. By running the minimal model program over B starting from the induced morphism $g := f^s \circ \mu : Y \to B$, we obtain a minimal fibration $h : Z \to B$ and a dominating rational map $\lambda : Y - \to Z$ over B. Since X^s has only log terminal singularity, there exists an effective $\mathbf{Q}$-divisor Δ with $\lfloor \Delta \rfloor = 0$ on Y such that

$$K_Y + \Delta = \mu^* K_{X^s}.$$

Since $K_Y + \Delta$, $K_Z + \lambda_*\Delta$ and K_Z are all numerically trivial over B, there exists a non-negative rational number $\mu_p \in \boldsymbol{Q}$ such that

$$\lambda_*\Delta_p = \mu_p h^*(p), \tag{4.1}$$

where Δ_p denotes the restriction of Δ in a neighbourhood of the fibre over $p \in B$. When B is complete, a canonical bundle formula can be calculated by using Lemma 4.1 as follows

$$K_Z = h^*(K_B + \frac{1}{b}L^{ss}_{Z/B} + \sum_{p \in B}(\frac{\ell_p - 1}{\ell_p} - \mu_p)p). \tag{4.2}$$

Define $s_p \in \boldsymbol{Q}$ by

$$s_p := b(\frac{\ell_p - 1}{\ell_p} - \mu_p).$$

We can check Mori's estimate of s_p in Theorem 4.1 as in the following way. Firstly, we note that we may assume that there exists a prime μ-exceptional divisor over $p \in B$ such that $E \subset \mathrm{Supp}\,\Delta$ and $\lambda_*E \neq 0$, since otherwise, $\mu_p = 0$ and hence $s_p = (\ell_p - 1)/\ell_p$. Put

$$I_p := \mathrm{Min}\{n \in \boldsymbol{N} | n(K_{X^s} + \Theta^s_p) \text{ is Cartier in a neighbourhood of the fibre } f^{s*}(p)\}.$$

Since (X^s, Θ^s) is log canonical, we have

$$0 \le I_p a_l(E; X^s, \Theta^s_p) = I_p(a(E; X^s) - \mathrm{mult}_E \mu^*\Theta^s_p + 1) \imath n \boldsymbol{Z}.$$

Thus if we put $v'_p := I_p(-a(E; X^s) + \mathrm{mult}_E \mu^*\Theta^s_p)$, we have $v'_p \in \boldsymbol{N}$ and $0 < v'_p \le I_p$. Put

$$u'_p := \ell_p \mathrm{mult}_E \mu^*\Theta^s_p = \mathrm{mult}_E g^*(p) \in \boldsymbol{N}.$$

Then we have $\mu_p = -a(E; X^s)/u'_p$ and $s_p = b(I_p u'_p - v'_p)/I_p u'_p$. So the estimation of s_p reduces to the estimation of I_p in our case. By passing to a divisorially log terminal model of (X^s, Θ^s) and using Proposition 4.2 below and [41], Proof of Theorem 2.1, we get $I_p | N(21)$.

Consider the following conditions assuming $\dim X \geq 2$.

$(M1)$ Δ is a standard $\mathbf{Q}$-boundary.

$(M2)$ (X, Δ) is divisorially log terminal.

$(M2)^*$ $(M2)^\alpha$ (X, Δ) is divisorially log terminal and $\{\Delta\} = 0$ or $(M2)^\beta$ (X, Δ) is purely log terminal.

$(M3)$ There exists an irreducible component Γ of $\lfloor \Delta \rfloor$ passing through $p \in X$ such that $K_X + \Gamma$ is $\mathbf{Q}$-Cartier.

Remark 4.3 $(M2)^*$ is a slightly stronger condition than $(M2)$.

Proposition 4.2 *Assume the conditions* $(M1)$, $(M2)$ *and* $(M3)$. *Then*

$$\mathrm{ind}_p(K_X + \Delta) = \mathrm{ind}_p(K_\Gamma + \mathrm{Diff}_\Gamma(\Delta - \Gamma)).$$

Remark 4.4 We should note that $\ell_p \in \boldsymbol{N}$ and $\mu_p \in \boldsymbol{Q}$ depends on the choice of strictly log minimal model. For example, consider a degeneration of elliptic curve whose singular fibre is of type ${}_m\mathrm{I}_1$. Obviously, the minimal model is a strictly log minimal model in this case and we obtain $\ell_p = m$ and $\mu_p = 0$. But blowing up the node of the singular fibre and blowing down the exceptional divisor we obtain another strictly log minimal model. When we use this model, we get $\ell_p = 2m$ and $\mu_p = 1/(2m)$.

To avoid this indeterminacy, we shall introduce the notion of moderate log canonical singularity.

Definition 4.1 Let (X, Δ) be a normal log variety and let Δ' be a boundary on X such that $\Delta' \leq \Delta$. Assume that (X, Δ) is log canonical and that $K_X + \Delta'$ is $\mathbf{Q}$-Cartier. (X, Δ) is said to be *moderately log canonical with respect* $K_X + \Delta'$ if for any exceptional prime divisor E of the function field of X with $a_l(E; X, \Delta) = 0$, the inequality $a_l(E; X, \Delta') > 1$ holds.

Remark 4.5 If X is $\mathbf{Q}$-Gorenstein and (X, Δ) is divisorially log terminal, then (X, Δ) is moderately log canonical with respect to K_X by [40], Divisorial Log Terminal Theorem. Thus it is easy to see that for a strictly log minimal fibration (or degeneration) $f^s : X^s \to B$ constructed in such a way as explained in Remark 1.1, (X^s, Θ^s) is moderate with respect to K_{X^s}, where $\Theta^s := \sum_{p \in \Sigma} \Theta^s_p$.

Lemma 4.2 *Let* $f^s_i : X^s_i \to B$ $(i = 1, 2)$ *be two strictly log minimal fibration* (*or degeneration*) *of surfaces with Kodaira dimension zero which are birationally equivalent to each other over* B *and let* $l^{(i)}_p$ *be a positive integer such that* $f^{s*}_i(p) = \ell^{(i)}_p \Theta^s_{i,p}$, *where* $\Theta^s_{i,p} := f^{s*}_i(p)_{\mathrm{red}}$ *for* $i = 1, 2$. *Assume that* $(X^s_1, \Theta^s_{1,p})$ *is moderately log canonical with respect to* $K_{X^s_1}$. *Then* $\ell^{(1)}_p \leq \ell^{(2)}_p$. *Moreover, if* $(X^s_2, \Theta^s_{2,p})$ *is also moderately log canonical with respect to* $K_{X^s_2}$, *then* $f^s_1 : X^s_1 \to B$ *and* $f^s_2 : X^s_2 \to B$ *are isomorphic in codimension one to each other over a neighbourhood of* $p \in B$ *and* $\ell^{(1)}_p = \ell^{(2)}_p$.

Proof. Take a desingularization $\alpha_i : W \to X^s_i$ of X^s_i and let $\omega : W \to B$ be the induced morphism. Let $G^{(i)}$ be a α_i-exceptional effective $\mathbf{Q}$-divisor defined by $G^{(i)} := K_W + \Theta^s_{i,W} - \alpha^*_i(K_{X^s_i} + \Theta^s_{i,p})$. We note that $G^{(1)} - G^{(2)} \in \omega^*(\mathrm{Div}(B) \otimes \boldsymbol{Q})$ by their definitions. Since $G^{(i)}$ is effective and Supp $G^{(i)}$ does not contain the support of the fibre $\omega^*(p)$ entirely for $i = 1, 2$, we have $G^{(1)} = G^{(2)}$. If we assume that any α_1-exceptional divisor contained in a fibre $\omega^{-1}(p)$ is α_2-exceptional, then we have $\ell^{(1)}_p = \ell^{(2)}_p$ obviously, so we may assume that there exists a α_1-exceptional prime divisor $E \subset \omega^{-1}(p)$ which is not α_2-exceptional. If we assume that $a_l(E; X^s_1, \Theta^s_{i,p}) > 0$, then $E \subset$ Supp $G^{(1)} =$ Supp $G^{(2)}$, which is a contradiction. Thus we have $a_l(E; X_1^s, \Theta_{i,p}^s) = 0$ and hence $a(E; X_1^s) > 0$ by the assumption. Therefore we deduce that

$$\ell_p^{(2)} = \mathrm{mult}_E \omega^*(p) = \ell_p^{(1)} \mathrm{mult}_E \alpha_1^* \Theta_{1,p}^s = \ell_p^{(1)}(a(E; X_1^s) + 1) > \ell_p^{(1)}.$$

The last assertion also follows from the above argument. ■

Proposition 4.3 (c.f. [20], Theorem 4.9) *Let $f_i^s : X_i^s \to B$ $(i = 1, 2)$ be two strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B which are birationally equivalent to each other over B. Assume that $(X_i^s, \Theta_{i,p}^s)$ is moderately log canonical with respect to $K_{X_i^s}$ for $i = 1, 2$. Then $f_1^s : X_1^s \to B$ and $f_2^s : X_2^s \to B$ are connected by a sequence of log flops over a neighbourhood of $p \in B$, that is, there exist birational morphisms between normal threefolds over a neighbourhood of $p \in B$ which are isomorphic in codimension one:*

$$X_1^s := X^{(0)} \to Z^{(0)} \leftarrow X^{(1)} \to Z^{(1)} \cdots \leftarrow X^{(n)} =: X_2^s,$$

where $X^{(k)}$ is $\mathbf{Q}$-factorial for $k = 0, 1, \ldots n$.

Proof. Take a relatively ample effective divisor H on X_2^s over B and let H' be the strict transform of H on X_1^s. Applying the log minimal model program on X_1^s over B with respect to $K_{X_1^s} + \varepsilon H'$, where ε is sufficiently small positive rational number, we may assume at first that H' is f_1^s-nef since contraction morphisms appearing in the log minimal model program do not contract divisors. By the Base Point Free Theorem ([29]), some multiple of H' defines a birational morphism $\gamma : X_1^s \to X_2^s$ over B which is isomorphic in codimension one. Since X_1^s and X_1^s are both $\mathbf{Q}$-factorial, γ is an isomorphism and thus we get the assertion. ■

Definition 4.2 Let $f^s : X^s \to B$ be a strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B such that (X^s, Θ_p^s) is moderately log canonical with respect to K_{X^s} and let $\ell_p \in \boldsymbol{N}$ and $\mu_p \in \boldsymbol{Q}$ be as defined in Remark 4.2. We define $\ell_p^* \in \boldsymbol{N}$, $\mu_p^* \in \boldsymbol{Q}$ and $s_p^* \in \boldsymbol{Q}$ as $\ell_p^* := \ell_p$, $\mu_p^* := \mu_p$ and

$$s_p^* := b(\frac{\ell_p^* - 1}{\ell_p^*} - \mu_p^*).$$

Proposition 4.3 give the following:

Corollary 4.1 *$\ell_p^* \in \boldsymbol{N}$ and $\mu_p^* \in \boldsymbol{Q}$ are birational (or bimeromorphic) invariants of germs of singular fibres over $p \in B$ and hence so is s_p^*.*

Example 4.1 For degenerations of elliptic curves, one can define invariants ℓ_p^*, μ_p^* and s_p^*in the same way and it can be checked that ℓ_p^* coincides with the multiplicty if the singular fibre is of type ${}_m\mathrm{I}_b$ or otherwise, with the order of the semisimple part of the monodromy group around the singular fibre. We can also obtain the following well known table:

Table V

	${}_m\mathrm{I}_b$	I_b^*	II	II*	III	III*	IV	IV*
ℓ_p^*	m	2	6	6	4	4	3	3
μ_p^*	0	0	2/3	0	1/2	0	1/3	0
s_p^*	(m-1)/m	1/2	1/6	5/6	1/4	3/4	1/3	2/3

Here we are using the Kodaira's notation ([19]). See also [10].

Lemma 4.3 $s_p^* = 0$ *if and only if* $\ell_p^* = 1$.

Proof. Firstly, assume that $\ell_p^* = 1$, then we have obviously $s_p^* = 0$ since $s_p^* \geq 0$ and $\mu_p^* \geq 0$. Secondarily, assume that $s_p^* = 0$. Let $f^s : X^s \to B$ be a strictly log minimal fibration (or degeneration) of surfaces with Kodaira dimension zero projective over B such that (X^s, Θ_p^s) is moderately log canonical with respect to K_{X^s}. We may assume that the singularity of X^s is worse than canonical and let $\mu : Y \to X^s$ and E be as in Remark 4.2. Then from the assumption, we have

$$s_p^* = b(\frac{\ell_p^* - 1}{\ell_p^*} + \frac{a(E; X^s)}{l_p^* \mathrm{mult}_E \mu^* \Theta_p^s}) = 0,$$

hence $a(E; X^s, \Theta_p^*) = 1 - l_p^* \mathrm{mult}_E \mu^* \Theta_p^s$. Since $a(E; X^s, \Theta_p^*) \geq 0$ and $l_p^* \mathrm{mult}_E \mu^* \Theta_p^s \in \boldsymbol{N}$, we have $a(E; X^s, \Theta_p^*) = 0$. From the definition of moderately log canonical singularity, we have $a(E; X^s) > 0$ and hence $\mu_p^* < 0$, which is a contradiction. ∎

Lemma 4.4 *Let* $f^s : X^s \to B$ *be a strictly log minimal fibration* (*or degeneration*) *of surfaces with Kodaira dimension zero projective over* B. *Then* $\mu_p = 0$ *if and only if* X^s *has only canonical singularity over a neighbourhood of* $p \in B$.

Proof. We shall use notation in Remark 4.2. Assume that $\mu_p = 0$ and that singularity of X^s is worse than canonical over a neighbourhood of $p \in B$. Let W be a resolution of the graph of λ and let $\alpha : W \to Y$ and $\beta : W \to Z$ be projections. Since $\beta_* \Delta^W = \lambda_* \Delta = 0$ over $p \in B$, we have $K_W + \Delta^W = \alpha^*(K_Y + \Delta) = \beta^* K_Z$ over $p \in B$. Since Z has only terminal singularity by its construction, we obtain $-\Delta^W \geq 0$, which is absurd. ∎

Lemma 4.5 *Let* $f^s : X^s \to \mathcal{D}$ *be a strictly log minimal degeneration of surfaces with Kodaira dimension zero projective over a unit complex disk* $\mathcal{D}$ *with the origin* $p := 0 \in \mathcal{D}$ *and let* $\pi : \tilde{\mathcal{D}} \to \mathcal{D}$ *be a cyclic covering from another unit disk* $\tilde{\mathcal{D}}$ *with the order* ℓ_p *which is étale over* $\mathcal{D}^* := \mathcal{D} \setminus \{p\}$. *Let* $\tilde{f} : \tilde{X} \to \tilde{\mathcal{D}}$ *be a relatively minimal degeneration which is bimeromorphic to* $X^s \times_{\mathcal{D}} \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ *over* $\tilde{\mathcal{D}}$. *Then* $\tilde{f}^*(\tilde{p})$ *is reduced and* $(\tilde{X}, \tilde{f}^*(\tilde{p}))$ *is log canonical and in particular* $s_{\tilde{p}}^* = 0$, *where* $\tilde{p} := 0 \in \tilde{\mathcal{D}}$.

Proof. Let $\tilde{X}'$ be a $\mathbf{Q}$-factorization of $(X^s \times_{\mathcal{D}} \tilde{\mathcal{D}})^\nu$ (see [14], §6, page 120). Then, as in the same way as in the proof of Lemma 3.1, we see that the induced degeneration $\tilde{f}' : \tilde{X}' \to \tilde{\mathcal{D}}$ is strictly log minimal, projective over $\tilde{\mathcal{D}}$ and that $f'(\tilde{p})$ is reduced, which imply that $\tilde{X}'$ has only canonical singularity. Let $\tilde{X}$ be a minimal model over $\tilde{X}'$, then the induced degeneration $\tilde{f} : \tilde{X} \to \tilde{\mathcal{D}}$ turns out to be a minimal degeneration projective over $\tilde{\mathcal{D}}$ and it is easily seen that $\tilde{f}^*(\tilde{p})$ is reduced and $(\tilde{X}, \tilde{f}^*(\tilde{p}))$ is log canonical . Since minimal models are unique up to flops, we get the first assertion. As for the last assertion, we only have to check that $(\tilde{X}, \tilde{f}^*(\tilde{p}))$ is moderately log canonical with respect to $K_{\tilde{X}}$, but which is trivial. ∎

The degree of the moduli contribution to a canonical bundle formula can be calculated in a certain condition.

Proposition 4.4 *Let* $f : X \to B$ *be a proper surjective morphism from a normal algebraic threefolds* X *with only canonical singularity onto a smooth projective curve* B *whose general fibre is a surface with Kodaira dimension zero. Assume that* $s_p^* = 0$ *for any* $p \in B$. *Then* $\deg L_{X/B}^{ss} = \deg f_* \mathcal{O}_X(bK_{X/B})$.

Proof. Let $f^s : X^s \to B$ be a strictly log minimal model of f such that (X^s, Θ_p^s) is moderately log canonical with respect to K_{X^s} for any $p \in B$. By Lemma 4.3, we have $\ell_p^* = 1$ and hence $\mu_p^* = 0$ for any $p \in B$, from which we infer that $f^{s*}(p)$ is reduced and that X^s has only canonical singularity by Lemma 4.4. By the argument in [27], proof of Definition-Theorem (1.11), there exists a Cartier divisor $\delta \in \mathrm{Div}B$ such that $bK_{X^s/B} \sim f^{s*}\delta$. Thus we have

$$f_*\mathcal{O}_X(bK_{X/B}) = f^s_*\mathcal{O}_{X^s}(bK_{X^s/B}) \simeq \mathcal{O}_B(\delta).$$

On the other hand, since $bK_{X^s/B} \sim_{\boldsymbol{Q}} f^{s*}L^{ss}_{X^s/B}$, we have $\deg \delta = \deg L^{ss}_{X^s/B} = \deg L^{ss}_{X/B}$. Thus we get the assertion. ■

Lemma 4.6 *Let $f : X \to B$ be a proper surjective morphism from a normal algebraic threefolds X onto a smooth projective curve B whose general fibre is a surface with Kodaira dimension zero. Assume that $B \simeq \boldsymbol{P}^1$. Then there exists a Kummer covering $\pi : B' \to B$ from a smooth projective curve B' with* Gal $(B'/B) \simeq \oplus_{p\in B}\boldsymbol{Z}/\ell^*_p\boldsymbol{Z}$ *such that $s^*_{p'} = 0$ for any $p' \in B'$ for the second projection $p_2 : (X \times_B B')^\nu \to B'$*

Proof. The assertion follows from Lemma 4.5 using the argument in [30], §4, page 112. ■

Definition 4.3 Let $f : X \to B$ be a proper surjective morphism from a normal algebraic threefolds X onto $B \simeq \boldsymbol{P}^1$ whose general fibre is a surface with Kodaira dimension zero. Let $\mathcal{C}_f$ be the set of all the pair (B', π) which consists of a smooth projective curve B' and a finite surjective morphism $\pi : B' \to B$ such that $s^*_{p'} = 0$ for any $p' \in B'$ for the second projection $p_2 : (X \times_B B')^\nu \to B'$. For f, we define a positive integer $d(f) \in \boldsymbol{N}$ as $d(f) := \mathrm{Min}\{\deg \pi | (B', \pi) \in \mathcal{C}_f\}$.

Definition 4.4 Let $\mathcal{CY}^3_B$ be the set of all the triple (X, f, B) where X is a normal projective threefold X with only canonical singularity whose canonical divisor K_X is numerically trivial and $f : X \to B$ is a projective connected morphism onto $B \simeq \boldsymbol{P}^1$.

The following conjecture is important for the bounding problem of Calabi-Yau threefolds.

Conjecture 4.1 There exists $d \in \boldsymbol{N}$ such that $d(f) \leq d$ for any $(X, f, B) \in \mathcal{CY}^3_B$.

By Lemma 4.6, Conjecture 4.1 reduces to the following:

Conjecture 4.2 There exists $\ell \in \boldsymbol{N}$ such that $\prod_{p\in B} \ell^*_p \leq \ell$ for all $(X, f, B) \in \mathcal{CY}^3_B$.

The following proposition is an important step toward Conjecture 4.2, which can be deduced from Theorem 4.1.

Proposition 4.5 *There exists a finite subset $\mathcal{S} \subset \boldsymbol{Q}$ and a positive integer ν such that for all $(X, f, B) \in \mathcal{CY}^3_B$, $\{s^*_p | p \in B\} \subset \mathcal{S}$ and* Card $\{p \in B | s^*_p > 0\} \leq \nu$.

By Proposition 4.5, Conjecture 4.2 reduces to the following conjecture on degenerations:

Conjecture 4.3 Put $c^*_p := \mu^*_p \ell^*_p$. There exists a finite subset $\mathcal{C} \subset \boldsymbol{Q}$ such that for any degeneration of algebraic surfaces with Kodaira dimension zero over a one-dimensional complex disk, $c^*_p \in \mathcal{C}$.

Remark 4.6 As we have seen in Example 4.1, for any degenerations of elliptic curves, we have $c^*_p \in \{0, 1, 2, 4\}$. One can state the analogue of conjecture 4.3 in higher dimensional cases using the conjectural higher dimensional Log Minimal Model Program. In this case, the problem involves another problem such as the boundedness of varieties with Kodaira dimension zero.

4.3 Abelian Fibred Case

In this section, we prove the following theorem and apply this to Conjecture 4.1 in abelian fibred cases.

Theorem 4.2 *For any degeneration of abelian surfaces over a one-dimensional complex disk, all the possible values of the invariant μ_p^* for the singular fibres can be listed in Table* VI *and* VII *except the case $\mu_p^* = 0$. In particular,we have*

$$c_p^* \in \{0, 1/5, 1/4, 1/3, 2/5, 1/2, 2/3, 1, 3/2, 2, 3, 4, 5, 6\}.$$

Definition 4.5 Let $\mathcal{CY}^3_{B,\mathrm{ab}}$ be the set of all the triple (X, f, B) where X is a normal projective threefold X with only canonical singularity whose canonical divisor K_X is numerically trivial and $f : X \to B$ is a projective connected morphism onto $B \simeq \boldsymbol{P}^1$ whose geometric generic fibre is an abelian surface.

By Theorem 4.2, we can give a positive answer to Conjecture 4.1 using the argument in the previous section.

Corollary 4.2 *There exists $d \in \boldsymbol{N}$ such that $d(f) \le d$ for any $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$.*

For the proof of Theorem 4.2, we need the following:

Lemma 4.7 *Let $f : X \to \mathcal{D}$ be a log minimal Type* II *degeneration of abelian surfaces. Then local fundamental groups of X at any point in X is cyclic. The same holds also for a strictly log minimal model $f^s : X^s \to \mathcal{D}$ obtained by applying the log minimal model program on f with respect to K_X.*

Proof. We use notation in Lemma 3.1. Since X^σ is smooth and the support of the singular fibre f^σ has only normal crossing singularity with all of the components being relatively elliptic ruled surfaces, Galois group $G := \mathrm{Gal}\ (\mathcal{D}^\sigma/\mathcal{D})$ acts biregulary on X^σ and $\pi : X^\sigma \to X$ factors into $\pi = \pi_1 \circ \pi_2$ where $\pi_1 : X^\sigma/G \to X$ is a bimeromorphic morphism and $\pi_2 : X^\sigma \to X^\sigma/G$ is the quotient map. Let $\{E_j | j \in J\}$ be the set of all the π_1-exceptional divisors. Since both of X^σ/G and X are $\mathbf{Q}$-factorial, we have $\mathrm{Exc}\ \pi_1 = \cup_{j\in J} E_j$, hence π_1 induces an isomorphism $X^\sigma/G \backslash \cup_{j\in J} E_j \to X \backslash \cup_{j\in J} \mathrm{Center}_X(E_j)$. So we only have to check that $\mathrm{Center}_X(E_j)$ is contained in a double curve of Θ for any $j \in J$ but which is immediate by [40], Divisorial Log Terminal Theorem, since we have $a_l(E_j; X, \Theta) = 0$ obviously. Thus we get the assertion. ∎

Proof of Theorem 4.2. If there exists a strictly log minimal model $f^s : X^s \to \mathcal{D}$ of the degeneration to be considered such that (X^s, Θ^s) is moderately log canonical with respect to K_{X^s} and that X^s has only canonical singularity, then we have $\mu_p^* = 0$ by its definition and there is nothing we have to prove. In particular, we do not have to care about the Type III degeneration by Theorem 1.2. Consider a strictly log minimal model $f^s : X^s \to \mathcal{D}$ obtained by applying the log minimal model program on a type I or II log minimal degeneration $f : X \to \mathcal{D}$ with respect to K_X. Assume that there exists a point $x \in X^s$ such that (X^s, x) is not canonical. By Theorem 1.2, Lemma 4.7 and [30], Proposition 3.6 and 3.8,types of singularity of (X^s, Θ^s) at $x \in X^s$ are of type $V_1(r; a_0, a_1, a_2)$ with $r = 3, 4, 5, 6, 8, 10, 12$ or $V_2(r; a_0, a_1, a_2)$ with $r = 3, 4, 6$. In what follows, we use notation in Remark 4.2. By [36], §4, E corresponds to some primitive vector $(1/r)(\overline{ka_0}, \overline{ka_1}, \overline{ka_2})$ and we have

$$a(E; X^s) = (1/r)(\overline{ka_0} + \overline{ka_1} + \overline{ka_2}) - 1, \quad \mathrm{mult}_E \mu^* \Theta^s = (1/r)\overline{ka_2},$$

hence

$$\mu_p^* = \frac{r - (\overline{ka_0} + \overline{ka_1} + \overline{ka_2})}{\ell_p^* \overline{ka_2}}$$

in the case $V_1(r; a_0, a_1, a_2)$. In the same way, we obtain

$$\mu_p^* = \frac{r - (\overline{ka_0} + \overline{ka_1} + \overline{ka_2})}{\ell_p^* (\overline{ka_0} + \overline{ka_1})}$$

in the case $V_2(r; a_0, a_1, a_2)$. Determination of possible primitive vectors in the case $V_1(r; a_0, a_1, a_2)$ with $r = 5, 8, 10, 12$ was essentially done in [30], since degenerations in these cases are Type I in our terminology which coincides "moderate" in Oguiso's terminology. When we determine possible primitive vectors in other cases, we only have to note that $\sum_{i=0}^{2} \overline{ka_i} < r$ which is obvious restriction from the assumption and that $(r, \overline{ka_2}) = 1$ since Sing $X^s \subset \Theta^s$ in the case $V_1(r; a_0, a_1, a_2)$ and X^s is smooth at the generic point of double curves in the case $V_2(r; a_0, a_1, a_2)$. ■

Remark 4.7 Theorem 4.2 implies the inequality $s_p \geq 1/6$ holds for any degeneration of abelian surfaces which has been obtained in the case of Type I degenerations by Oguiso ([30], Main Theorem).

Table VI, Case $V_1(r; a_0, a_1, a_2)$

	primitive vectors	μ_p^*	s_p^*	divisibility of ℓ_p^*
(1)	(1/3)(1,0,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$3 \vert \ell_p^*$
(2)	(1/4)(1,1,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$4 \vert \ell_p^*$
(3)	(1/4)(0,1,1)	$2/\ell_p^*$	$(\ell_p^* - 3)/\ell_p^*$	$4 \vert \ell_p^*$
(4)	(1/5)(1,2,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$5 \vert \ell_p^*$
(5)	(1/6)(3,1,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$6 \vert \ell_p^*$
(6)	(1/6)(2,1,1)	$2/\ell_p^*$	$(\ell_p^* - 3)/\ell_p^*$	$6 \vert \ell_p^*$
(7)	(1/6)(1,1,1)	$3/\ell_p^*$	$(\ell_p^* - 4)/\ell_p^*$	$6 \vert \ell_p^*$
(8)	(1/6)(1,0,1)	$4/\ell_p^*$	$(\ell_p^* - 5)/\ell_p^*$	$6 \vert \ell_p^*$
(9)	(1/8)(5,1,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$8 \vert \ell_p^*$
(10)	(1/8)(3,1,1)	$3/\ell_p^*$	$(\ell_p^* - 4)/\ell_p^*$	$8 \vert \ell_p^*$
(11)	(1/8)(3,1,3)	$1/(3\ell_p^*)$	$(3\ell_p^* - 4)/(3\ell_p^*)$	$8 \vert \ell_p^*$
(12)	(1/10)(7,1,1)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$10 \vert \ell_p^*$
(13)	(1/10)(3,1,1)	$5/\ell_p^*$	$(\ell_p^* - 6)/\ell_p^*$	$10 \vert \ell_p^*$
(14)	(1/10)(3,1,3)	$1/\ell_p^*$	$(\ell_p^* - 2)/\ell_p^*$	$10 \vert \ell_p^*$
(15)	(1/12)(7,1,1)	$3/\ell_p^*$	$(\ell_p^* - 4)/\ell_p^*$	$12 \vert \ell_p^*$
(16)	(1/12)(4,3,1)	$4/\ell_p^*$	$(\ell_p^* - 5)/\ell_p^*$	$12 \vert \ell_p^*$
(17)	(1/12)(5,1,1)	$5/\ell_p^*$	$(\ell_p^* - 6)/\ell_p^*$	$12 \vert \ell_p^*$
(18)	(1/12)(3,2,1)	$6/\ell_p^*$	$(\ell_p^* - 7)/\ell_p^*$	$12 \vert \ell_p^*$
(19)	(1/12)(5,1,5)	$1/(5\ell_p^*)$	$(5\ell_p^* - 6)/(5\ell_p^*)$	$12 \vert \ell_p^*$
(20)	(1/12)(3,2,5)	$2/(5\ell_p^*)$	$(5\ell_p^* - 7)/(5\ell_p^*)$	$12 \vert \ell_p^*$

Table VII, Case $V_2(r; a_0, a_1, a_2)$

	primitive vectors	μ_p^*	s_p^*	divisibility of ℓ_p^*
(21)	(1/3)(1,0,1)	$1/\ell_p^*$	$(\ell_p^*-2)/\ell_p^*$	$3\|\ell_p^*$
(22)	(1/4)(1,0,1)	$2/\ell_p^*$	$(\ell_p^*-3)/\ell_p^*$	$4\|\ell_p^*$
(23)	(1/4)(1,1,1)	$1/(2\ell_p^*)$	$(2\ell_p^*-3)/(2\ell_p^*)$	$2\|\ell_p^*$
(24)	(1/6)(1,0,1)	$4/\ell_p^*$	$(\ell_p^*-5)/\ell_p^*$	$6\|\ell_p^*$
(25)	(1/6)(1,1,1)	$3/(2\ell_p^*)$	$(2\ell_p^*-5)/(2\ell_p^*)$	$3\|\ell_p^*$
(26)	(1/6)(1,2,1)	$2/(3\ell_p^*)$	$(3\ell_p^*-5)/(3\ell_p^*)$	$2\|\ell_p^*$
(27)	(1/6)(1,3,1)	$1/(4\ell_p^*)$	$(4\ell_p^*-5)/(4\ell_p^*)$	$3\|\ell_p^*$

5 Bounding the number of singular fibres of Abelian Fibered Calabi Yau threefolds

Let B be a smooth projective irreducible curve defined over the complex number field and let K denote the rational function field of B.

Definition 5.1 We define the loci of singular fibres Σ_f, Σ_φ as follows.

(1) Let $f : X \to B$ is a projective connected morphism from a normal $\mathbf{Q}$-factorial projective variety X with only canonical singularity onto a B. We define the subset of closed points of B, Σ_f by

$$\Sigma_f := \{p \in B |\ f \text{ is not smooth over a neighbourhood of } p\ \}.$$

(2) Let A_K be an abelian variety over K, and let $\varphi : A \to B$ be the Néron model of A_K. We define the subset of closed points of B, Σ_φ by

$$\Sigma_\varphi := \{p \in B |\ A_K \text{ does not have good reduction at } p\ \}.$$

Remark 5.1 Consider two morphisms $f_i : X_i \to B$ $(i = 1, 2)$ as in Definition 5.1, (1), such that $\dim X_i = 3$ and the that the geometric generic fiber of f_i is an abelian variety for $i = 1, 2$. Assume that K_{X_i} is f_i-nef for $i = 1, 2$ and that X_1 is birationally equivalent to X_2 over B. Then we see that $\Sigma_{f_1} = \Sigma_{f_2}$ and hence Card Σ_1 = Card Σ_2, for, as is well known, birationally equivalent minimal models are connected by a sequence of flops while abelian varieties do not contain rational curves. Moreover, both of the defintions of (1) and (2) are compatible in the case $\dim A_K = 2$, that is, for a birational model $f : X \to B$ of A_K such as in Definition 5.1, (1), Σ_f coincides with Σ_φ.

The aim of this section is to prove the following theorem which gives a positive answer to Oguiso's question communicated to the author; Are the numbers of singular fibres of abelian fibered Calabi Yau threefolds are bounded?

Theorem 5.1 *There exists $s \in \boldsymbol{N}$, such that for any triple $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$,*

$$s_f := \text{Card } \Sigma_f \le s.$$

Let $f : X \to B$ be a projective connected morphism from a normal $\mathbf{Q}$-factorial projective variety X with only canonical singularity onto B. Put $B_\circ := B \setminus \Sigma_f$ and let $f_\circ : X_\circ \to B_\circ$ be the restriction of f to $X_\circ := f^{-1}(B_\circ)$. According to the theory of relative Picard schemes and relative Albanese maps due to Grothendiek (see [11] and see also [7] working on the analytic category), there exists the relative Picard schemes Pic $(X_\circ/B_\circ) \to B_\circ$ representing the Picard functor whose connected component containing the unit section, denoted by $\mathrm{Pic}^\circ(X_\circ/B_\circ) \to B_\circ$, is a projective abelian scheme. Moreover, there exists a morphism $\alpha : X_\circ \to \mathrm{Alb}^1(X_\circ/B_\circ)$ over $B_\circ$ where $\mathrm{Alb}^1(X_\circ/B_\circ)$ is $B_\circ$-torsor under the dual projective abelian scheme of $\mathrm{Pic}^\circ(X_\circ/B_\circ)$ denoted by $\mathrm{Alb}^0(X_\circ/B_\circ)$ satisfying the universal mapping property. α is called *the relative Albanese map associated with* f. If we assume that the geometric generic fibre of f is an abelian variety, then α is an isomorphism over $B_\circ$ by the universal mapping property.

Definition 5.2 Let X_η be the generic fibre of f. The Néron Model $\varphi : \mathcal{A} \to B$ of $\mathrm{Alb}^0(X_\eta)$ is an extension of the abelian scheme $\varphi_\circ : \mathrm{Alb}^0(X_\circ/B_\circ) \to B_\circ$. We call φ, *the Albanese group scheme associated with* f.

Corollary 5.1 *There exists $s \in \boldsymbol{N}$, such that for any triple $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$,*

$$s_\varphi := \mathrm{Card}\ \Sigma_\varphi \leq s,$$

where $\varphi : \mathcal{A} \to B$ is the Albanese group scheme associated with f.

Proof. Since $\mathcal{A} \times_B B_\circ \simeq \mathrm{Alb}^0(X_\circ/B_\circ)$, we have $\Sigma_\varphi \subset \Sigma_f$ and hence $s_\varphi \leq s_f$. Thus the assertion follows immediately from Theorem 5.1.

■

The following Lemma can be deduced immediately from the property of torsors.

Lemma 5.1 *Let $f : X \to B$ be a projective connected morphism from a normal $\mathbf{Q}$-factorial projective variety X with only canonical singularity onto B whose geometric generic fiber is an abelian variety and let $\varphi : \mathcal{A} \to B$ be the Albanese group scheme associated to f.*

(i) *we have $L^{ss}_{X/B} \sim_{\boldsymbol{Q}} L^{ss}_{\mathcal{A}/B}$ and*

(ii) *moreover, if we assume that $\dim X = 3$ and that $f : X \to B$ admits an analytic local section in a neighbourhood of any closed points $p \in B$, then we have $\Sigma_f = \Sigma_\varphi$ and $s^*_p(f) = s^*_p(\varphi)$, where $s^*_p(f)$ and $s^*_p(\varphi)$ are the analytic local bimeromorphic invariants s^*_p of the fibres of f and φ over $p \in B$ defined in Lemma 4.3 respectively. In particular, the Kodaira dimensions of X and $\mathcal{A}$ are the same.*

By Corollary 4.2, there exists $d \in \boldsymbol{N}$ such that $d(f) \leq d$ for any $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$ and in what follows, we fix such $d \in \boldsymbol{N}$. Firstly, take any $(X, f, B) \in \mathcal{CY}^3_{B,\mathrm{ab}}$. Then there exists a finite surjective morphism $\tau : B' \to B$ from a projective smooth irreducible curve B' with $\deg \tau \leq d$ which is étale over any closed points $p \in B$ with $s^*_p(f) = 0$ and $p \neq p_\circ$, where $p_\circ$ is a certain closed point of B which is not contained in Σ_f, such that $s^*_{p'}(f') = 0$ for any closed points $p' \in B'$, where $f' : X' \to B'$ is a minimal model of $p_2 : (X \times_B B')^\nu \to B'$. Since the number of closed points of B with $s^*_p(f) > 0$ is bounded by Proposition 4.5, we only have to bound $s_{f'} :=$ Card $\Sigma_{f'}$ to prove Theorem 5.1. Let $X'_{\eta'}$ be the generic fibre of f' over the generic point η' of B' and let $\varphi' : \mathcal{A}' \to B'$ be the Néron model of $\mathrm{Alb}^0(X'_{\eta'})$. Since $\ell^*_{p'}(f') = 1$ for any closed points $p' \in B'$ by Lemma 4.3, $f' : X' \to B'$ admits an analytic local section in a neighbourhood of any closed points $p' \in B'$, so we only have to bound $s_{\varphi'} :=$ Card $\Sigma_{\varphi'}$ by Lemma 5.1. From Proposition 4.4, we see that

$$\deg e'^* \det \Omega^1_{\mathcal{A}'/B'} = \deg L^{ss}_{\mathcal{A}'/B'} = \deg L^{ss}_{X'/B'} = (\deg \tau)(\deg L^{ss}_{X/B}) \leq 2d,$$

where $e' : B' \to \mathcal{A}'$ is the unit section of φ'.

Lemma 5.2 *$\varphi' : \mathcal{A}' \to B'$ has semi-abelian reduction at all closed points $p' \in B'$.*

Proof. From Proposition 4.4, we see that, for $\mathrm{Alb}^0(X'_{\eta'})$, the stable height equals to the differential height, hence by [25], Ch.X, proposition 2.5 (ii), we get the result. ■

Let K' be the rational function field of B'. Define an abelian variety Z' over K', by

$$Z' := (\mathrm{Alb}^0(X'_{\eta'}) \times_{K'} xAlb^0(X'_{\eta'})^t)^4,$$

where t denotes its dual. Let $\zeta' : \mathcal{Z}' \to B'$ be the Néron model of Z'.

Lemma 5.3 *Put $B'_\circ := B' \setminus \Sigma_{\varphi'}$ and let $\zeta'_\circ : \mathcal{Z}'_\circ \to B'_\circ$ denote the restriction of ζ' to $\zeta'^{-1}(B'_\circ)$. Then,*

(i) $\Sigma_{\zeta'} = \Sigma_{\varphi'}$ *and $\zeta'_\circ : \mathcal{Z}'_\circ \to B'_\circ$ is a principally polarized abelian scheme.*

(ii) $\zeta' : \mathcal{Z}' \to B'$ *has semi-abelian reduction at all closed points $p' \in B'$ and*

(iii) $\deg \zeta'_* \det \Omega^1_{\mathcal{Z}'/B'} = 8 \deg e'^* \det \Omega^1_{\mathcal{A}'/B'} \leq 16d$

Proof. Since $\varphi'_\circ : \mathrm{Alb}^0(X'_\circ/B'_\circ) \to B'_\circ$ is a projective abelian scheme, there exists a polarization $\lambda'_\circ : \mathrm{Alb}^0(X'_\circ/B'_\circ) \to \mathrm{Alb}^0(X'_\circ/B'_\circ)^t$ which is an isogeny over $B'_\circ$. Let $\varphi'^t : \mathcal{A}'^t \to B'$ be the Néron Model of $\mathrm{Alb}^0(X'_{\eta'})^t$. Then $\lambda'_\circ$ extends to an isogeny $\lambda' : \mathcal{A}' \to \mathcal{A}'^t$ over B', hence we have $\Sigma_{\varphi'} = \Sigma_{\varphi'^t}$ and $\mathcal{A}'^t \to B'$ also has semi-abelian reduction at all closed points $p' \in B'$ (see [1], §7.3, Proposition 6, Corollary 7). Therefore, using [1], §7.4 Proposition 3, we conclude that $\mathcal{Z}'^\circ = (\mathcal{A}'^\circ \times_{B'} \mathcal{A}'^{t\circ})^4 \to B'$ is a semi-abelian sheme and $\Sigma_{\zeta'} = \Sigma_{\varphi'}$, where $\mathcal{Z}'^\circ$, $\mathcal{A}'^\circ$ and $\mathcal{A}'^{t\circ}$ denotes connected components of $\mathcal{Z}'$, $\mathcal{A}'$ and $\mathcal{A}'^t$ containing their unit respectively. It is well known that $\zeta'_\circ : \mathcal{Z}'_\circ \to B'_\circ$ has a principal polarization and that we have the equality $\deg \zeta'_* \det \Omega^1_{\mathcal{Z}'/B'} = 8 \deg e'^* \det \Omega^1_{\mathcal{A}'/B'}$ (see [6], Ch I, §5, Lemma 5.5, [25], Ch IX and [45]). ■

Let H_g, $\mathrm{Sp}(2g, \boldsymbol{Z})$ and $\Gamma_{g,n}$ be the Siegel's upper half-plane, integral symplectic group of degree $2g$ and the congruence subgroup of level n of $\mathrm{Sp}(2g, \boldsymbol{Z})$ respectively as usual. For $n \geq 3$, $\mathcal{A}_{g,n} := H_g/\Gamma_{g,n}$ is known to be smooth quasi projective scheme over the complex number field $\boldsymbol{C}$. According to [5], there exists a normal projective variety $\mathcal{A}^*_{g,n}$ over $\boldsymbol{C}$ which contains $\mathcal{A}_{g,n}$ as a open subsheme and an ample invertible $\omega_{\mathcal{A}^*_{g,n}}$ such that for any g-dimensional principally polarized abelian variety A_K defined over K with a level n structure, the period map $\phi : B_\circ := B \setminus \Sigma_\varphi \to \mathcal{A}_{g,n}$ associated with $\varphi_\circ : \mathcal{A}_\circ := \varphi^{-1}(B_\circ) \to B_\circ$ extends to a morphism $\phi : B \to \mathcal{A}^*_{g,n}$ which gives an isomorphism $e^* \det \Omega^1_{\mathcal{A}/B} \simeq \phi^* \omega_{\mathcal{A}^*_{g,n}}$, where $\varphi : \mathcal{A} \to B$ is the Néron Model of A_K and e is its unit section. Take a natural number $N = N(g,n) \in \boldsymbol{N}$ depending only on g and n such that $\omega^{\otimes N}_{\mathcal{A}^*_{g,n}}$ is very ample. We may assume that there exists an effective divisor H on $\mathcal{A}^*_{g,n}$ such that $\mathcal{O}_{\mathcal{A}^*_{g,n}}(H) \simeq \omega^{\otimes N}_{\mathcal{A}^*_{g,n}}$ and that $\mathcal{A}^*_{g,n} \setminus \mathcal{A}_{g,n} \subset$ Supp H. Take any g-dimensional principally polarized abelian variety A_K defined over K with a level n structure. Under the assumption that $n \geq 3$, its Néron Model $\varphi : \mathcal{A} \to B$ is known to have semi-abelian reduction at any closed points of B so we have

$$\Sigma_\varphi \subset \phi^{-1}(\mathcal{A}^*_{g,n} \setminus \mathcal{A}_{g,n}) \subset \phi^{-1}(\mathrm{Supp}\ H)$$

and hence,

$$s_\varphi := \mathrm{Card}\ \Sigma_\varphi \leq \deg \phi^* H = N(g,n) \deg \phi^* \omega_{\mathcal{A}^*_{g,n}} = N(g,n) \deg e^* \det \Omega^1_{\mathcal{A}/B}. \tag{5.3}$$

If g-dimensional principally polarized abelian variety A_K does not have a level n structure, there exists an finite extension K' of K with $[K' : K] \leq \mathrm{Card}\ \mathrm{Sp}(2g, \boldsymbol{Z}/n\boldsymbol{Z})$ such that $A_{K'} := A_K \times_K K'$ has a level n-structure. Let $\tau : B' \to B$ be the finite surjective morphism from the smooth projective model B' of K' and let $\varphi' : \mathcal{A}' \to B'$ be the Néron Model of $A_{K'}$. If we assume that $\varphi : \mathcal{A} \to B$

has semi-abelian reduction at any closed points of B, then we have $\mathcal{A}' = \mathcal{A} \times_B B'$ and hence, $\Sigma_{\varphi'} = \tau^{-1}(\Sigma_\varphi)$. Thus, by (5.3), we have

$$\begin{aligned} s_\varphi \le s_{\varphi'} &\le N(g,n) \deg e'^* \det \Omega^1_{\mathcal{A}'/B'} && (5.4)\\ &= N(g,n) \deg \tau \deg e^* \det \Omega^1_{\mathcal{A}/B} && (5.5)\\ &= N(g,n) \text{ Card Sp}(2g, \boldsymbol{Z}/n\boldsymbol{Z}) \deg e^* \det \Omega^1_{\mathcal{A}/B}. && (5.6) \end{aligned}$$

Proof of Theorem 5.1. As we preiviously remarked, we only have to bound $s_{\varphi'}$. From Lemma 5.3, we have

$$\begin{aligned} s_{\varphi'} &= s_{\zeta'} && (5.7)\\ &\le N(16,3) \text{ Card Sp}(32, \boldsymbol{Z}/3\boldsymbol{Z}) \deg \zeta'_* \det \Omega^1_{\mathcal{Z}'/B'} && (5.8)\\ &\le 16dN(16,3) \text{ Card Sp}(32, \boldsymbol{Z}/3\boldsymbol{Z}). && (5.9) \end{aligned}$$

Thus we get the assertion. ∎

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