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ANTICANONICAL DIVISORS OF A MODULI SPACE OF PARABOLIC VECTOR BUNDLES OF HALF WEIGHT ON \mathbb{P}^1

TAKESHI ABE

ABSTRACT. This is a revised version of my preprint Kyoto-Math 2002-11, on which I talked at Hodge theory and algebraic geometry workshop held at Hokkaido Univ., Oct. 2002. I am grateful to the organizers.

1. INTRODUCTION

Let \mathcal{M}_0 [resp. \mathcal{M}_1] be a coarse moduli space of rank 2 semistable vector bundles of even [resp. odd] degree with fixed determinant on a smooth projective curve X . The Picard group is infinite cyclic. Let L be the ample generator. The dimension of a vector space $H^0(\mathcal{M}_i, L^m)$ ($i = 0, 1$) is given by the Verlinde formula. For small $m > 0$, the meaning of this dimension can be explained in the framework of algebraic geometry. For example, we have

$$\dim H^0(\mathcal{M}_0, L) = 2^g,$$

where g is the genus of X . On the otherhand, we have

$$\dim H^0(\text{Jac}(X), \mathcal{O}(2\Theta)) = 2^g.$$

In fact we have a natural isomorphism between these two vector spaces (See [1]). In [2], the meaning of the two equations

$$\begin{aligned}\dim H^0(\mathcal{M}_0, L^2) &= 2^{g-1}(2^g + 1) \\ \dim H^0(\mathcal{M}_1, L) &= 2^{g-1}(2^g - 1)\end{aligned}$$

are clarified. The above dimensions are the number of even or odd theta characteristics on X . Beauville associated to an even [resp. odd] theta characteristic κ a divisor D_κ on \mathcal{M}_0 [resp. \mathcal{M}_1] that can be described from a moduli-theoretic viewpoint, and proved that they form a basis of $H^0(\mathcal{M}_0, L^2)$ [resp. $H^0(\mathcal{M}_1, L)$]. In [13], two vector spaces $H^0(\mathcal{M}_0, L^4)$ and $H^0(\mathcal{M}_1, L^2)$ are considered.

The purpose of this paper is to carry out a similar study for a moduli space $\overline{\mathcal{M}}^{\text{Par}}(\mathbb{P}^1; I)$ of rank 2 semistable parabolic vector bundles with half weights of degree zero on \mathbb{P}^1 .

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To be more precise, by Verlinde formula, we have

$$\dim H^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}^{-1}) = \frac{2^{2g+1} + 1}{3}$$

for $I = \{x_1, \dots, x_{2g+2}\}$, and

$$\dim H^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}^{-1}) = \frac{2^{2g} - 1}{3}$$

for $I = \{x_1, \dots, x_{2g+1}\}$. Let us consider the case when $|I|$ is even. Let C be the hyperelliptic curve whose branch locus is I . By a result of Bhosle in [4], $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ is isomorphic to a moduli space $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$ of rank 2 semistable vector bundles endowed with an involution action with trivial determinat. On $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$, we shall find as many effective anticanonical divisors as the dimension of $H^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}^{-1})$ that are described moduli-theoretically, and shall prove that they form a basis of $H^0(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}}^{-1})$ (Theorem3.4). In Section 4, we investigate a relation between these divisors and those divisors constructed by Beauville (Theorem4.1, 4.2). It would be desirable that the divisors constructed on $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$ in Section 3 are defined set-theoretically. So we shall prove the reducedness of these divisors in Section 5.

Notation. • Otherwise mentioned, all schemes are of finite type over \mathbb{C} .

- If a group $G = \langle g \mid g^2 = e \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on a coherent sheaf F on a scheme X , F^+ [resp. F^-] stands for the g -invariant [resp. $(-g)$ -invariant] subsheaf of F . For a vector space V acted on by G , we define V^+ and V^- similarly. $\chi(F)^+$ means $\sum (-1)^i H^i(X, F)^+$.
- If E is a locally free sheaf, $\mathcal{E}nd^0(E)$ is a sheaf of traceless endomorphisms.
- Otherwise mentioned, p_X, p_Y, p_Z etc. stand for projections to the factors X, Y, Z etc..

2. PRELIMINARIES

Let X be an smooth irreducible projective curve over \mathbb{C} and I a non-empty finite set of points of X . Let E be a vector bundle on X .

Definition 2.1. A parabolic structure on E is giving, at each point $x \in I$, a filtration $E_x = F_1(E)_x \supset \dots \supset F_{n_x(E)}(E) \supset F_{n_x(E)+1}(E) = 0$ and a sequence of real numbers, called parabolic weights, $0 \leq a_1(x) < \dots < a_{n_x(E)}(x) < 1$. The parabolic degree of E , denoted by $\text{pardeg}(E)$, is defined by

$$\text{pardeg}(E) := \deg E + \sum_{x \in I} \sum_{i=1}^{n_x(E)} a_i(x) \left(\dim F_i(E)_x - \dim F_{i+1}(E)_x \right)$$

2.2. Let E' be a subbundle of a parabolic vector bundle E . We can equip E' with the canonical parabolic structure: the filtration of E' consists of the distinct ones of $\{E'_x \cap F_i(E)_x\}$, and the parabolic weight $a'_j(x)$ of E' is given by $a'_j(x) = a_i(x)$, where i is the biggest integer satisfying $F_j(E')_x = E'_x \cap F_i(E)_x$.

Definition 2.3. A parabolic vector bundle E is said to be semistable [resp. stable] if for any subbundle E' of E with $0 < \text{rank} E' < \text{rank} E$, we have

$$\frac{\text{pardeg} E'}{\text{rank} E'} \leq \frac{\text{pardeg} E}{\text{rank} E} \quad [\text{resp. } <],$$

where E' is equipped with the canonical parabolic structure.

2.4. In this paper, we are concerned with only rank 2 parabolic bundles of parabolic weights $(0, \frac{1}{2})$ at each $x \in I$. Let $\overline{\mathcal{M}}^{Par}(X; I)$ [resp. $\mathcal{M}^{Par}(X; I)$] be the coarse moduli space of semistable [resp. stable] rank 2 parabolic vector bundles with trivial determinant of parabolic weights $(0, \frac{1}{2})$ at each point $x \in I$

2.5. Given a family of parabolic rank 2 vector bundles with trivial determinant of weight $(0, \frac{1}{2})$ at each point $x \in I$ parametrized by a scheme S , i.e. a rank 2 vector bundle \mathcal{E} on $S \times X$ such that $\det \mathcal{E}$ is a pull-back of a line bundle on S and a surjection $\mathcal{E}|_{S \times \{x\}} \rightarrow \mathcal{Q}_x$ for each $x \in I$, where \mathcal{Q}_x is a line bundle on $S \times \{x\}$ such that, for any $s \in S$, $\mathcal{E}|_s$ is parabolic vector bundle of parabolic weight $(0, \frac{1}{2})$ at each point $x \in I$, we denote the line bundle $(\det R p_{S*} \mathcal{E})^{-2} \otimes (\det R p_{S*} \mathcal{E}')^{-2} \otimes (\det \mathcal{E}|_{S \times \{y\}})^{4(1-g)-|I|}$ by $\Xi_{\mathcal{E}}$, where g is the genus of X and y is a point of X and $\mathcal{E}' := \text{Ker}(\mathcal{E} \rightarrow \bigoplus_{x \in I} \mathcal{Q}|_{S \times \{x\}})$.

In [14] Pauly proved the following theorem.

Theorem 2.6. *There exists a unique line bundle Ξ on $\overline{\mathcal{M}}^{Par}(X; I)$ such that for any family of semistable rank 2 parabolic vector bundles \mathcal{E} with trivial determinant of weight $(0, \frac{1}{2})$ at each point $x \in I$ parametrized by a scheme S , $f^* \Xi \simeq \Xi_{\mathcal{E}}$, where f is the natural morphism $f : S \rightarrow \overline{\mathcal{M}}^{Par}(X; I)$. Moreover Ξ is ample.*

Proposition 2.7. *Let Ξ be as above. Then $\Xi|_{\mathcal{M}^{Par}(X; I)} \simeq K_{\mathcal{M}^{Par}(X; I)}^{-1}$.*

Proof. Recall that $\mathcal{M}^{Par}(X; I)$ is constructed as a geometric quotient of a smooth variety R^s by a PGL -action, where R^s is an open subscheme of a $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ -bundle over an open subset of a quot scheme. Note that the quotient map $\pi : R^s \rightarrow \mathcal{M}^{Par}(X; I)$ is a principal PGL -bundle. In order to prove the proposition, we have to prove that $\pi^* K_{\mathcal{M}^{Par}(X; I)}^{-1}$ and $\Xi_{\mathcal{E}}$ are isomorphic as PGL -linearized line bundles on R^s , where \mathcal{E} is the universal parabolic bundle on $R^s \times X$. By Lemma 3.2 of [5], it suffices to prove that $\pi^* K_{\mathcal{M}^{Par}(X; I)}^{-1}$ and $\Xi_{\mathcal{E}}$ are isomorphic as line bundles. Since $\pi^* T_{\mathcal{M}^{Par}(X; I)} \simeq R^1 p_* \mathcal{E} \text{nd}^\circ(\mathcal{E}' \subset \mathcal{E})$, where $\mathcal{E} \text{nd}^\circ(\mathcal{E}' \subset \mathcal{E})$ is a sheaf of traceless endomorphism preserving $\mathcal{E}' :=$

$\text{Ker}(\mathcal{E} \rightarrow \bigoplus_{x \in I} \mathcal{Q}|_{R^s \times \{x\}})$, we need to prove that $\det R^1 p_* \mathcal{E} \text{nd}^\circ(\mathcal{E}' \subset \mathcal{E}) \simeq (\det R^1 p_* \mathcal{E}')^{-2} \otimes (\det R^1 p_* \mathcal{E})^{-2} \otimes (\det \mathcal{E}|_{S \times \{y\}})^{4(1-g)-|I|}$, where p is the projection $R^s \times C \rightarrow R^s$. A calculation using Riemann-Roch theorem implies that $\det R^1 p_* \mathcal{E} \text{nd}^\circ(\mathcal{E}' \subset \mathcal{E})$ and $(\det R^1 p_* \mathcal{E}')^{-2} \otimes (\det R^1 p_* \mathcal{E})^{-2} \otimes (\det \mathcal{E}|_{S \times \{y\}})^{4(1-g)-|I|}$ are isomorphic modulo torsion. Using Theorem 2.3 of [5], we know $\det R^1 p_* \mathcal{E} \text{nd}^\circ(\mathcal{E}' \subset \mathcal{E})$ also descends to $\overline{\mathcal{M}}^{\text{Par}}(X; I)$. Therefore the proof will be completed if we prove $\text{Pic}(\overline{\mathcal{M}}^{\text{Par}}(X; I))$ is torsion-free. This follows from the next lemma. \square

Lemma 2.8. $\text{Pic}(\overline{\mathcal{M}}^{\text{Par}}(X; I))$ is torsion-free.

Proof. Since $\overline{\mathcal{M}}^{\text{Par}}(X; I)$ is a GIT quotient of a smooth variety, it has rational singularities by [10]. On top of that, the canonical divisor of $\overline{\mathcal{M}}^{\text{Par}}(X; I)$ is Cartier, hence it has canonical singularities by Corollary 5.24 of [9]. Since $K_{\overline{\mathcal{M}}^{\text{Par}}(X; I)}^{-1}$ is ample, we have $H^i(\overline{\mathcal{M}}^{\text{Par}}(X; I), \mathcal{O}_{\overline{\mathcal{M}}^{\text{Par}}(X; I)}) = 0$ for any $i > 0$ by Theorem 1-2-5 in [8]. Given a finite étale morphism $f : Y \rightarrow \overline{\mathcal{M}}^{\text{Par}}(X; I)$, we also have $H^i(Y, \mathcal{O}_Y) = 0$ for any $i > 0$. Since $1 = H^0(Y, \mathcal{O}_Y) = \chi(\mathcal{O}_Y) = \deg f \chi(\mathcal{O}_{\overline{\mathcal{M}}^{\text{Par}}(X; I)}) = \deg f \dim H^0(\overline{\mathcal{M}}^{\text{Par}}(X; I), \mathcal{O}_{\overline{\mathcal{M}}^{\text{Par}}(X; I)}) = \deg f$, f is an isomorphism. This proves the torsion-freeness of $\text{Pic}(\overline{\mathcal{M}}^{\text{Par}}(X; I))$. \square

Remark 2.9. Lemma 2.8 also follows from the description of the Picard group of a moduli stack of quasi-parabolic bundles given in [11].

2.10. From now on throughout this paper, we treat the case when $X = \mathbb{P}^1$. When $I = \{x_1, \dots, x_{2g+2}\}$, put $B := I$. When $I = \{x_1, \dots, x_{2g+1}\}$, put $B := I \cup \{x_{2g+2}\}$, where x_{2g+2} is a point of $X - I$. Let $\pi : C \rightarrow X = \mathbb{P}^1$ be the hyperelliptic curve of genus g whose branch points are B . Let $i : C \rightarrow C$ be the involution and put $\{c_j\} := \pi^{-1}(x_j)$. These notations are used throughout this paper.

Definition 2.11. An involutorial vector bundle on C is a vector bundle F on C endowed with an i -action, i.e., an isomorphism $\alpha : i^* F \rightarrow F$ such that $\alpha \circ i^* \alpha$ is the identity. An involutorial vector bundle F is said to be involutorially semistable [resp. involutorially stable] if for any i -invariant subbundle G with $0 < \text{rank} G < \text{rank} F$ the inequality $\deg G / \text{rank} G \leq \deg F / \text{rank} F$ [resp. $\deg G / \text{rank} G < \deg F / \text{rank} F$] holds.

Remark 2.12. An involutorial vector bundle is involutorially semistable if and only if it is semistable as a vector bundle.

2.13. When I is even, we denote by $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}$ [resp. $\mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}$] the coarse moduli space of rank 2 involutorially semistable [resp. involutorially stable] involutorial vector bundles on C with trivial determinant such

that the eigen values of the i -action on the fiber over c_j are 1 and -1 for $1 \leq j \leq 2g + 2$.

When I is odd, we denote by $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$ [resp. $\mathcal{M}_{C/\mathbb{P}^1}^{invo}$] the coarse moduli space of rank 2 involutinally semistable [resp. involutinally stable] involutinal vector bundles on C with determinant $\mathcal{O}(c_{2g+2})$ such that the eigen values of the i -action on the fiber over c_j are 1 and -1 for $1 \leq j \leq 2g + 1$, and 1 with multiplicity two for $j = 2g + 2$ and when g is odd, and -1 with multiplicity two for $j = 2g + 2$ and when g is even.

Let $(E, \{E_{x_j} = F_1(E)_{x_j} \supset F_2(E)_{x_j}\})$ be an element of $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$. When $|I|$ is even [resp. odd], put $\tilde{E} := \text{Ker}(\pi^*E \rightarrow \bigoplus_{j=1}^{2g+2} F_1(E)_{x_j}/F_2(E)_{x_j})$ [resp. $\tilde{E} := \text{Ker}(\pi^*E \rightarrow \bigoplus_{j=1}^{2g+1} F_1(E)_{x_j}/F_2(E)_{x_j})$], where the morphism $\pi^*E \rightarrow F_1(E)_{x_j}/F_2(E)_{x_j}$ is given by $\pi^*E \rightarrow \pi^*E \otimes \mathbb{C}_{c_j} \simeq E \otimes \mathbb{C}_{c_j} \rightarrow F_1(E)_{x_j}/F_2(E)_{x_j}$. We endow \tilde{E} with the natural i -action. Then by Proposition 1.2 of [4], we obtain an isomorphism between $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ and $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$, taking $(E, \{E_{x_j} = F_1(E)_{x_j} \supset F_2(E)_{x_j}\})$ to $\tilde{E} \otimes \mathcal{O}_C((g+1)c_{2g+2})$, where $\mathcal{O}_C((g+1)c_{2g+2})$ is given the natural i -action.

3. A BASIS OF ANTICANONICAL SECTIONS

In the rest of this paper, we assume that $g \geq 2$ mainly because we want the codimension of the locus of non-stable bundles in the moduli to be greater than one.

3.1. In Section 2, we saw that, for $I = \{x_1, \dots, x_{2g+2}\}$ or $\{x_1, \dots, x_{2g+1}\}$, $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ and $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$ are isomorphic. Since $\text{codim}(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I) \setminus \mathcal{M}^{Par}(\mathbb{P}^1; I), \overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)) \geq 2$, Proposition 2.7 implies that the canonical divisor of $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ is a Cartier divisor and that $\Xi \simeq K_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}^{-1}$. We have isomorphisms of vector spaces

$$\begin{aligned} \mathrm{H}^0(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}}^{-1}) &\simeq \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}^{-1}) \\ &\simeq \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), \Xi). \end{aligned}$$

By Verlinde formula (cf. [3]), the dimension of these vector spaces is $\frac{2^{2g+1}+1}{3}$ if $|I|$ is even, and $\frac{2^{2g}-1}{3}$ if $|I|$ is odd. Our goal in this section is to find a basis of $\mathrm{H}^0(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}}^{-1})$. We define $\mathcal{S}^{\text{even}}$ and \mathcal{S}^{odd} to be the sets given by

$$\mathcal{S}^{\text{even}} := \{(\mathfrak{A}, \lambda) \in 2^{\{c_1, \dots, c_{2g+1}\}} \times \mathbb{Z} \mid 4|\mathfrak{A}| + 3\lambda = 4g - 2\}$$

and

$$\mathcal{S}^{\text{odd}} := \left\{ (\mathfrak{A}, \lambda) \in 2^{\{c_1, \dots, c_{2g+1}\}} \times \mathbb{Z} \mid \begin{array}{l} |\mathfrak{A}| \leq g-1 \\ 4|\mathfrak{A}| + 3\lambda = 4g-4 \end{array} \right\}.$$

3.2. Let \mathcal{F} be a family of involutively vector bundles on C parameterized by an irreducible scheme S . If $\chi(\mathcal{F}|_{\{s\} \times C})^+ = 0$ for any $s \in S$ and $H^0(C, \mathcal{F}|_{\{s\} \times C})^+ = 0$ for general $s \in S$, then we can construct an effective divisor denoted by $\text{Div}(R^1 p_{S*}(\mathcal{F})^+)$ whose support is $\{s \in S \mid H^0(C, \mathcal{F}|_{\{s\} \times C})^+ \neq 0\}$: for any $s \in S$ we can find a resolution $0 \rightarrow \mathcal{O}_U^{\oplus m} \xrightarrow{(\varphi_{ij})} \mathcal{O}_U^{\oplus m} \rightarrow R^1 p_{S*}(\mathcal{F})^+ \rightarrow 0$ in a neighborhood U of s , then on U $\text{Div}(R^1 p_{S*}(\mathcal{F})^+)$ is defined by the equation $\det(\varphi_{ij}) = 0$.

Recall that $\mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}$ is constructed as a geometric quotient of a smooth irreducible variety Z by an action of a reductive algebraic group G of the form $GL(a_1) \times GL(a_2) / \{(t\text{Id}, t\text{Id}) \mid t \in \mathbb{G}_m\}$, where Z is an open subscheme of an equivariant quot scheme with trivial determinant. Let $\mathcal{O}_{Z \times C}(-N) \otimes V \rightarrow \mathcal{Q} \rightarrow 0$ be the universal quotient, where $\mathcal{O}_{Z \times C}(-N) \otimes V$ and \mathcal{Q} have an i -action. For (\mathfrak{A}, λ) in $\mathcal{S}^{\text{even}}$ or \mathcal{S}^{odd} , we have the divisor $\text{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)$ on Z by 3.2. By construction, it is G -invariant, hence it is a pullback of a divisor, say $D_{(\mathfrak{A}, \lambda)}^\circ$, on $\mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}$.

Lemma 3.3. *We have an isomorphism $\mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}}(D_{(\mathfrak{A}, \lambda)}^\circ) \simeq K_{\mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}}^{-1}$.*

Proof. We have to prove that the two line bundles $\det(R^1 p_{Z*} \mathcal{E}nd^\circ(\mathcal{Q})^+)$ and $\mathcal{O}_Z(\text{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+))$ are isomorphic as G -linearized line bundles. We can check that

$$\begin{aligned} & \mathcal{O}_Z(\text{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)) \\ & \simeq \det(R^1 p_{Z*} \mathcal{E}nd^\circ(\mathcal{Q})^+) \otimes \otimes_{c_j \in \mathfrak{A}} \left(\det \mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_j\}}^- \right)^{-1} \\ & \otimes \left(\det(\mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_{2g+2}\}}^+) \otimes \det(\mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_{2g+2}\}}^-) \right)^{-\frac{\lambda}{2}} \end{aligned}$$

as G -linearized line bundles. Since $\otimes_{c_j \in \mathfrak{A}} \left(\det \mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_j\}}^- \right)^{-1} \otimes \left(\det(\mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_{2g+2}\}}^+) \otimes \det(\mathcal{E}nd^\circ(\mathcal{Q})|_{Z \times \{c_{2g+2}\}}^-) \right)^{-\frac{\lambda}{2}}$ is trivial as a G -linearized line bundle, the lemma is proved. \square

Since $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}} \simeq \overline{\mathcal{M}}^{\text{Par}}(X; I)$ and $K_{\overline{\mathcal{M}}^{\text{Par}}(X; I)}$ is a Cartier divisor by Proposition 2.7, $K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}}$ is also a Cartier divisor. Since $\text{codim}(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}} \setminus \mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}, \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}) \geq 2$, $D_{(\mathfrak{A}, \lambda)}^\circ$ extends uniquely to an effective divisor, denoted by $D_{(\mathfrak{A}, \lambda)}$, on $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}$. Now we can state the main theorem of this section.

Theorem 3.4. *If $|I|$ is even [resp. odd], $\{D_{(\mathfrak{A}, \lambda)}\}_{(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{even}}}$ [resp. $\{D_{(\mathfrak{A}, \lambda)}\}_{(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{odd}}}$] is a basis of $H^0(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}}^{-1})$.*

We give a proof only for the case when $|I|$ is even. The following lemmas in this section are valid only for this case. The proof of the case when $|I|$ is odd is similar.

We follow closely the proof given in [2]

3.5. Let \mathcal{F} be a family of involutively stable involutorial vector bundles with trivial determinant on C parameterized by a scheme by S . Then we have a natural map $\varphi : S \rightarrow \mathcal{M}_{C/\mathbb{P}^1}^{\text{invo}}$. We have $\chi(\mathcal{E}nd^\circ(\mathcal{F}|_{\{s\} \times C}) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$ for any $s \in S$ and any $(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{even}}$. Assume that S is irreducible and that $H^0(\mathcal{E}nd^\circ(\mathcal{F}|_{\{s\} \times C}) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$ for general $s \in S$. By the construction of $D_{(\mathfrak{A}, \lambda)}$, we have $\text{Div}(R^1 p_{S*}(\mathcal{E}nd^\circ(\mathcal{F}) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})))^+ = \varphi^* D_{(\mathfrak{A}, \lambda)}$ as effective divisors on S .

Let \mathcal{P} be a Poincaré line bundle on $\text{Jac}(C) \times C$. Since $\mathcal{P} \oplus (1 \times i)^* \mathcal{P}$ has a natural i -action, it is a family of rank 2 involutorial vector bundles with trivial determinant on C parameterized by $\text{Jac}(C)$.

Lemma 3.6. (i) $P \oplus i^* P$ is involutorially semistable for any $P \in \text{Jac}(C)$
 (ii) For $P \in \text{Jac}(C)$, $P \oplus i^* P$ is not involutorially stable if and only if $P \in \text{Jac}(C)[2]$, where $\text{Jac}(C)[2]$ is the subgroup of $\text{Jac}(C)$ consisting of 2-torsion points.

Proof. (i) is obvious by Remark 2.12. If $P \in \text{Jac}(C)[2]$, P and $i^* P$ are isomorphic. We can find an isomorphism $\alpha : P \rightarrow i^* P$ such that the composite of α and $i^* \alpha : i^* P \rightarrow i^* i^* P \simeq P$ is the identity. Then $(1, \alpha) : P \hookrightarrow P \oplus i^* P$ is an i -invariant line subbundle. Since $\deg P = 0$, $P \oplus i^* P$ is not involutorially stable. Conversely, if $P \oplus i^* P$ is not involutorially stable, we can find an i -invariant line subbundle A of $P \oplus i^* P$ with $\deg A = 0$. Since $A \simeq i^* A$, $A \in \text{Jac}(C)[2]$. The inclusion $A \hookrightarrow P \oplus i^* P$ gives rise to an isomorphism $A \xrightarrow{\sim} P$ or $A \xrightarrow{\sim} i^* P$. Hence $P \in \text{Jac}(C)[2]$. \square

3.7. By Lemma 3.6, $\mathcal{P} \oplus i^* \mathcal{P}$ gives rise to the natural morphism $\varphi : \text{Jac}(C) \rightarrow \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}$.

Lemma 3.8. Take $(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{even}}$.

- (i) If $|\mathfrak{A}| \neq g + 1$, then $H^0(C, \mathcal{E}nd^\circ(L \oplus i^* L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \neq 0$ for any $L \in \text{Jac}(C)$.
- (ii) If $|\mathfrak{A}| = g + 1$, then $H^0(C, \mathcal{E}nd^\circ(L \oplus i^* L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$ for general $L \in \text{Jac}(C)$ and $\varphi^* D_{(\mathfrak{A}, \lambda c_{2g+2})} = 2_{\text{Jac}}^* T_{(\mathfrak{A}, \lambda)}^* \Theta$, where Θ is the theta divisor on $\text{Jac}(C)^{g-1}$ and $2_{\text{Jac}} : \text{Jac}(C) \rightarrow \text{Jac}(C)$

is given by $L \mapsto L^{\otimes 2}$ and $T_{(\mathfrak{A}, \lambda)} : \text{Jac}(C) \rightarrow \text{Jac}(C)^{g-1}$ is given by $L \mapsto L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$.

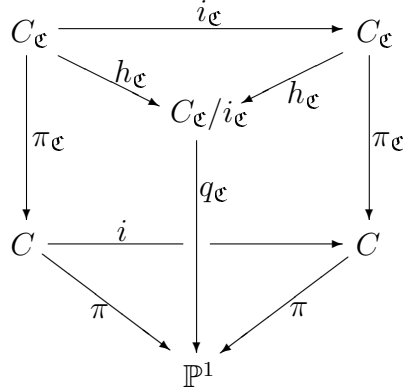
Proof. For $E = L \oplus i^*L$, where $L \in \text{Jac}(C)$ and E is endowed with the natural i -action, $\mathcal{E}nd^\circ(E)$ is isomorphic to $\mathcal{O}_C \oplus ((L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L))$, where i acts on \mathcal{O}_C by (-1) multiplication and on $(L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)$ by $i^*((L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)) \simeq (i^*L^{-1} \otimes L) \oplus (L^{-1} \otimes i^*L) \xrightarrow{\text{switch the factors}} (L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)$. Hence we have $H^0(C, \mathcal{E}nd^\circ(L \oplus i^*L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- \oplus H^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))$. If $|\mathfrak{A}| < g+1$, then $\deg i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) > g-1$, hence $H^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})) \neq 0$. Since $\dim H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- = \max\{\frac{-2-\lambda}{4}, 0\}$, if $|\mathfrak{A}| > g+1$ then $\dim H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- \neq 0$. These prove (i).

If $|\mathfrak{A}| = g+1$, then $\dim H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- = 0$ and $H^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})) = 0$ for general $L \in \text{Jac}(C)$ because $\deg i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) = g-1$. This implies the former statement of (ii). Since we are assuming that $g \geq 2$, in order to prove the latter statement of (ii), it suffices to see that $\varphi^*D_{(\mathfrak{A}, \lambda c_{2g+2})} = 2_{\text{Jac}}^* T_{(\mathfrak{A}, \lambda)}^* \Theta$ over $\text{Jac}(C) \setminus \text{Jac}(C)[2]$. Over $\text{Jac}(C) \setminus \text{Jac}(C)[2]$, $\mathcal{P} \oplus i^*\mathcal{P}$ gives a family of involutively stable involutorial vector bundles, hence we have $\varphi^*D_{(\mathfrak{A}, \lambda)} = \text{Div}(R^1 p_{\text{Jac}(C) \setminus \text{Jac}(C)[2]}(\mathcal{E}nd^\circ(\mathcal{P} \oplus i^*\mathcal{P}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)$ by 3.5. As above $R^1 p_{\text{Jac}(C) \setminus \text{Jac}(C)[2]}(\mathcal{E}nd^\circ(\mathcal{P} \oplus i^*\mathcal{P}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq R^1 p_{\text{Jac}(C) \setminus \text{Jac}(C)[2]}(i^*\mathcal{P}^{-1} \otimes \mathcal{P} \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))$. The latter statement follows from this. \square

We define the set \mathcal{T} by

$$\mathcal{T} := \left\{ \mathfrak{C} \subseteq_{\neq} \{c_1, \dots, c_{2g+2}\} \mid \begin{array}{l} |\mathfrak{C}| \text{ is even.} \\ \mathfrak{C} \neq \phi \end{array} \right\}.$$

Since $\mathcal{O}(\mathfrak{C} - |\mathfrak{C}|c_{2g+2}) \in \text{Jac}(C)[2]$ for any $\mathfrak{C} \in \mathcal{T}$, from this we can construct an étale double cover $\pi_{\mathfrak{C}} : C_{\mathfrak{C}} \rightarrow C$. Moreover, since $\mathcal{O}(\mathfrak{C} - |\mathfrak{C}|c_{2g+2})$ has the natural i -action, we get an involution $i_{\mathfrak{C}} : C_{\mathfrak{C}} \rightarrow C_{\mathfrak{C}}$ such that $\pi_{\mathfrak{C}} \circ i_{\mathfrak{C}} = i \circ \pi_{\mathfrak{C}}$. By $h_{\mathfrak{C}} : C_{\mathfrak{C}} \rightarrow C_{\mathfrak{C}}/i_{\mathfrak{C}}$ we mean the quotient of $C_{\mathfrak{C}}$ by this involution. There is a unique map $q_{\mathfrak{C}} : C_{\mathfrak{C}}/i_{\mathfrak{C}} \rightarrow \mathbb{P}^1$ such that $q_{\mathfrak{C}} \circ h_{\mathfrak{C}} = \pi \circ \pi_{\mathfrak{C}}$.



The branch locus of $q_{\mathfrak{e}}$ is $p(\mathfrak{e})$ and the genus of $C_{\mathfrak{e}}/i_{\mathfrak{e}}$ is $\frac{|\mathfrak{e}|}{2} - 1$. Put $\{c_j^{\mathfrak{e}}, c_j^{\dagger\mathfrak{e}}\} := \pi_{\mathfrak{e}}^{-1}(c_j)$. For $N \in \text{Jac}(C_{\mathfrak{e}}/i_{\mathfrak{e}})$, the line bundle $h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})$ on $C_{\mathfrak{e}}$ has the natural $i_{\mathfrak{e}}$ -action, which induces an i -action on $\pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$. Note that $\det\left(\pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})\right) \simeq \mathcal{O}_C$ and the eigen values of the i -action on the fiber of $\pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$ over every c_j are 1 and -1 .

Lemma 3.9. *The above involutorial vector bundle $\pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$ is involutorially stable.*

Proof. Given a line subbundle $L \subset \pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$, we have a nonzero homomorphism $\pi_{\mathfrak{e}}^*L \rightarrow h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}}) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}(c_{2g+2}^{\mathfrak{e}} + c_{2g+2}^{\dagger\mathfrak{e}}))$. Since $\deg h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}}) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}(c_{2g+2}^{\mathfrak{e}} + c_{2g+2}^{\dagger\mathfrak{e}})) = 0$, we have $\deg L \leq 0$, which implies the semistability of $\pi_{\mathfrak{e}*}(h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$.

Suppose that the above L is i -invariant and of degree zero. Then there exists a subset U of $\{c_1, \dots, c_{2g+2}\}$ with $|U|$ even such that L is isomorphic to $\mathcal{O}(\sum_{c_j \in U} c_j - |U|c_{2g+2})$ as involutorial bundles. Then $h_{\mathfrak{e}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}}) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{e}|+2|U|}{2}(c_{2g+2}^{\mathfrak{e}} + c_{2g+2}^{\dagger\mathfrak{e}})) \otimes \mathcal{O}_C(-\sum_{c_j \in U} (c_j^{\mathfrak{e}} + c_j^{\dagger\mathfrak{e}}))$ is a trivial line bundle with a trivial $i_{\mathfrak{e}}$ -action. For $c_j \notin (\mathfrak{e} \cup \{c_{2g+2}\})$, however, either $c_j^{\mathfrak{e}}$ or $c_j^{\dagger\mathfrak{e}}$ appears with an odd multiplicity. This means that the above line bundle cannot be trivial as an involutorial bundle. \square

3.10. Let $\mathcal{P}_{\mathfrak{e}}$ be a Poincaré line bundle on $\text{Jac}(C_{\mathfrak{e}}/i_{\mathfrak{e}}) \times C_{\mathfrak{e}}/i_{\mathfrak{e}}$. By Lemma 3.9, $(1 \times \pi_{\mathfrak{e}})_* \left((1 \times h_{\mathfrak{e}})^* \mathcal{P}_{\mathfrak{e}} \otimes p_{C_{\mathfrak{e}}}^* \mathcal{O}(-\sum_{c_j \notin \mathfrak{e}} c_j^{\mathfrak{e}}) \right) \otimes p_C^* \mathcal{O}(\frac{2g+2-|\mathfrak{e}|}{2}c_{2g+2})$ is a family of involutorially stable involutorial vector bundles with trivial determinant, hence gives rise to a natural map $\varphi_{\mathfrak{e}} : \text{Jac}(C_{\mathfrak{e}}/i_{\mathfrak{e}}) \rightarrow \mathcal{M}_{C/\mathbb{P}^1}^{invo} \subset \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$.

Lemma 3.11. *Take $(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{even}}$.*

(i) *If $2|\mathfrak{A} \cap \mathfrak{C}| + \lambda - |\mathfrak{C}| + 2 \neq 0$, then*

$$H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{C}*} \left(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right)^+ \neq 0$$

for any $N \in \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$.

(ii) *If $2|\mathfrak{A} \cap \mathfrak{C}| + \lambda - |\mathfrak{C}| + 2 = 0$, then*

$$H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{C}*} \left(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right)^+ = 0$$

for general $N \in \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$, and we have $\varphi_{\mathfrak{C}}^(D_{(\mathfrak{C}, \lambda)}) = 2_{\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}^* \left(T_{\tau((\mathfrak{A}, \lambda); \mathfrak{C})}^* \Theta \right)$,*

where Θ is the theta divisor on $\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})^{\frac{|\mathfrak{C}|}{2}-2}$, $2_{\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})} : \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}}) \rightarrow \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$ is defined by $L \mapsto L^{\otimes 2}$ and $T_{\tau((\mathfrak{A}, \lambda); \mathfrak{C})} : \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}}) \rightarrow \text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})^{\frac{|\mathfrak{C}|}{2}-2}$ by $L \mapsto L \otimes \tau((\mathfrak{A}, \lambda); \mathfrak{C})$, here

$$\begin{aligned} & \tau((\mathfrak{A}, \lambda), \mathfrak{C}) \\ & := \mathcal{O} \left(2 \sum_{c_j \in \mathfrak{C} \cap \mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) + \sum_{c_j \in \mathfrak{A} \setminus \mathfrak{C}} h_{\mathfrak{C}}(c_j^{\dagger \mathfrak{C}}) \right. \\ & \quad \left. + \sum_{c_j \in \mathfrak{C} \setminus \mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) - \sum_{c_j \notin \mathfrak{C} \cup \mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) + \frac{\lambda - |\mathfrak{C}|}{2} \left(h_{\mathfrak{C}}(c_{2g+2}^{\mathfrak{C}}) + h_{\mathfrak{C}}(c_{2g+2}^{\dagger \mathfrak{C}}) \right) \right). \end{aligned}$$

Proof. On $C_{\mathfrak{C}}$, we have the exact sequence

$$\begin{aligned} 0 & \rightarrow h_{\mathfrak{C}}^*(N^{-1}) \otimes \mathcal{O} \left(- \sum_{c_j \in \mathfrak{C}} (c_j^{\mathfrak{C}} + c_j^{\dagger \mathfrak{C}}) - \sum_{c_j \notin \mathfrak{C}} c_j^{\dagger \mathfrak{C}} + |\mathfrak{C}|(c_{2g+2}^{\mathfrak{C}} + c_{2g+2}^{\dagger \mathfrak{C}}) \right) \\ & \rightarrow \pi_{\mathfrak{C}*} \pi_{\mathfrak{C}*} \left(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \right) \rightarrow h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \rightarrow 0. \end{aligned}$$

From this, we have the exact sequence on C ,

$$\begin{aligned} 0 & \rightarrow \mathcal{O}(\mathfrak{A} + \mathfrak{C} + (\lambda - |\mathfrak{C}|)c_{2g+2}) \\ & \rightarrow \mathcal{E}nd^\circ \left(\pi_{\mathfrak{C}*} \left(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \\ & \rightarrow \pi_{\mathfrak{C}*} (h_{\mathfrak{C}}^*(N^2) \otimes Q) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \rightarrow 0, \end{aligned}$$

where

$$Q := \mathcal{O}_{C_{\mathfrak{C}}} \left(\sum_{c_j \in \mathfrak{C}} (c_j^{\mathfrak{C}} + c_j^{\dagger \mathfrak{C}}) + \sum_{c_j \notin \mathfrak{C}} (c_j^{\dagger \mathfrak{C}} - c_j^{\mathfrak{C}}) - |\mathfrak{C}|(c_{2g+2}^{\mathfrak{C}} + c_{2g+2}^{\dagger \mathfrak{C}}) \right).$$

This gives rise to the long exact sequence

$$\begin{aligned} 0 & \rightarrow H^0(C, \mathcal{O}(\mathfrak{A} + \mathfrak{C} + (\lambda - |\mathfrak{C}|)c_{2g+2}))^+ \\ & \rightarrow H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{C}*} \left(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}} \right) \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right)^+ \\ & \rightarrow H^0(C, \pi_{\mathfrak{C}*} (h_{\mathfrak{C}}^*(N^2) \otimes Q) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \\ & \rightarrow H^1(C, \mathcal{O}(\mathfrak{A} + \mathfrak{C} + (\lambda - |\mathfrak{C}|)c_{2g+2}))^+ \rightarrow \dots \end{aligned}$$

Since $\dim H^0(C, \mathcal{O}(\mathfrak{A} + \mathfrak{E} + (\lambda - |\mathfrak{E}|)c_{2g+2}))^+ = \max\{0, (2|\mathfrak{A} \cap \mathfrak{E}| + \lambda - |\mathfrak{E}| + 2)/2\}$, we have

$$H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{E}} c_j^{\mathfrak{E}} \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right) \right)^+ \neq 0$$

for any $N \in \text{Jac}(C_{\mathfrak{E}/i_{\mathfrak{E}}})$ if $2|\mathfrak{A} \cap \mathfrak{E}| + \lambda - |\mathfrak{E}| + 2 > 0$. If $2|\mathfrak{A} \cap \mathfrak{E}| + \lambda - |\mathfrak{E}| + 2 < 0$, applying the above argument to $K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}$ in place of $\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ and using Serre duality, we obtain

$$H^1 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{E}} c_j^{\mathfrak{E}} \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right) \right)^+ \neq 0$$

for any $N \in \text{Jac}(C_{\mathfrak{E}/i_{\mathfrak{E}}})$. Since

$$\chi \left(\mathcal{E}nd^\circ \left(\pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{E}} c_j^{\mathfrak{E}} \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right) \right)^+ = 0,$$

this proves (1).

Suppose that $2|\mathfrak{A} \cap \mathfrak{E}| + \lambda - |\mathfrak{E}| + 2 = 0$. Then $H^0(C, \mathcal{O}(\mathfrak{A} + \mathfrak{E} + (\lambda - |\mathfrak{E}|)c_{2g+2}))^+ = 0$. Since $\chi(\mathcal{O}(\mathfrak{A} + \mathfrak{E} + (\lambda - |\mathfrak{E}|)c_{2g+2}))^+ = (2|\mathfrak{A} \cap \mathfrak{E}| + \lambda - |\mathfrak{E}| + 2)/2 = 0$, $H^1(C, \mathcal{O}(\mathfrak{A} + \mathfrak{E} + (\lambda - |\mathfrak{E}|)c_{2g+2}))^+$ also vanishes. We obtain isomorphisms

$$\begin{aligned} & H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{E}} c_j^{\mathfrak{E}} \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right) \right)^+ \\ & \simeq H^0 \left(C, \pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N^2) \otimes Q \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right)^+ \\ & \simeq H^0 \left(C_{\mathfrak{E}}, h_{\mathfrak{E}}^*(N^{\otimes 2}) \otimes \mathcal{O}_{c_{\mathfrak{E}}} \left(2 \sum_{c_j \in \mathfrak{E} \cap \mathfrak{A}} (c_j^{\mathfrak{E}} + c_j^{\dagger \mathfrak{E}}) + 2 \sum_{c_j \in \mathfrak{A} \setminus \mathfrak{E}} c_j^{\dagger \mathfrak{E}} \right. \right. \\ & \quad \left. \left. + \sum_{c_j \in \mathfrak{E} \setminus \mathfrak{A}} (c_j^{\mathfrak{E}} + c_j^{\dagger \mathfrak{E}}) + \sum_{c_j \notin \mathfrak{E} \cup \mathfrak{A}} (c_j^{\dagger \mathfrak{E}} - c_j^{\mathfrak{E}}) + (\lambda - |\mathfrak{E}|)(c_{2g+2}^{\mathfrak{E}} + c_{2g+2}^{\dagger \mathfrak{E}}) \right) \right)^+ \\ & \simeq H^0 \left(C_{\mathfrak{E}/i_{\mathfrak{E}}}, N^{\otimes 2} \otimes \mathcal{O} \left(2 \sum_{c_j \in \mathfrak{E} \cap \mathfrak{A}} h_{\mathfrak{E}}(c_j^{\mathfrak{E}}) + \sum_{c_j \in \mathfrak{A} \setminus \mathfrak{E}} h_{\mathfrak{E}}(c_j^{\dagger \mathfrak{E}}) \right. \right. \\ & \quad \left. \left. + \sum_{c_j \in \mathfrak{E} \setminus \mathfrak{A}} h_{\mathfrak{E}}(c_j^{\mathfrak{E}}) - \sum_{c_j \notin \mathfrak{E} \cup \mathfrak{A}} h_{\mathfrak{E}}(c_j^{\mathfrak{E}}) + \frac{\lambda - |\mathfrak{E}|}{2} \left(h_{\mathfrak{E}}(c_{2g+2}^{\mathfrak{E}}) + h_{\mathfrak{E}}(c_{2g+2}^{\dagger \mathfrak{E}}) \right) \right) \right). \end{aligned}$$

Therefore, $H^0 \left(C, \mathcal{E}nd^\circ \left(\pi_{\mathfrak{E}*} \left(h_{\mathfrak{E}}^*(N) \otimes \mathcal{O} \left(- \sum_{c_j \notin \mathfrak{E}} c_j^{\mathfrak{E}} \right) \right) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right) \right)^+ \neq 0$ if and only if $H^0(C_{\mathfrak{E}/i_{\mathfrak{E}}}, N^{\otimes 2} \otimes \tau((\mathfrak{A}, \lambda); \mathfrak{E})) \neq 0$. This proves that, set-theoretically, $\varphi_{\mathfrak{E}}^*(D(\mathfrak{E}, \lambda)) = 2_{\text{Jac}(C_{\mathfrak{E}/i_{\mathfrak{E}}})}^* \left(T_{\tau((\mathfrak{A}, \lambda); \mathfrak{E})}^* \Theta \right)$. The proof that this holds scheme-theoretically is similar to the counterpart in the proof of Lemma 3.8. \square

Proof of Theorem 3.4. Since $|\mathcal{S}^{\text{even}}| = \sum_{\substack{0 \leq k \leq 2g+1 \\ k \equiv g+1 \pmod{3}}} \binom{2g+1}{k} = \frac{2^{2g+1}+1}{3} = \dim H^0 \left(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{\text{invo}}}^{-1} \right)$, we have only to prove that $D_{(\mathfrak{e}, \lambda)}$ are linearly independent. Suppose that we have

$$(\clubsuit) \quad \sum_{(\mathfrak{A}, \lambda) \in \mathcal{S}^{\text{even}}} a_{(\mathfrak{A}, \lambda)} D_{(\mathfrak{e}, \lambda)} = 0$$

for $a_{(\mathfrak{A}, \lambda)} \in \mathbb{C}$. Pulling back this equation to $\text{Jac}(C)$ by φ in 3.7, we obtain $\sum_{|\mathfrak{A}|=g+1} a_{(\mathfrak{A}, \lambda)} \left(2_{\text{Jac}(C)}^* T_{(\mathfrak{A}, \lambda)}^* \Theta \right) = 0$ by Lemma 3.8. $\left\{ 2_{\text{Jac}(C)}^* T_{\kappa}^* \Theta \right\}_{\kappa: \text{theta characteristics}}$ are linearly independent by Proposition A.8 of [2]. $\left\{ \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \right\}_{|\mathfrak{A}|=g+1}$ is a subset of the set of theta characteristics of C by Lemma 3 of Chapter VIII of [6]. Thus $a_{(\mathfrak{A}, \lambda)} = 0$ if $|\mathfrak{A}| = g+1$. Fix $(\mathfrak{A}_0, \lambda_0) \in \mathcal{S}^{\text{even}}$ with $|\mathfrak{A}_0| < g+1$. We choose an element $\mathfrak{C} \in \mathcal{T}$ such that $\mathfrak{A}_0 \subset \mathfrak{C}$ and $c_{2g+2} \in \mathfrak{C}$ and $|\mathfrak{C}| = 2|\mathfrak{A}_0| + \lambda_0 + 2$. Pulling back (\clubsuit) to $\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$ by $\varphi_{\mathfrak{C}}$ of 3.10, we obtain

$$\sum_{2|\mathfrak{A} \cap \mathfrak{C}| + \lambda + 2 = |\mathfrak{C}|} a_{(\mathfrak{A}, \lambda)} 2_{\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}^* T_{\tau((\mathfrak{A}, \lambda); \mathfrak{C})}^* \Theta = 0$$

by Lemma 3.11. For $(\mathfrak{A}, \lambda), (\mathfrak{A}', \lambda') \in \mathcal{S}^{\text{even}}$ such that $2|\mathfrak{A} \cap \mathfrak{C}| + \lambda + 2 = |\mathfrak{C}|$ and $2|\mathfrak{A}' \cap \mathfrak{C}| + \lambda' + 2 = |\mathfrak{C}|$,

$$\begin{aligned} \tau((\mathfrak{A}, \lambda), \mathfrak{C}) \otimes \tau((\mathfrak{A}', \lambda'), \mathfrak{C})^{-1} &\simeq \mathcal{O}_{C_{\mathfrak{C}}/i_{\mathfrak{C}}} \left(\sum_{c_j \in \mathfrak{C} \cap \mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) - \sum_{c_j \in \mathfrak{C} \cap \mathfrak{A}'} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) \right. \\ &\quad \left. + (\lambda - \lambda' + 2|\mathfrak{A} \setminus (\mathfrak{C} \cup \mathfrak{A}')| - 2|\mathfrak{A}' \setminus (\mathfrak{C} \cup \mathfrak{A})|) h_{\mathfrak{C}}(c_{2g+2}^{\mathfrak{C}}) \right) \end{aligned}$$

Claim. If $\tau((\mathfrak{A}_0, \lambda_0); \mathfrak{C}) \simeq \tau((\mathfrak{A}, \lambda); \mathfrak{C})$, then $(\mathfrak{A}_0, \lambda_0) = (\mathfrak{A}, \lambda)$.

Proof of Claim. If $\tau((\mathfrak{A}_0, \lambda_0); \mathfrak{C}) \simeq \tau((\mathfrak{A}, \lambda); \mathfrak{C})$, we have $\mathfrak{C} \cap \mathfrak{A}_0 = \mathfrak{C} \cap \mathfrak{A}$ or $\mathfrak{C} \cap \mathfrak{A}_0 = \mathfrak{C} \setminus (\mathfrak{A} \cup \{c_{2g+2}\})$. If $\mathfrak{C} \cap \mathfrak{A}_0 = \mathfrak{C} \cap \mathfrak{A}$, we get $\lambda = \lambda_0$. Hence $|\mathfrak{A}_0| = |\mathfrak{A}|$. Since $\mathfrak{A}_0 = \mathfrak{C} \cap \mathfrak{A}_0 = \mathfrak{C} \cap \mathfrak{A} \subset \mathfrak{A}$, we have $\mathfrak{A}_0 = \mathfrak{A}$. Let us prove that $\mathfrak{C} \cap \mathfrak{A}_0 = \mathfrak{C} \setminus (\mathfrak{A} \cup \{c_{2g+2}\})$ does not occur. If it did occur, we have $\lambda = -\lambda_0 - 2$. Since $\lambda + \lambda_0 \equiv 0 \pmod{4}$, we obtain a contradiction. \square

Taking account of the above claim and the fact that $\tau((\mathfrak{A}, \lambda), \mathfrak{C}) \otimes \tau((\mathfrak{A}', \lambda'), \mathfrak{C})^{-1}$ is a 2-torsion point of $\text{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$, we have $a_{(\mathfrak{A}_0, \lambda_0)} = 0$ again by Proposition A.8 of [2]. For $(\mathfrak{A}_0, \lambda_0) \in \mathcal{S}^{\text{even}}$ with $|\mathfrak{A}_0| > g+1$, we choose an element \mathfrak{C} of \mathcal{T} such that $\{c_1, \dots, c_{2g+1}\} \setminus \mathfrak{A}_0 \subset \mathfrak{C}$ and $c_{2g+2} \in \mathfrak{C}$ and $|\mathfrak{C}| = 4g+2 - \lambda_0 - 2|\mathfrak{A}_0|$. Then a similar argument as above implies that $a_{(\mathfrak{A}_0, \lambda_0)} = 0$. \square

4. RELATION WITH BEAUVILLE'S BASIS

By Remark 2.12, we have a natural homomorphism $f : \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo} \rightarrow \overline{\mathcal{M}}(2, \mathcal{O}_C)$ when $|I|$ is even, and $f : \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo} \rightarrow \overline{\mathcal{M}}(2, \mathcal{O}_C(c_{2g+2}))$ when $|I|$ is odd, where $\overline{\mathcal{M}}(2, \mathcal{O}_C)$ [resp. $\overline{\mathcal{M}}(2, \mathcal{O}_C(c_{2g+2}))$] is the coarse moduli space of rank 2 semistable vector bundles with the determinant \mathcal{O}_C [resp. $\mathcal{O}_C(c_{2g+2})$]. In this section, we compare the basis $\{D_{(\mathfrak{a}, \lambda)}\}_{(\mathfrak{a}, \lambda) \in \mathcal{S}^{\text{even}}}$ [resp. $\{D_{(\mathfrak{a}, \lambda)}\}_{(\mathfrak{a}, \lambda) \in \mathcal{S}^{\text{odd}}}$] of $H^0(\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}, K_{\overline{\mathcal{M}}_{C/\mathbb{P}^1}}^{-1})$ with the basis $\{D_\kappa\}_{\kappa: \text{even theta characteristic of } C}$ of $H^0(\overline{\mathcal{M}}(2, \mathcal{O}_C), K_{\overline{\mathcal{M}}(2, \mathcal{O}_C)}^{-\frac{1}{2}})$ [resp. $\{D_\kappa\}_{\kappa: \text{odd theta characteristic of } C}$ of $H^0(\overline{\mathcal{M}}(2, \mathcal{O}_C(c_{2g+2})), K_{\overline{\mathcal{M}}(2, \mathcal{O}_C(c_{2g+2}))}^{-\frac{1}{2}})$]. Here D_κ is the divisor constructed by Beauville in [2]. It is the unique effective divisor that satisfies the relation $2D_\kappa = \text{Div}(Rp_*(\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^*\kappa))$ over the stable locus of the moduli space, where \mathcal{E} is the universal vector bundle and p is the projection to the moduli space. (See [2] for details.)

Put

$$\mathcal{V} := \{V \subset \{c_1, \dots, c_{2g+1}\} \mid |V| \equiv g + 1 \pmod{2}\}.$$

The map from \mathcal{V} to the set of theta characteristics of C that sends V to $\mathcal{O}_C(V + (g - 1 - |V|)c_{2g+2})$ is a bijection. We have

$$\dim H^0(C, \mathcal{O}_C(V + (g - 1 - |V|)c_{2g+2})) = \frac{|g + 1 - |V||}{2}.$$

Theorem 4.1. *Assume that $|I|$ is even. Let κ be an even theta characteristic of C , say $\mathcal{O}_C(V + (g - 1 - |V|)c_{2g+2})$.*

- (i) *If $H^0(C, \kappa) = 0$, i.e. $|V| = g + 1$, then $f^*D_\kappa = D_{(V, -2)}$.*
- (ii) *If $H^0(C, \kappa) \neq 0$, i.e. $|V| \neq g + 1$, then $f^*D_\kappa = 0$.*

Theorem 4.2. *Assume that $|I|$ is odd. Let κ be an odd theta characteristic of C , say $\mathcal{O}_C(V + (g - 1 - |V|)c_{2g+2})$.*

- (i) *If $|V| = g - 1$, then $f^*D_\kappa = D_{(V, 0)}$.*
- (ii) *If $|V| \neq g - 1$, then $f^*D_\kappa = 0$.*

We give a proof only for Theorem 4.1.

Proof of Theorem 4.1. Suppose that $H^0(C, \kappa) = 0$. Then $f^*(2D_\kappa) = \text{Div}(R^1p_*(\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^*\kappa))$ over $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$, where \mathcal{E} is the universal involutorial bundle on $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo} \times C$ and p is the projection $\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo} \times C \rightarrow \overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$.

Claim. $\text{Div}(R^1p_*(\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^*\kappa)^+) = \text{Div}(R^1p_*(\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^*\kappa)^-)$.

Proof of Claim. Grothendieck duality (cf. Chapter VII Theorem 3.3 in [7]) says that

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}}} \left(\mathbb{R}p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa), \mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}} \right) \\ \rightarrow \mathbb{R}p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* (\kappa^{-1} \otimes K_C)) [1] \end{aligned}$$

is a quasi-isomorphism. Taking cohomologies, we obtain an isomorphism

$$\begin{aligned} \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}}}^1 \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa), \mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}} \right) \\ \rightarrow R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* (\kappa^{-1} \otimes K_C)). \end{aligned}$$

Since this isomorphism is compatible with the i -action, we get an isomorphism

$$\begin{aligned} \mathcal{E}xt_{\mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}}}^1 \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa)^+, \mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}} \right) \\ \rightarrow R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* (\kappa^{-1} \otimes K_C))^+. \end{aligned}$$

$\kappa^{-1} \otimes K_C$ and κ are isomorphic as line bundles on C , but their i -actions differ by (-1) -multiplication. Hence

$$R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* (\kappa \otimes K_C))^+ \simeq R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa)^-.$$

This isomorphism and the equality

$$\begin{aligned} \text{Div} \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa)^+ \right) \\ = \text{Div} \left(\mathcal{E}xt^1 \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa)^+, \mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}} \right) \right) \end{aligned}$$

implies the claim. \square

This claim implies that

$$\text{Div} \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa) \right) = 2 \text{Div} \left(R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \kappa)^+ \right).$$

This proves (i).

Suppose that $H^0(C, \kappa) \neq 0$. We shall show that $H^0(C, \mathcal{E}nd^\circ(E) \otimes \kappa) \neq 0$ for any $E \in \mathcal{M}_{C/\mathbb{P}^1}^{inv\circ}$. Since $\chi(\mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2}))^+ = (-g-1+|V|)/2 \neq 0$, either $H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2}))^+$ or $H^1(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2}))^+$ is nonzero. A similar argument as in the proof of the above claim implies that $\dim H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2})) = \dim H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2}))^+ + H^1(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2}))^+$. Hence $H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}_C(V + (g-1-|V|)c_{2g+2})) \neq 0$. \square

5. REDUCEDNESS OF THE DIVISORS $D_{(\mathfrak{A}, \lambda)}$

In this section, we shall show that the divisors $D_{(\mathfrak{A}, \lambda)}$ are reduced.

For a non-negative integer l , let Z_l be the closed subset of $\mathcal{M}_{C/\mathbb{P}^1}^{inv}$ consisting of involutively stable involutorial vector bundle $E \in \mathcal{M}_{C/\mathbb{P}^1}^{inv}$ that satisfies $\dim H^0(\mathcal{E}nd^\circ(E) \otimes (\mathfrak{A} + \lambda c_{2g+2}))^+ \geq l$. We give Z_l a scheme structure as follows. For some open neighborhood U of $E \in Z_l$, we can take a resolution of involutorial vector bundles $0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}(-Nc_{2g+2}) \otimes V \rightarrow \mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \rightarrow 0$ on $U \times C$, where \mathcal{E} is the universal involutorial vector bundle (strictly speaking, only $\mathcal{E}nd^\circ(\mathcal{E})$ exists on $\mathcal{M}_{C/\mathbb{P}^1}^{inv} \times C$), N is a sufficiently large positive integer and V is a vector space with an i -action. This resolution gives rise to a short exact sequence

$$0 \rightarrow R^1 p_* \mathcal{G}^+ \xrightarrow{g} R^1 p_* (\mathcal{O}(-Nc_{2g+2}) \otimes V)^+ \rightarrow R^1 p_* (\mathcal{E}nd^\circ(\mathcal{E}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \rightarrow 0,$$

where p is the projection $p : U \times C \rightarrow U$. By trivializing $R^1 p_* \mathcal{G}^+$ and $R^1 p_* (\mathcal{O}(-Nc_{2g+2}) \otimes V)^+$ over an open neighborhood W of $E \in \mathcal{M}_{C/\mathbb{P}^1}^{inv}$, g can be expressed by an $r \times r$ matrix $\{g_{ij}\}$ with $g_{ij} \in \mathcal{O}_{\mathcal{M}_{C/\mathbb{P}^1}^{inv}}(W)$,

where $r = \text{rank } R^1 p_* \mathcal{G}^+ = \text{rank } R^1 p_* (\mathcal{O}(-Nc_{2g+2}) \otimes V)^+$. The ideal sheaf of Z_l is generated, over W , by all the $l \times l$ minors of this matrix. Note that, with this scheme structure, Z_1 is nothing but $D_{(\mathfrak{A}, \lambda)}^\circ$.

For $E \in \mathcal{M}_{C/\mathbb{P}^1}^{inv}$, let ν_E be the map

$$\begin{aligned} H^0(\mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \otimes H^0(\mathcal{E}nd^\circ(E) \otimes K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ \\ \rightarrow H^0(\mathcal{E}nd^\circ(E) \otimes K_C)^+, \end{aligned}$$

taking $\phi \otimes \psi$ to $[\phi, \psi]$ ($:= \phi \circ \psi - \psi \circ \phi$).

Proposition 5.1. *Assume that $|I|$ is even [resp. odd]. There exists a closed subset \mathcal{Y} of $\mathcal{M}_{C/\mathbb{P}^1}^{inv}$ of dimension at most $\lfloor \frac{4g-5}{3} \rfloor$ [resp. $\lfloor \frac{4g-7}{3} \rfloor$] such that if E is not in \mathcal{Y} then $\dim \text{Ker } \nu_E \leq l^2 - 2l + 1$, where $l = \dim H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$.*

Proof. We give a proof for the case when $|I|$ is even. The case when $|I|$ is odd is very similar.

Suppose that $\dim \text{Ker } \nu_E > l^2 - 2l + 1$. Then we can find a non-zero element $\phi \otimes \psi$ in $\text{Ker } \nu_E$.

Claim 5.1.1. Either ϕ or ψ is of rank one.

Proof of Claim 5.1.1. For c_j , $1 \leq j \leq 2g+1$, fix an isomorphism $\alpha : \mathbb{C}^{\oplus 2} \xrightarrow{\sim} E \otimes \mathbb{C}_{c_j}$ such that $\alpha^t(1, 0)$ is, for the i -action, an eigen vector with the eigen value 1 and $\alpha^t(0, 1)$ is an eigen vector with eigen value -1 . With this trivialization, at c_j , ϕ and ψ are expressed, respectively, by $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ with $b, c \in \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})_{c_j}^{-1}$ and $a \in K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})_{c_j}^{-1}$ if $c_j \in \mathfrak{A}$ because ϕ and ψ are traceless and compatible with the i -action. The relation $\phi \circ \psi = \psi \circ \phi$ implies that $\begin{pmatrix} 0 & -ab \\ ac & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab \\ -ac & 0 \end{pmatrix}$.

Hence either ϕ or ψ is a zero map on the fiber over c_j . If $c_j \notin \mathfrak{A}$, switching the role of $\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ and $K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}$, we know that ϕ or ψ is zero map on the fiber over c_j . Therefore $\phi \circ \psi$ is a zero map on the fiber over c_j for $1 \leq \forall j \leq 2g+1$. If $\phi \circ \psi$ is injective, then $\deg E + 2(2g-2) = \deg E \otimes K_C \geq \deg E + 2(2g+1)$, which is a contradiction. Hence $\phi \circ \psi$ is not injective, which implies that either ϕ or ψ is of rank one. \square

Without loss of generality we may assume that ϕ is of rank one. Put $L := \text{Ker}\phi$. Since L is i -invariant, L is isomorphic to $\mathcal{O}(\mathfrak{B} + \mu c_{2g+2})$, for some $\mathfrak{B} \subset \{c_1, \dots, c_{2g+1}\}$ and $\mu \in \mathbb{Z}$, as involutorial bundles. E/L is isomorphic to $\mathcal{O}(\mathfrak{B}' - (2g+1+\mu)c_{2g+2})$, where $\mathfrak{B}' := \{c_1, \dots, c_{2g+1}\} \setminus \mathfrak{B}$.

Claim 5.1.2. We have $|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1 \geq 0$ if $H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$. We have $|\mathfrak{B}| - |\mathfrak{A}| + \mu - \frac{\lambda}{2} + g - 1 \geq 0$ if $H^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$.

Proof of Claim 5.1.2. First note that either $H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$ or $H^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+$ is zero because $H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = \max\{\frac{\lambda}{2} + 1, 0\}$ and $H^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = \max\{-\frac{\lambda}{2} - 1, 0\}$. If $H^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$, then ϕ induces a non-zero i -equivariant morphism $E/L \rightarrow L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ because the composite $E/L \rightarrow E \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \rightarrow E/L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ is zero. This nonzero i -equivariant morphism corresponds to a non-zero element of $H^0(C, L \otimes (E/L)^{-1} \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$. Since $H^0(C, L \otimes (E/L)^{-1} \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq H^0(C, \mathcal{O}(\mathfrak{B} + \mathfrak{A} - \mathfrak{B}' + (2g+1+2\mu+\lambda)c_{2g+2}))^+ \simeq H^0(\mathbb{P}^1, \mathcal{O}(|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1))$, we have $|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1 \geq 0$. Next let us consider the case $H^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$. Since $\phi \circ \psi = \psi \circ \phi$ and $L = \text{Ker}\phi$, the restriction of ψ to L factors through $L \otimes K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}$, hence is a zero map because $H^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$. This means that $L = \text{Ker}\psi$. Then a similar argument as above implies that $|\mathfrak{B}| - |\mathfrak{A}| + \mu - \frac{\lambda}{2} + g - 1 \geq 0$. \square

In order to complete the proof, it suffices to prove that $\dim H^1(C, \mathcal{H}om(E/L, L))^+ - 1 \leq \lfloor \frac{4g-5}{3} \rfloor$ because $H^1(C, \mathcal{H}om(E/L, L))^+$ parameterizes extensions of E/L by L that are compatible with the i -actions. Since $H^1(C, \mathcal{H}om(E/L, L))^+ \simeq H^1(C, \mathcal{O}(\mathfrak{B} - \mathfrak{B}' + (2g+1+2\mu)c_{2g+2}))^+ \simeq H^1(\mathbb{P}^1, \mathcal{O}(|\mathfrak{B}| + \mu - g - 1))$, $\dim H^1(C, \mathcal{H}om(E/L, L))^+ = \max\{g - \mu - |\mathfrak{B}|, 0\}$. By the above claim, $\dim H^1(C, \mathcal{H}om(E/L, L))^+ \leq \max\{|\mathfrak{A}| + \frac{\lambda}{2} - 1, 0\} \leq \lfloor \frac{4g-2}{3} \rfloor$. \square

Before stating and proving the main theorem of this section, we recall an important property of ν_E .

Proposition 5.2. *For $E \in Z_l \setminus Z_{l+1}$, the codimension of the irreducible component of Z_l containing E is greater than or equal to $\dim \text{Im}\nu_E$. If ν_E is injective, Z_l is smooth at E and of codimension l^2 .*

This proposition is essentially proved in [12] and we omit the proof.

Theorem 5.3. *All the divisors $D_{(\mathfrak{a},\lambda)}$ are reduced.*

Proof. Note that the closed subset \mathcal{Y} of Proposition 5.1 is of codimension greater than one. Therefore we have only to prove that $D_{(\mathfrak{a},\lambda)}|_{\mathcal{M}_{C/\mathbb{P}^1}^{inv}\setminus\mathcal{Y}}$ is reduced. For $E \in D_{(\mathfrak{a},\lambda)}\setminus\mathcal{Y}$, if $\dim H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{a} + \lambda c_{2g+2}))^+ = 1$, then by Proposition 5.1 ν_E is injective, hence $D_{(\mathfrak{a},\lambda)}$ is smooth at E by Proposition 5.2. Therefore it suffices to prove that $\text{codim}(Z_2 \setminus \mathcal{Y}, \mathcal{M}_{C/\mathbb{P}^1}^{inv} \setminus \mathcal{Y}) \geq 2$. For $E \in Z_2 \setminus \mathcal{Y}$, $\dim \text{Im} \nu_E \geq 2l - 1$ by Proposition 5.1, where $l = \dim H^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{a} + \lambda c_{2g+2}))^+$. By Proposition 5.2, the codimension of Z_l at E is greater than or equal to $2l - 1 > 2$. \square

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