# THE NASH PROBLEM ON ARC FAMILIES OF SINGULARITIES

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Abstract. Nash proved that every irreducible component of the space of arcs through a singularity corresponds to an exceptional divisor that occurs on every resolution. He asked if the converse also holds: does every such exceptional divisor correspond to an arc family? We prove that the converse holds for toric singularities but fails in general.

## 1. Introduction

In a 1968 preprint, later published as [20], Nash introduced arc spaces and jet schemes for algebraic and analytic varieties. The problems raised by Nash were studied by Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others, see [3, 10, 11, 16, 17, 21].

The study of these spaces was further developed by Kontsevich, Denef and Loeser as the theory of motivic integration, see [15, 7]. Further interesting applications of jet spaces are given by Mustaţă [19].

The main subject of the paper of Nash is the map from the set of irreducible components of the families of arcs at singular points to the set of essential components of the resolutions of the singularities. These are the exceptional divisors of a resolution that appear on every possible resolution whose exceptional set is a divisor, see Definition 2.3.

We call this map the *Nash map*, see Theorem 2.15 for a precise definition. The Nash map is always injective and Nash asked if it is always bijective. This problem remained open even for 2-dimensional singularities.

In this paper we prove that the Nash map is bijective for toric singularities in any dimension, see Theorem 3.18. On the other hand we also show that the Nash map is not bijective in general. For instance, the 4-dimensional hypersurface singularity

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$$

has only 1 irreducible family of arcs but 2 essential exceptional divisors over any algebraically closed field of characteristic $\neq 2, 3$. See Example 4.5.

In §2 we define the Nash map and show its injectivity. This is essentially taken from [20] with some scheme theoretic details filled in. The Nash map for toric singularities is studied in §3. Counter examples are given in §4.

In this paper, the ground field $k$ is an algebraically closed field of arbitrary characteristic. A $k$-scheme is not necessarily of finite type unless we state otherwise. A variety means a separated, irreducible and reduced scheme of finite type over $k$. Every variety $X$ that we consider is assumed to have a resolution of singularities $f : Y \longrightarrow X$ which is an isomorphism over the smooth locus and whose exceptional set is purely one codimensional. Without this or similar assumptions the definition of essential components would not make sense. The existence of resolutions is known in characteristic zero and for toric varieties in any characteristic.

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## 2. The space of arcs and the Nash problem

**Definition 2.1.** Let $X$ be a normal scheme, $g : X_1 \longrightarrow X$ a proper birational morphism and $E \subset X_1$ an irreducible exceptional divisor of $g$. Let $f : X_2 \longrightarrow X$ be another proper birational morphism. We say that $E$ appears in $f$ (or in $X_2$), if the birational map $f^{-1} \circ g : X_1 \dashrightarrow X_2$ is a local isomorphism at the generic point of $E$. In this case we denote the proper transform of $E$ on $X_2$ again by $E$. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. (Strictly speaking, we should be talking about the corresponding valuation instead.) Such an equivalence class is called an *exceptional divisor over* $X$.

**Definition 2.2.** Let $X$ be a variety over $k$. A morphism $f : Y \longrightarrow X$ is called a *divisorial resolution* of the singularities of $X$, if $f$ is proper, the exceptional set is of pure codimension one and $Y$ is non-singular.

**Definition 2.3.** An exceptional divisor $E$ over the variety $X$ is called an *essential divisor* (resp. an *essential component*) if $E$ appears on every resolution (resp. on every divisorial resolution) of the singularities of $X$. This is equivalent to the corresponding notions in [3].

**Example 2.4.** Let $(X, x)$ be a normal 2-dimensional singularity. Then the set of the essential divisors over $X$ coincides with the set of the exceptional curves appearing on the minimal resolution. Every essential divisor is also an essential component.

**Example 2.5.** Let $X = (xy - uv = 0) \subset \mathbb{A}^4_k$. Blowing up at $(x = u = 0)$ gives a resolution without exceptional divisors. Thus there are no essential divisors over $X$. On the other hand, it is easy to see that the divisor corresponding to the blowing up of the origin is an essential component. Examples like these led to the introduction of the concept of essential component.

**Example 2.6.** Let $(X, x)$ be a canonical singularity which admits a crepant divisorial resolution. A quite large group of such singularities is known (see, for example, [6] and the references there). Then the set of the essential components over $X$ coincides with the set of the crepant exceptional divisors. Indeed, an essential component should be one of the crepant exceptional divisors because of the existence of a crepant resolution. Conversely a crepant exceptional component cannot be contracted on a non-singular model of $X$, because if it could be contracted, the discrepancy of the crepant component would have to be positive.

**Definition 2.7.** Let $X$ be a scheme of finite type over $k$ and $K \supset k$ a field extension. A morphism $\operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ is called an $m$-jet of $X$ and $\operatorname{Spec} K[[t]] \longrightarrow X$ is called an *arc* of $X$. We denote the closed point of $\operatorname{Spec} K[[t]]$ by 0 and the generic point by $\eta$.

**2.8.** Let $X$ be a scheme of finite type over $k$. Let $\mathcal{S}ch/k$ be the category of $k$-schemes and $\mathcal{S}et$ the category of sets. Define a contravariant functor $F_m : \mathcal{S}ch/k \longrightarrow \mathcal{S}et$ by

$$F_m(Y) = \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

Then, $F_m$ is representable by a scheme $X_m$ of finite type over $k$, that is

$$\operatorname{Hom}_k(Y, X_m) \simeq \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[t]/(t^{m+1}), X).$$

This $X_m$ is called the *scheme of $m$-jets* of $X$. The canonical surjection $k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m)$ induces a morphism $\phi_m : X_m \longrightarrow X_{m-1}$. Define $\pi_m = \phi_1 \circ \cdots \circ \phi_m : X_m \longrightarrow X$. A point $x \in X_m$ gives an

$m$-jet $\alpha_x : \operatorname{Spec} K[t]/(t^{m+1}) \longrightarrow X$ and $\pi_m(x) = \alpha_x(0)$, where $K$ is the residue field at $x$.

Let $X_\infty = \varprojlim_m X_m$ and call it the *space of arcs* of $X$. $X_\infty$ is not of finite type over $k$ but it is a scheme, see [7]. Denote the canonical projection $X_\infty \longrightarrow X_m$ by $\eta_m$ and the composite $\phi_m \circ \eta_m$ by $\pi$. A point $x \in X_\infty$ gives an arc $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ and $\pi(x) = \alpha_x(0)$, where $K$ is the residue field at $x$.

Using the representability of $F_m$ we obtain the following universal property of $X_\infty$:

**Proposition 2.9.** *Let $X$ be a scheme of finite type over $k$. Then*

$$\operatorname{Hom}_k(Y, X_\infty) \simeq \operatorname{Hom}_k(Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]], X)$$

*for an arbitrary $k$-scheme $Y$.*

**Corollary 2.10.** *There is a universal family of arcs*

$$X_\infty \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X.$$

**Definition 2.11.** Let $X$ be a $k$-variety with singular locus $\operatorname{Sing} X \subset X$. Every point $x$ of the inverse image $\pi^{-1}(\operatorname{Sing} X) \subset X_\infty$ corresponds to an arc $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha_x(0) \in \operatorname{Sing} X$, where $K$ is the residue field at $x$. $\pi^{-1}(\operatorname{Sing} X)$ is the space of arcs through $\operatorname{Sing} X$.

Decompose $\pi^{-1}(\operatorname{Sing} X)$ into its irreducible components

$$\pi^{-1}(\operatorname{Sing} X) = (\bigcup_{i \in I} C_i) \cup (\bigcup_{j \in J} C'_j),$$

where the $C_i$'s are the components with a point $x$ corresponding to an arc $\alpha_x$ such that $\alpha_x(\eta) \notin \operatorname{Sing} X$, while the $C'_j$'s are the components without such points. We call the $C_i$'s the *good components* of the space of arcs through $\operatorname{Sing} X$.

The next lemma shows that in characteristic zero every irreducible component of $\pi^{-1}(\operatorname{Sing} X) \subset X_\infty$ is good. This can be viewed as a strong form of Kolchin's irreducibility theorem [12, Chap.IV,Prop.10]. See also [9]. It is also interesting to compare this with the results of [19] according to which the jet spaces $X_m$ are usually reducible.

**Lemma 2.12.** *Let $k$ be a field of characteristic zero and $X$ a $k$-variety. Then every arc through* $\operatorname{Sing} X$ *is a specialization of an arc through* $\operatorname{Sing} X$ *whose generic point maps into* $X \setminus \operatorname{Sing} X$.

Proof. We may assume that $X$ is affine. Pick any arc $\phi : \operatorname{Spec} k'[[s]] \longrightarrow X$ such that $\phi(0) \in \operatorname{Sing} X$. Let $Y$ be the Zariski closure of the image of $\phi$. Then $\mathcal{O}_Y$ is an integral domain and $\phi$ corresponds to an injection $\mathcal{O}_Y \longrightarrow k'[[s]]$, where we can take $k'$ to be algebraically closed. We are done if $Y \not\subset \operatorname{Sing} X$. Otherwise we write $\phi$ as a specialization in two steps.

First we prove that $\phi$ is a specialization of an arc $\Phi : \operatorname{Spec} K[[s]] \longrightarrow Y \subset X$ such that $\Phi(0)$ is the generic point of $Y$.

We have an embedding $k'[[s]] \hookrightarrow k'[[S,T]]$ which sends $s$ to $S+T$. It is easy to check that the composite

$$k'[[s]] \hookrightarrow k'[[S,T]] \longrightarrow k'[[S,T]]/(T) \cong k'[[S]]$$

is an isomorphism. Thus we obtain $\Phi$ as the composite

$$\mathcal{O}_Y \xrightarrow{\phi} k'[[s]] \hookrightarrow k'[[S,T]] \hookrightarrow k'((T))[[S]].$$

Set $K = k'((T^{1/n} : n = 1, 2, \dots))$, the algebraic closure of $k'((T))$. The generic point of $\operatorname{Spec} K[[S]]$ maps to the ideal $\Phi^{-1}(S)$, but the pull back of $(S)$ to $k'[[s]]$ is already the zero ideal. Thus the closed point of $\operatorname{Spec} K[[S]]$ maps to the generic point of $Y$.

Repeatedly cutting with hypersurfaces containing $Y$ we obtain a subvariety $Y \subset Z \subset X$ such that $\dim Z = \dim Y + 1$ and $X$ is smooth along the generic points of $Z$. Let $n : \bar{Z} \longrightarrow Z$ be the normalization and $\bar{Y} \subset \bar{Z}$ the preimage of $Y$ with reduced scheme structure. $\bar{Y} \longrightarrow Y$ is finite, surjective, and so generically étale in characteristic zero. Thus the arc $\Phi : \operatorname{Spec} K[[S]] \longrightarrow Y$ can be lifted to $\bar{\Phi} : \operatorname{Spec} K[[S]] \longrightarrow \bar{Y}$. $\bar{Z}$ is normal, so smooth along the generic point of $\bar{Y}$. Thus $\bar{\Phi}$ is the specialization of an arc through $\bar{Y}$ whose generic point maps to the generic point of $\bar{Z}$. Projecting to $Z$ we obtain $\Phi$ and hence $\phi$ as the specialization of an arc through $\operatorname{Sing} X$ whose generic point maps into $X \setminus \operatorname{Sing} X$. □

**Example 2.13.** Let $k$ have characteristic $p$ and consider the surface $S = (x^p = y^p z) \subset \mathbb{A}^3$ with singular locus $Y = (x = y = 0)$. The normalization is $\bar{S} \cong \mathbb{A}^2$ with $(u, v) \mapsto (uv, v, u^p)$. The preimage of $Y$ is $\bar{Y} = (v = 0)$ and $\bar{Y} \longrightarrow Y$ is purely inseparable. Thus a smooth arc in $Y$ can not be lifted to $\bar{Y}$ and it is also not the specialization of an arc through $Y$ whose generic point maps into $S \setminus \operatorname{Sing} S$. In this case $\pi^{-1}(\operatorname{Sing} S) \subset S_\infty$ has a component which is not good.

**Lemma 2.14.** *Let $f : Y \longrightarrow X$ be a divisorial resolution which is an isomorphism outside the singular locus of $X$ and $E_1, \dots, E_r$ the irreducible exceptional divisors on $Y$. For a good component $C_i$, let $C_i^o$ denote the open subset of $C_i$ consisting of arcs $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha_x(\eta) \notin \operatorname{Sing} X$. Then, for every $x \in C_i^o$ the arc $\alpha_x$ can be lifted to an arc $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$.*

*Proof.* As $f$ is isomorphic outside of $\operatorname{Sing} X$ and $\alpha_x(\eta) \notin \operatorname{Sing} X$, we obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K((t)) & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \operatorname{Spec} K[[t]] & \xrightarrow{\alpha_x} & X. \end{array}$$

Since $f$ is proper, there exists a unique morphism $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $f \circ \tilde{\alpha}_x = \alpha_x$ by the valuative criterion of properness. □

This $\tilde{\alpha}_x$ is called the lifting of $\alpha_x$. Now we have a map

$$\varphi : \text{points of } (\bigcup_i C_i^o) \longrightarrow \text{points of } (\bigcup_l E_l)$$

given by $x \mapsto \tilde{\alpha}_x(0)$. We emphasize that this map is not a continuous map of schemes. In fact, the image of an irreducible subset is not necessarily irreducible.

**Theorem 2.15** (Nash [20]). *Let $X$ be a $k$-variety and $C_i$ a good component of the space of arcs through* $\operatorname{Sing} X$. *Let $z_i$ be the generic point of $C_i$. Then:*

(i) *$\varphi(z_i)$ is the generic point of a divisor $E_{l_i}$ for some $l_i$.*
(ii) *For every $i \in I$, $E_{l_i}$ is an essential component over $X$.*
(iii) *The resulting Nash map*

$$\mathcal{N} : \left\{ \begin{array}{c} \textit{good components} \\ \textit{of the space of arcs} \\ \textit{through } \operatorname{Sing} X \end{array} \right\} \longrightarrow \{\textit{essential components of } X\}$$

*given by $C_i \mapsto E_{l_i}$ is injective. In particular, there are only finitely many good components of the space of arcs through* $\operatorname{Sing} X$.

*Proof.* The divisorial resolution $f : Y \longrightarrow X$ induces a morphism $f_\infty : Y_\infty \longrightarrow X_\infty$ of schemes. Let $\pi^Y : Y_\infty \longrightarrow Y$ be the canonical projection. As $Y$ is non-singular, $(\pi^Y)^{-1}(E_l)$ is irreducible for every $l$. Denote by $(\pi^Y)^{-1}(E_l)^o$ the open subset of $(\pi^Y)^{-1}(E_l)$ consisting of the points $y$ corresponding to arcs $\beta_y : \operatorname{Spec} K[[t]] \longrightarrow Y$ such that $\beta_y(\eta) \notin \bigcup_l E_l$. By restriction $f_\infty$ gives $f'_\infty : \bigcup_{l=1}^r (\pi^Y)^{-1}(E_l)^o \longrightarrow \bigcup_{i\in I} C_i^o$. For a point $x \in C_i^o$, let $\alpha_x : \operatorname{Spec} K[[t]] \longrightarrow X$ be the corresponding arc, where $K$ is the residue field at $x$. The lifting $\tilde{\alpha}_x : \operatorname{Spec} K[[t]] \longrightarrow Y$ of $\alpha_x$ obtained in Lemma 2.14 determines a $K$-valued point $\beta : \operatorname{Spec} K \longrightarrow Y_\infty$. Denote the image of $\beta$ by $y$, then $f_\infty(y) = x$. Therefore, $f'_\infty$ is surjective. Hence, for each $i \in I$ there is $1 \leq l_i \leq r$ such that the generic point $y_{l_i}$ of $(\pi^Y)^{-1}(E_{l_i})^o$ is mapped to the generic point $z_i$ of $C_i^o$. Let $\tilde{\alpha}_{z_i}$ be the lifting of the arc $\alpha_{z_i}$ corresponding to $z_i$ and let $\beta_{y_{l_i}}$ be the arc of $Y$

corresponding to $y_{l_i}$. Let $L$ and $K$ be the residue fields at $y_{l_i}$ and $z_i$, respectively and $g : \operatorname{Spec} L[[t]] \longrightarrow \operatorname{Spec} K[[t]]$ be the canonical morphism induced from the inclusion $K \longrightarrow L$. Then $\beta_{y_{l_i}} = \tilde{\alpha}_{z_i} \circ g$. From this, we have $K = L$ and therefore $\beta_{y_{l_i}} = \tilde{\alpha}_{z_i}$. Note that $\beta_{y_{l_i}}(0) = \pi^Y(y_{l_i})$, which is the generic point of $E_{l_i}$. To finish the proof of (i), just recall $\varphi(z_i) = \tilde{\alpha}_{z_i}(0) = \beta_{y_{l_i}}(0)$.

Next, we can see that the map $C_i \mapsto E_{l_i}$ is injective. Indeed, if $E_{l_i} = E_{l_j}$ for $i \neq j$, then $z_i = f'_\infty(y_{l_i}) = f'_\infty(y_{l_j}) = z_j$, a contradiction.

To prove that the $E_{l_i}$ $(i \in I)$ are essential components, let $Y' \longrightarrow X$ be another divisorial resolution and $\tilde{Y} \longrightarrow X$ a divisorial resolution which factors through both $Y$ and $Y'$. Let $E'_{l_i} \subset Y'$ be the exceptional component corresponding to $C_i$. Since the morphisms $\tilde{Y} \longrightarrow Y$ and $\tilde{Y} \longrightarrow Y'$ are isomorphic at the generic points of $E_{l_i}$ and $E'_{l_i}$, the proper transforms of $E_{l_i}$ and $E'_{l_i}$ on $\tilde{Y}$ must correspond to $C_i$, therefore they should coincide with each other in $\tilde{Y}$. Hence, $E_{l_i}$ $(i \in I)$ appears in every divisorial resolution. □

Nash poses the following problem in his paper [20, p.36].

**Problem 2.16.** *Is the Nash map bijective?*

## 3. The Nash problem for toric singularities

**3.1.** We use the notation and terminology of [8]. Let $M$ be the free abelian group $\mathbb{Z}^n$ $(n \geq 2)$ and $N$ its dual $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote $M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N \otimes_{\mathbb{Z}} \mathbb{R}$ by $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$, respectively. The canonical pairing $\langle\ ,\ \rangle : N \times M \longrightarrow \mathbb{Z}$ extends to $\langle\ ,\ \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \longrightarrow \mathbb{R}$. For a finite fan $\Delta$ in $N_{\mathbb{R}}$, the corresponding toric variety is denoted by $X = X(\Delta)$. For the primitive vector $v$ in a one-dimensional cone $\tau \in \Delta$, denote the invariant divisor $\overline{orb(\tau)}$ in $X$ by $D_v$.

For a cone $\tau \in \Delta$ denote by $U_\tau$ the invariant affine open subset which contains $orb\ \tau$ as the unique closed orbit. A cone $\tau$ is called regular or non-singular, if its generators can be extended to a basis of $N$. A cone is called singular, if it is not regular. Note that a cone $\tau$ is regular, if and only if $U_\tau$ is non-singular.

We can write $k[M]$ as $k[x^u]_{u \in M}$, where we use the shorthand $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ for $u = (u_1, \ldots, u_n) \in M$.

**Definition 3.2.** An exceptional divisor $E$ over a toric variety $X$ is called a toric essential component over $X$ if it appears in every toric divisorial resolution of the singularities of $X$.

The following is obvious by the definition.

**Proposition 3.3.** *For a toric variety $X$ an essential component over $X$ is a toric essential component over $X$.*

At this moment the converse of the above proposition is not clear. But later on, as a corollary of our theorem we obtain the converse.

**3.4.** In what follows, we consider an affine toric variety $X = X(\Delta)$, therefore the fan $\Delta$ consists of all faces of a cone $\sigma$. Let $\sigma = \langle e_1, \dots, e_s \rangle$, where the right hand side means the cone generated by primitive vectors $e_1, \dots, e_s$. Let $T$ be the open orbit in $X$. Let $W$ be the singular locus of $X$, then $W = \bigcup_{\tau:\text{singular}} orb(\tau)$. Let $S = N \cap (\bigcup_{\tau:\text{singular}} \tau^o)$, where $^o$ means the relative interior.

**Proposition 3.5.** *If $D_v$ is a toric essential component for $v \in N \cap \sigma$, then $v$ belongs to $S$.*

*Proof.* If $D_v$ is a toric essential component, then the image of $D_v$ must be in the singular locus $W = \bigcup_{\tau:\text{singular}} orb(\tau)$. Therefore $v \in S$. □

**3.6** (Sketch of the proof)**.** We prove that all maps in the following diagram are injective and that composite of all maps is the identity. This shows that all maps are bijective.

$$\begin{array}{ccc}
\{\text{minimal elements in } S\} & \xrightarrow{\mathcal{F}} & \left\{\begin{array}{c}\text{good components of}\\ \text{arcs through Sing}\, X\end{array}\right\} \\
\mathcal{G}\uparrow & & \downarrow\mathcal{N} \\
\left\{\begin{array}{c}\text{toric essential}\\ \text{components over } X\end{array}\right\} & \supset & \left\{\begin{array}{c}\text{essential components}\\ \text{over } X\end{array}\right\}
\end{array}$$

First we define an order in $N \cap \sigma$.

**Definition 3.7.** For two elements $v, v' \in N \cap \sigma$ we define $v \le v'$, if $v' \in v + \sigma$.

For a subset $A \subset N \cap \sigma$, $a \in A$ is called minimal in $A$, if there is no other element $a' \in A$ such that $a' \le a$.

Note that $v \le v'$ if and only if $\langle v, u \rangle \le \langle v', u \rangle$ for every $u \in M \cap \sigma^\vee$. It is clear that $\le$ is a partial order, i.e.,

(1) $v \le v$,
(2) if $v \le v'$ and $v' \le v$, then $v = v'$,
(3) if $v \le v'$ and $v' \le v''$, then $v \le v''$.

**Proposition 3.8.** (i) *The set of minimal elements in $N \cap \sigma \setminus \{0\}$ is a finite set.*
(ii) *The set $N \cap \sigma \setminus (\bigcup_{i=1}^{s} a_i e_i + \sigma)$ is a finite set, where $a_i \ge 0$ $(i = 1, \dots, r)$.*

*Proof.* (ii) is obvious, since the domain $\sigma \setminus (\bigcup_{i=1}^{s} a_i e_i + \sigma) \subset N_{\mathbb{R}}$ is bounded. (i) follows from (ii), because the minimal elements in $N \cap \sigma \setminus \{0\}$ are $e_1, \dots, e_s$ and some elements from $N \cap \sigma \setminus (\bigcup_{i=1}^{s} e_i + \sigma)$ which is a finite set by (ii). □

**Definition 3.9.** For an arc $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $\alpha(\eta) \in T$, define $v_\alpha \in N \cap \sigma$ as follows:

By the condition of $\alpha$, we have a commutative diagram of ring homomorphisms:

$$\begin{array}{ccc} k[M \cap \sigma^\vee] & \xrightarrow{\alpha^*} & K[[t]] \\ \cap & & \cap \\ k[M] & \xrightarrow{\alpha^*} & K((t)). \end{array}$$

The map $M \longrightarrow \mathbb{Z}$, $u \mapsto \operatorname{ord}(\alpha^* x^u)$ is a group homomorphism, therefore it determines an element $v_\alpha \in N$ such that $\langle v_\alpha, u \rangle = \operatorname{ord}(\alpha^* x^u)$ for every $u \in M$. By the commutative diagram it follows that $v_\alpha|_{M \cap \sigma^\vee} \geq 0$, hence $v_\alpha \in N \cap \sigma$.

**Proposition 3.10.** (i) *Let $\alpha$ be an arc of $X$ such that $\alpha(\eta) \in T$ and $\tau$ a face of $\sigma$. Then $\alpha(0) \in orb(\tau)$, if and only if $v_\alpha \in \tau^o$. In particular, $\alpha(0) \in T$, if and only if $v_\alpha = 0$.*

(ii) *Let $\Sigma$ be a subdivision of the fan $\Delta$ and $f : Y \longrightarrow X$ be the toric morphism corresponding to this subdivision. Then, an arc $\alpha$ of $X$ such that $\alpha(\eta) \in T$ is lifted to an arc $\tilde{\alpha}$ of $Y$. Let $\tau \in \Sigma$. Then, $\tilde{\alpha}(0) \in orb(\tau)$, if and only if $v_\alpha = v_{\tilde{\alpha}} \in \tau^o$.*

*Proof.* The first statement of (ii) follows immediately from the properness of $f$ and the condition $\alpha(\eta) \in T$ (see Lemma 2.14). The second statement of (ii) follows from the result (i) with replacing $X$ by $U_\tau$.

For the proof of (i) it is sufficient to prove one direction in the equivalence, since the orbits $orb(\tau)$ (resp. the interiors of the faces $\tau^o$) are disjoint. Assume $v = v_\alpha \in \tau^o$. Then, $\langle v, u \rangle \geq 0$ for every $u \in \tau^\vee$. Therefore the ring homomorphism $\alpha^* : k[M \cap \sigma^\vee] \longrightarrow K[[t]]$ corresponding to $\alpha$ factors through $k[M \cap \tau^\vee]$, which implies that $\operatorname{Im} \alpha \subset U_\tau$. On the other hand, $v \in \tau^o$ yields the equivalence for $u \in \tau^\vee$: $u \notin \tau^\perp$ if and only if $\langle v, u \rangle > 0$. Here, noting that $\langle v, u \rangle = \operatorname{ord}(\alpha^* x^u)$, we have $\operatorname{ord}(\alpha^* x^u) > 0$ for $u \in \tau^\vee \setminus \tau^\perp$. Then, it follows that $x^u(\alpha(0)) = 0$ for every $u \in \tau^\vee \setminus \tau^\perp$. This implies that $x^u(\alpha(0)) = 0$ for every monomial $x^u \in \mathcal{I}_{orb(\tau)}$, where $\mathcal{I}_{orb(\tau)}$ is the ideal of $orb(\tau)$ in $U_\tau$. Since the ideal is generated by the monomials, we obtain that $\alpha(0) \in orb(\tau)$. □

**Proposition 3.11.** *For every point $v \in S$, there exists an arc $\alpha : \operatorname{Spec} k[[t]] \longrightarrow X$ such that $\alpha(0) \in W$, $\alpha(\eta) \in T$ and $v = v_\alpha$.*

*Proof.* Define the ring homomorphism $\alpha^* : k[M] \longrightarrow k((t))$ by $\alpha^*(x^u) = t^{\langle v,u \rangle}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} k[M\cap\sigma^\vee] & \xrightarrow{\alpha^*} & k[[t]] \\ \cap & & \cap \\ k[M] & \xrightarrow{\alpha^*} & k((t)), \end{array}$$

because $\langle v, u\rangle \geq 0$ for every $u \in M \cap \sigma^\vee$. Let $\alpha : \operatorname{Spec} k[[t]] \longrightarrow X$ be the morphism corresponding to $\alpha^*$, then $v = v_\alpha$ and we obtain $\alpha(\eta) \in T$ by the diagram. On the other hand, as $v \in S$, there is a singular face $\tau < \sigma$ such that $v = v_\alpha \in N \cap \tau^0$. By Proposition 3.10 $\alpha(0) \in orb(\tau) \subset W$. □

**Proposition 3.12** (Upper semi-continuity)**.** *Let $C$ be a $k$-scheme, $\alpha : C \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ a family of arcs on $C$ and $\alpha_c : \{c\} \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ the arc induced from $\alpha$ for each point $c \in Y$. Assume $\alpha(C \times_{\operatorname{Spec} k} \{\eta\}) \subset T$. Then the map $C \longrightarrow N \cap \sigma$, $c \mapsto v_{\alpha_c}$ is upper semi-continuous, i.e., for every $v \in N \cap \sigma$ the subset $F_v := \{c \in C \mid v \leq v_{\alpha_c}\}$ is closed in $C$.*

*Proof.* It is sufficient to prove the assertion in the affine case $C = \operatorname{Spec} A$. Let $\alpha^* : k[M \cap \sigma^\vee] \longrightarrow A[[t]]$ be the ring homomorphism corresponding to $\alpha$. Let $\alpha^*(x^u)$ be $a_0^u + a_1^u t + a_2^u t^2 + \ldots$, where $a_i^u \in A$ for $i \geq 0$. By the definition of $F_v$, a point $c \in C$ belongs to $F_v$, if and only if $\langle v, u\rangle \leq \langle v_{\alpha_c}, u\rangle$ for every $u \in M \cap \sigma^\vee$. This is equivalent to $a_i^u(c) = 0$ for every $i < \langle v, u\rangle$ and every $u \in M \cap \sigma^\vee$. This is also equivalent to $a_i^u(c) = 0$ for every $i < \langle v, u\rangle$ and every generator $u$ of $M \cap \sigma^\vee$. Now, we see that $F_v$ is the zero locus of a finite number of functions on $C$. □

**Proposition 3.13.** *Let $C$ be a $k$-scheme, $\alpha : C \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ a family of arcs on $C$ and $\alpha_c : \{c\} \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ the arc induced from $\alpha$ for each point $c \in C$. Assume that $\alpha(C \times_{\operatorname{Spec} k} \{0\}) \subset W$ and $\alpha(C \times_{\operatorname{Spec} k} \{\eta\}) \subset T$. If there is a point $z \in C$ such that $v_{\alpha_z}$ is minimal in $S$, then there is a non-empty open subset $U \subset C$ such that $v_{\alpha_c} = v_{\alpha_z}$ holds for every $c \in U$.*

*Proof.* First note that $F_{v'} \subset F_v$ if $v \leq v'$. As $\alpha_c(0) \in W = \bigcup_{\tau:\text{singular}} orb(\tau)$ for $c \in C$, it follows that $v_{\alpha_c} \in \tau^o$ for a singular cone $\tau$ by Proposition 3.10. Hence, $v_{\alpha_c} \in S$ for every $c \in C$. Therefore, $C = \bigcup_{v \in S} F_v = \bigcup_{v \in S \text{ minimal}} F_v$. Choose $a_i > 0$ $(i = 1, \ldots, s)$ such that $v_{\alpha_z} \notin \bigcup_{i=1}^s (a_i e_i + \sigma)$. By Proposition 3.8, there is only a finite number of minimal elements of $S$ in the domain $\sigma \setminus \bigcup_{i=1}^s (a_i e_i + \sigma)$, say $v_0 = v_{\alpha_z}, v_1, \ldots, v_m$.

Then the subset $U = \{c \in C \mid v_{\alpha_c} = v_{\alpha_z}\}$ is represented as

$$U = C \setminus \left\{ \left( \bigcup_{i=1}^{s} F_{a_i e_i} \right) \cup \left( \bigcup_{j=1}^{m} F_{v_j} \right) \cup \left( \bigcup_{v:\text{minimal in } N\cap\sigma\setminus\{0\}} F_{v_{\alpha_z}+v} \right) \right\}.$$

Since the number of minimal elements of $N \cap \sigma \setminus \{0\}$ is finite by Proposition 3.8, $U$ is an open subset. □

**3.14.** Let $\{C_i\}_{i\in I}$ be the good components of the space of arcs through $W$. For each component $C_i \subset X_\infty$, there exists the canonical family $\alpha_i : C_i \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ of arcs by Corollary 2.10.

**Lemma 3.15.** *Under the notation as used above, for a minimal element $v \in S$ there are a component $C_i$ and a non-empty open subset $U \subset C_i$ such that $v_{\alpha_{ic}} = v$ for every $c \in U$, where $\alpha_{ic} : \{c\} \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]] \longrightarrow X$ is an arc induced from $\alpha_i$.*

*For a minimal element $v \in S$, take one of these components $C_i$ and define $\mathcal{F}(v) := C_i$, then the map* $\{\text{minimal elements in } S\} \xrightarrow{\mathcal{F}} \{C_i\}$ *is injective.*

*Proof.* For a given minimal element $v \in S$ there is an arc $\alpha : \operatorname{Spec} k[[t]] \longrightarrow X$ such that $\alpha(0) \in W$, $\alpha(\eta) \in T$ and $v_\alpha = v$ by Proposition 3.11. Then, by Proposition 2.9, there exist a component $C_i$ and its $k$-valued point $z$ such that $\alpha = \alpha_{iz}$. As $\alpha_i(C_i \times_{\operatorname{Spec} k} \{0\}) \subset W$ and $\alpha_i(C_i \times_{\operatorname{Spec} k} \{\eta\}) \cap T \neq \emptyset$, there exists a non-empty open subset $V \subset C_i$ such that both the conditions $\alpha_i(V \times_{\operatorname{Spec} k} \{0\}) \subset W$ and $\alpha_i(V \times_{\operatorname{Spec} k} \{\eta\}) \subset T$ hold. Then, by Proposition 3.13, there exists a non-empty open subset $U \subset V$ such that $v_{\alpha_{ic}} = v$.

The second assertion is obvious from the first statement. □

**Lemma 3.16.** *Let $\mathcal{N} : \{C_i\}_{i\in I} \longrightarrow$* {essential components}*, $C_i \mapsto E_{l_i}$ be the Nash map in Theorem 2.15. Then the composite $\mathcal{N} \circ \mathcal{F}$ :* {minimal elements in $S$} $\longrightarrow$ {essential components} *satisfies $\mathcal{N} \circ \mathcal{F}(v) = D_v$.*

*Proof.* By Lemma 3.15, the generic point $z$ of $\mathcal{F}(v)$ corresponds to an arc $\alpha : \operatorname{Spec} K[[t]] \longrightarrow X$ such that $v_\alpha = v$. Let $\tilde{\alpha}$ be the lifting of $\alpha$ as an arc of a toric divisorial resolution $Y$. By the definition of $\mathcal{N}$, $\mathcal{N} \circ \mathcal{F}(v)$ is an exceptional divisor containing $\tilde{\alpha}(0)$ as the generic point. By Proposition 3.10, the exceptional divisor $\overline{orb(\tau)}$ containing $\tilde{\alpha}(0)$ satisfies $v = v_\alpha = v_{\tilde{\alpha}} \in \tau^o$. Therefore this component is $D_v$. □

The injectivity of $\mathcal{G}$ was proved in [3], which in turn builds on [4, Théorème 1.10]. Indeed, note that condition (3) of the theorem in [3, Sec.2.3] is equivalent to the definition of minimal elements of $S$.

**Lemma 3.17.** [4, 3] *Let* $\mathcal{G}$ : {toric essential components over $X$} $\longrightarrow$ $S$ *be the map by* $\mathcal{G}(D_v) = v$. *Then, this map is injective and the image of this map is in* {minimal elements of $S$}. ☐

**Theorem 3.18.** *Let* $X$ *be an affine toric variety. Then the Nash map*

$$\mathcal{N} : \{C_i\}_{i\in I} \longrightarrow \{\text{essential components over } X\}$$

*is bijective.*

*Proof.* In the diagram 3.6, we obtain that $\mathcal{F}$ is injective by Lemma 3.15, $\mathcal{N}$ is injective by Nash's theorem 2.15 and $\mathcal{G}$ is injective by Lemma 3.17. We also have that $\mathcal{G} \circ \mathcal{N} \circ \mathcal{F}$ is the identity map on {minimal elements in $S$} by Lemma 3.16 and 3.17. Hence, $\mathcal{G}, \mathcal{N}, \mathcal{F}$ are all bijective. ☐

By the proof of the above theorem, the following are obvious.

**Corollary 3.19.** *For a toric variety* $X$, $E$ *is an essential component over* $X$, *if and only if* $E$ *is a toric essential component over* $X$.

The analogous result for essential divisors is proved in [3].

**Corollary 3.20.** *For a cone* $\sigma$ *in* $N$ *the number of the minimal elements in* $S = \bigcup_{\tau:\text{singular}} \tau^o$ *is finite. More precisely this number is the number of essential components and also the number of the components* $C_i$*'s.*

**Corollary 3.21.** *For a general point* $c \in C_i$ $(i \in I)$, *the corresponding arc* $\alpha_{ic}$ *satisfies* $\alpha_{ic}(\eta) \in T$.

**Example 3.22.** Let $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (1, 1, e) \in N \simeq \mathbb{Z}^3$ and $\sigma = \langle e_1, e_2, e_3\rangle$. Then all proper faces of $\sigma$ are regular and $\sigma$ itself is not regular, therefore the affine toric variety $X$ corresponding to $\sigma$ has an isolated singularity at the closed orbit. We can also see that $S = \sigma^o$. By simple calculations we obtain that the minimal elements in $S$ are $(1, 1, d)$ $(1 \leq d \leq e - 1)$. Therefore, by our theorem the number of $C_i$'s and the number of the essential components are both $e - 1$.

## 4. Counter examples to the Nash problem

The basic idea of our counter examples to the Nash problem is the following:

Take a singularity $x \in X$ and a partial resolution $p : Y \longrightarrow X$ with exceptional divisor $F \subset Y$. Assume that $Y$ has a singular point $y \in Y$ such that every general arc $g : \operatorname{Spec} k[[s]] \longrightarrow (y \in Y)$ is contained in an embedded surface germ $G : \operatorname{Spec} k[[s, t]] \longrightarrow (y \in Y)$.

Let $E \subset B_yY$ be the exceptional divisor. The arcs through $y \in Y$ that should correspond to the exceptional divisor $E$ are the arcs of $Y$

through $y$. If such an arc is contained in an embedded smooth surface germ, then this arc can be moved in $Y$ along the curve $G^{-1}(F)$, hence the arcs through $E$ are all limits of arcs through some component of $F$.

This implies that $E$ does not correspond to an irreducible component of the family of arcs through $x \in X$. If we can also arrange $E$ to be essential, we have a counter example to the Nash problem.

**4.1.** Algebraically, a smooth formal curve through $0 \in Y$ is equivalent to a surjection $\phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s]]$, where $\hat{\mathcal{O}}_Y$ denotes the completion of $\mathcal{O}_Y$ at the ideal $m_0$ of 0. Similarly, a smooth surface germ is equivalent to a surjection $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[t,s]]$. The induced maps $m_0/m_0^2 \longrightarrow (s)/(s)^2$ and $m_0/m_0^2 \longrightarrow (s,t)/(s,t)^2$ correspond to a point and a line in the exceptional divisor of the blow up $B_0Y \longrightarrow Y$.

**Lemma 4.2.** *Let $0 \in Y \subset \mathbb{A}^n$ be a hypersurface singularity of multiplicity $m$ defined by an equation $F = 0$ where $F = F_m + F_{m+1} + \dots$ are the homogeneous pieces. Set $Z = (F_m = 0) \subset \mathbb{P}^{n-1}$ and let $z \in Z$ be a point and $z \in L \subset Z$ a line such that $Z$ is smooth along $L$ and $H^1(L, N_{L|Z}) = 0$.*

*Let $\phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s]]$ be a smooth formal curve through 0 with tangent direction $z$. Then $\phi$ can be extended to a surjection $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[t,s]]$ with tangent direction $L$.*

*Proof.* The line $L$ can be identified with a map $\Phi_1 : k[y_1,\dots,y_n] \longrightarrow k[s,t]$ such that the $\Phi_1(y_i)$ are linear in $s,t$ and $\Phi_1(F) \in (s,t)^{m+1}$. Our aim is to find inductively maps

$$\Phi_r : k[y_1,\dots,y_n] \longrightarrow k[s,t] \quad \text{such that} \quad \Phi_r(F) \in (s,t)^{m+r}$$

and $\Phi_r$ is congruent to $\Phi_{r+1}$ modulo $(s,t)^{r+1}$. If this can be done then the inverse limit of the maps

$$k[y_1,\dots,y_n] \xrightarrow{\Phi_r} k[s,t] \longrightarrow k[s,t]/(s,t)^{r+1}$$

gives $\Phi : k[[y_1,\dots,y_n]] \longrightarrow k[[s,t]]$ such that $\Phi(F) = 0$. Thus it descends to $\Phi : \hat{\mathcal{O}}_Y \longrightarrow k[[s,t]]$

A map $g : k[y_1,\dots,y_n] \longrightarrow$ (any ring) can be identified with the vector $(g(y_1),\dots,g(y_n))$. Using this convention, by changing coordinates we may assume that $\phi = (s,0,\dots,0)$ and $L = (y_2 = \dots = y_n = 0)$. The first condition implies that no power of $y_1$ appears in $F$ and the second means that we can choose $\Phi_1 = (s,t,0,\dots,0)$.

Assume that we already have $\Phi_r$ which we assume to be of the form

$$\Phi_r = (s, t, tA_{3,r-1}(s,t), \dots, tA_{n,r-1}(s,t))$$

where the $A_{i,r-1}$ are polynomials of degree $\leq r-1$ without constant terms. The vanishing of the constant term comes from extending the map $\Phi_1$ and the divisibility by $t$ comes from the requirement of extending $\phi$. We are looking for $\Phi_{r+1}$ of the form

$$\Phi_{r+1} = (s, t, tA_{3,r-1}(s,t) + tB_{3,r}(s,t), \dots, tA_{n,r-1}(s,t) + tB_{n,r}(s,t)),$$

where the $B_{i,r}$ are homogeneous of degree $r$. Let us compute $\Phi_{r+1}(F)$. Using the Taylor expansion, we get that

$$\begin{aligned}\Phi_{r+1}(F) &= \Phi_r(F) + t \cdot \sum_{i=3}^{n} \frac{\partial F_m}{\partial y_i}(s,t,0,\dots,0) \cdot B_{i,r}(s,t) \\ &\quad +(\text{terms of multiplicity} \geq m+r+1).\end{aligned}$$

By the inductive assumption,

$$\Phi_r(F) = t \cdot C_{m+r-1}(s,t) + (\text{terms of multiplicity} \geq m+r+1),$$

where $C_{m+r-1}$ has degree $m+r-1$. In order to achieve that $\Phi_{r+1}(F) \in (s,t)^{m+r+1}$, we need to find polynomials $B_{i,r}$ such that

$$C_{m+r-1}(s,t) = -\sum_{i=3}^{n} \frac{\partial F_m}{\partial y_i}(s,t,0,\dots,0) \cdot B_{i,r}(s,t). \qquad (*)$$

Since we know nothing about $C_{m+r-1}$, we need to guarantee that the ideal generated by the partials $\partial F_m/\partial y_i(s,t,0,\dots,0)$ contains all homogeneous polynomials of degree $m+r-1$ in $s,t$ for every $r \geq 1$. The critical case is $r=1$.

The normal bundles of $L$ in $Z$ and in $\mathbb{P}^{n-1}$ are related by an exact sequence

$$0 \longrightarrow N_{L|Z} \longrightarrow N_{L|\mathbb{P}^{n-1}} \cong \mathcal{O}(1)^{n-2} \stackrel{dF_m}{\longrightarrow} N_{Z|\mathbb{P}^{n-1}}|_L \cong \mathcal{O}(m) \longrightarrow 0,$$

and $dF_m$ is the map $\mathcal{O}(1)^{n-2} \longrightarrow \mathcal{O}(m)$ given by multiplication by the partials $\partial F_m/\partial y_i$ for $i=3,\dots,n$. We have assumed that $H^1(L, N_{L|Z}) = 0$, thus the induced map

$$\begin{aligned} dF_m : \sum_{i=3}^{n} H^0(L, \mathcal{O}(1)) &\longrightarrow H^0(L, \mathcal{O}(m)), \quad \text{given by} \\ (l_3,\dots,l_n) &\mapsto \sum_{i=3}^{n} l_i \frac{\partial F_m}{\partial y_i}(s,t,0,\dots,0)\end{aligned}$$

is surjective. Thus the equation (*) always has a solution. □

**Theorem 4.3.** *Let $Z \subset \mathbb{P}^{n-1}$ be a smooth hypersurface. Assume that $Z$ is covered by lines but it is not birationally ruled.*

*Let $0 \in X$ be any singularity with a partial resolution $p : Y \longrightarrow X$ and $y \in Y$ a point such that*

(1) *$y \in Y$ is a hypersurface singularity whose projectivised tangent cone is isomorphic to $Z$, and*
(2) *$p^{-1}(0) \subset Y$ is a Cartier divisor.*

*Then the blow up $B_yY$ gives an essential exceptional divisor $Z \cong E \subset B_yY$ over $0 \in X$ which does not correspond to an irreducible family of arcs on $X$.*

*Proof.* A lemma of [1] asserts that if $E$ is an exceptional divisor of a birational morphism $Y \longrightarrow Y'$ with $Y'$ smooth then $E$ is (birationally) ruled. As noted by Nash, this implies that any nonruled exceptional divisor of a resolution $Y \longrightarrow X$ is essential.

Consider the family $W$ of arcs in $B_yY$ through $E$. These correspond to a subset $W_y$ of arcs on $Y$ through $y$ and to a subset $W_x$ of arcs in $X$ through $x$. We claim that $W_x$ is not an irreducible component of the family of all arcs on $X$ through $x$.

In order to see this, it is enough to show that a general arc in $W_y$ is a limit of arcs in $Y$ through $p^{-1}(0)$ but not passing through $y$.

A general arc in $W$ is transversal to $E$, so the general member of $W_y$ is a smooth arc in $Y$ with general tangent direction. By assumption $Z$ is covered by lines and the general line has semi positive normal bundle by [14, II.3.10]. Hence by (4.2), a general arc in $W_y$ is contained in a smooth surface germ. Thus it is a limit of arcs which do not pass through $y$. Hence $W_x$ is not an irreducible component of the space of arcs on $X$ through $x$. □

**Remark 4.4.** A hypersurface $Z \subset \mathbb{P}^{n-1}$ is covered by lines if and only if $\deg Z \leq n-2$. Thus the key condition is to check that $Z$ is not birationally ruled. This can not happen if $n \leq 4$. In higher dimensions there are two known sets of examples:

(1) $Z \subset \mathbb{P}^4$ is a smooth cubic. Then $Z$ is not rational. This was proved by [5] over $\mathbb{C}$ and by [18] in characteristic $\neq 2$. This implies that $Z$ is not ruled. Indeed, assume that $Z$ is birational to $S \times \mathbb{P}^1$. $Z$ is unirational, so we get a dominant separable map $\mathbb{P}^3 \dashrightarrow S \times \mathbb{P}^1 \longrightarrow S$. Thus $S$ is separably unirational hence rational by Castelnuovo's theorem. Therefore $Z$ is birational to $\mathbb{P}^2 \times \mathbb{P}^1$ and so rational, a contradiction.

(2) $Z \subset \mathbb{P}^{n-1}$ is a very general hypersurface with $n \geq \deg Z \geq \frac{2n}{3}+2$. These are nonruled in characteristic zero by [13].

**Example 4.5.** The 4-dimensional hypersurface singularity over an algebraically closed field of characteristic $\neq 2, 3$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$$

has only 1 irreducible family of arcs but 2 essential exceptional components.

*Proof.* Apply Theorem 4.3 to $X = (x_1^3+x_2^3+x_3^3+x_4^3+x_5^6 = 0)$. Blowing up the origin produces $Y$. The exceptional divisor $F \subset Y$ is Cartier and $Y$ has a unique singular point which is the cone over the cubic 3–fold $Z := (x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0)$. $Z$ is not birationally ruled by (4.4.1).

Blowing up the unique singular point of $Y$ we get a resolution of $X$ with 2 exceptional divisors. One is $E \cong Z$ and the other is $F'$, the birational transform of $F$.

$F'$ is birationally ruled, but it is still essential. Indeed, the family of arcs on $X$ has to correspond to some exceptional divisor, and $F'$ is the only possibility. Thus $F'$ has to be essential. Another way to see this is to note that $X$ is terminal and $F'$ has minimal discrepancy, namely 1. □

## References


1. S. Abhyankar, *On the valuations centered in a local domain.* Amer. J. Math. 78 (1956) 321–348.
2. M. Artin, *On the solutions of analytic equations*, Invent. Math. 5 1968 277–291.
3. C. Bouvier, *Diviseurs essentiels, composantes essentielles des variétés toriques singulières*, Duke Math. J. 91 (1998) 609–620
4. C. Bouvier and G. Gonzalez-Sprinberg, *Système générateure minimal, diviseurs essentiels et G-désingularisations de variétés toriques*, Tohoku Math. J. **47**, (1995) 125–149.
5. C.H. Clemens and P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. 95 (1972), 281–356
6. D. I. Dais, M. Henk and G. M. Ziegler, *All toric local complete intersection singularities admit projective crepant resolutions*, Tohoku Math. J. (2) **53**, (2001) 95–107.
7. J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration,* Invent. Math. **135**, (1999) 201–232.
8. W. Fulton, *Introduction to Toric Varieties*, Annals of Math. St. **131**, (1993) Princeton University Press.
9. H. Gillet, *Differential Algebra - a scheme theory approach*, preprint, http://www.math.uic.edu/ henri
10. G. Gonzalez-Sprinberg and M. Lejeune-Jalabert, *Families of smooth curves on surface singularities and wedges*, Annales Polonici Mathematici, LXVII.2, (1997) 179–190.
11. M. Hickel, *Fonction de Artin et germes de courbes tracées sur un germe d'espace analytique*, Amer. J. Math. **115**, (1993) 1299–1334.
12. E. R. Kolchin *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. 54. Academic Press, New York-London, 1973.
13. J. Kollár, *Nonrational hypersurfaces*, J. Amer. Math. Soc. 8 (1995) 241–249
14. J. Kollár, *Rational Curves on Algebraic Varieties*, Springer, 1996

15. M. Kontsevich, *Lecture at Orsay* (December 7, 1995)
16. M. Lejeune-Jalabert, *Courbes tracées sur un germe d'hypersurface.* Amer. J. Math. **112**, (1990) 525–568.
17. M. Lejeune-Jalabert and A. J. Reguera-Lopez, *Arcs and wedges on sandwiched surface singularities*, Amer. J. Math. **121**, (1999) 1191–1213.
18. J. P. Murre, *Reduction of the proof of the non-rationality of a non-singular cubic threefold to a result of Mumford*, Compositio Math. 27 (1973, 63–8
19. M. Mustaţă, *Singularities of Pairs via Jet Schemes*, J. Amer. Math. Soc. 15 (2002), 599-615.
20. J. F. Nash, *Arc structure of singularities*, Duke Math. J. **81**, (1995) 31–38.
21. A. Nobile, *On Nash theory of arc structure of singularities.* Ann. Mat. Pura Appl. 160 (1991), 129–146

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