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SUBVARIETIES OF GENERIC HYPERSURFACES IN A NONSINGULAR PROJECTIVE TORIC VARIETY

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1. Introduction

Bounding the geometric genus of subvarieties of generic hypersurface started form the following result by H. Clemens [3, Theorem 1.1].

Theorem 1.1 (H. Clemens)**.** *Let X be a generic hypersurface of degree d in $\mathbf{P}^n$. Let C be a nonsingular projective curve, and $C \to X$ be a morphism of degree e. If $d \geq 2n-1$, then the genus satisfies $g(C) \geq \frac{1}{2} \cdot e \cdot (d-2n+1)+1$.*

This is generalized by L. Ein [7, Theorem 2.1] to the following form.

Theorem 1.2 (L. Ein)**.** *Let $\mathbf{P}$ be a nonsingular projective variety of dimension n, $\mathcal{L}$ be an ample invertible sheaf which is generated by global sections, and $X \in |\mathcal{L}^d|$ be a generic hypersurface. Let Y be a nonsingular complete variety of dimension l, and $Y \to X$ be a morphism which is birational to the image.*

1. *If $d \geq 2n-l$, then the geometric genus $p_g(Y)$ is positive.*
2. *If $d \geq 2n-l+1$, then Y is of general type.*

The purpose of this paper is to improve the bound in the case when $\mathbf{P}$ is a nonsingular projective toric variety. The theorem of L. Ein is stated when a morphism to a projective space is fixed, but we want to state the result independent from an embedding to a projective space. More clearly, we describe the bound in the Picard group using the intersection numbers with invariant curves on the toric variety. Theorem 3.5 is a generalization of Theorem 1.1, and Theorem 3.7 is a generalization of Theorem 1.2. Of course, our bound is not sharp. In the case when $\mathbf{P}$ is a projective space, there is a better result by C. Voisin [16, Theorem 1].

Theorem 1.3 (C. Voisin)**.** *Let X be a generic hypersurface of degree d in $\mathbf{P}^n$. Let Y be a nonsingular complete variety of dimension $l \leq n-3$, and $Y \to X$ be a morphism which is birational to the image.*

1. *If $d \geq 2n-l-1$, then the geometric genus $p_g(Y)$ is positive.*
2. *If $d \geq 2n-l$, then Y is of general type.*

In the same way as this case, we need another study to approach the sharp bound, but it remains in our case.

By the similar arguments, C. Voisin gives a following results concerning about zero cycles on a generic hypersurface in a projective space [15, Theorem 1.11].

Theorem 1.4 (C. Voisin)**.** *Let X be a generic hypersurface of degree d in $\mathbf{P}^n$. If $d \geq 2n+1$, then any two distinct closed points of X are not rationally equivalent.*

In Theorem 3.8, we generalize this theorem to the case of toric hypersurfaces, describing the bound in the Picard group by the intersection numbers with invariant curves.

In order to work without fixing an embedding to a projective space, we want to treat a projective toric variety like a projective space. Since a projective toric variety $\mathbf{P}$ has similar characters to a projective space, it is appropriate to consider as an ambient space. For example, it has the homogeneous coordinate ring S, and quasi-coherent sheaves on $\mathbf{P}$ are corresponding to $\mathrm{Pic}(\mathbf{P})$-graded S-modules [4]. Based on these study, our results are proved by the same way as the case of a projective space. But, in the toric case, we need another argument about the surjectivity of the multiplications of linear systems on $\mathbf{P}$. Although it is clear in projective space case, it is not found a fine condition for the surjectivity in the case of a nonsingular projective toric variety. In Theorem 2.5, we give the condition, when the toric variety is a toric projective space bundle over a projective toric variety on which the condition is known. So our results about hypersurfaces needs certain assumption $*$-2.2 on the toric variety. Without assuming $*$-2.2, we can state the results by modifying the bounds, but we omit these, because it is not found a nonsingular projective toric variety which does not satisfy the condition $*$-2.2 yet.

As an application of the surjectivity of the multiplications of linear systems on $\mathbf{P}$, in Theorem 3.2, we treat the infinitesimal Torelli problem on a nonsingular hypersurface in $\mathbf{P}$.

2. Multiplications of linear systems on a nonsingular projective toric variety

We consider the following conditions on a nonsingular projective toric variety $\mathbf{P}$ of dimension n over a field k.

$*$- 2.1. For any invertible shaves $\mathcal{L}_1$ and $\mathcal{L}_2$ which are generated by global sections, the multiplication map

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

is surjective.

$*$- 2.2. For any $\mathcal{L}_1$ generated by global sections and for any ample $\mathcal{L}_2$, the multiplication map

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

is surjective.

It is well-known that a projective space $\mathbf{P}^n$ satisfies the condition $*$-2.1. There is a nonsingular projective toric variety which does not satisfy the condition $*$-2.1. Recently, in [8, Theorem 1], N. Fakhruddin shows that any nonsingular projective toric surface satisfies the condition $*$-2.2. It is not known that there exists a nonsingular projective toric variety which does not satisfy the condition $*$-2.2. This problem have been proposed in [14].

We set the following subsets of the Néron-Severi group $\mathrm{NS}(\mathbf{P})$;

$$\begin{aligned} A^{(l)}(\mathbf{P}) &= \{\eta \in \mathrm{NS}(\mathbf{P}) \mid (\eta \,.\, V(\tau)) \geq l \ \mathit{for}\ \tau \in \Sigma(n-1)\}, \\ \mathrm{Nef}(\mathbf{P}) &= A^{(0)}(\mathbf{P}), \\ \mathrm{Ample}(\mathbf{P}) &= A^{(1)}(\mathbf{P}), \end{aligned}$$

where Σ is a fan defining $\mathbf{P}$, and $V(\tau)$ is the closure of the torus-orbit corresponding to a $(n-1)$-dimensional cone $\tau \in \Sigma$.

Remark 2.3. On a complete toric variety, an invertible sheaf is generated by global sections if and only if it is nef [11, Theorem 1.6]. If $\mathbf{P}$ is nonsingular projective toric variety, then an ample invertible sheaf is very ample [13, Corollary 2.15], and it is characterized by the toric Nakai criterion [13, Theorem 2.18]. We note that the Picard group $\mathrm{Pic}(\mathbf{P})$ is isomorphic to the Néron-Severi group $\mathrm{NS}(\mathbf{P})$, which is a finitely generated free abelian group.

For a pair of nonnegative integers (l_1, l_2), we consider a condition

$*$- 2.4. For any $\mathcal{L}_1 \in A^{(l_1)}(\mathbf{P})$ and for any $\mathcal{L}_2 \in A^{(l_2)}(\mathbf{P})$, the multiplication map

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

is surjective.

The condition $*$-2.4 for $(0,0)$ is the condition $*$-2.1, and the condition $*$-2.4 for $(0,1)$ is the condition $*$-2.2. We prove the following statement.

Theorem 2.5. *Let $\mathbf{P}$ be a toric projective space bundle over a nonsingular projective toric variety $\mathbf{P}'$. If $\mathbf{P}'$ satisfies the condition $*$-2.4 for (l_1, l_2), then $\mathbf{P}$ satisfies the condition $*$-2.4 for (l_1, l_2).*

Corollary 2.6. *Let $\mathbf{P}$ be a toric variety defined by a splitting fan. Then $\mathbf{P}$ satisfies the condition $*$-2.1. In particular, any nonsingular projective toric variety with Picard number 2 satisfies the condition $*$-2.1.*

Corollary 2.7. *Let $\mathbf{P}$ be a toric variety produced from a nonsingular projective toric surface by a sequence of toric projective space bundles. Then $\mathbf{P}$ satisfies the condition $*$-2.2.*

Remark 2.8. A toric variety is defined by a splitting fan if and only if it is produced from a projective space by a sequence of toric projective space bundles [1, Section 4]. A nonsingular projective toric variety with Picard number 2 is a toric projective space bundle over a projective space [10, Theorem 1].

Let $\mathbf{P}'$ be a nonsingular projective toric variety defined by a fan (N', Σ'), and let $\{D'_1, \cdots, D'_s\}$ be the set of all torus-invariant prime divisors on $\mathbf{P}'$. Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}'}(E'_0) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}'}(E'_r)$ be a locally free sheaf defined by torus-invariant divisors $E'_0, \cdots, E'_r$ on $\mathbf{P}'$. The projective space bundle $\mathbf{P} = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} \mathbf{P}'$ is an equivariant morphism of toric varieties [13, Section 1.7]. If we denote $D_j = \pi^{-1}(D'_j)$ and $E_i = \mathbf{P}(\mathcal{E}/\mathcal{O}_{\mathbf{P}'}(E'_i))$, then the set of all torus-invariant prime divisors on $\mathbf{P}$ is $\{D_1, \cdots, D_s, E_0, \cdots, E_r\}$.

Let ξ be the class of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ in the Néron-Severi group $\mathrm{NS}(\mathbf{P})$. By the injection $\mathrm{NS}(\mathbf{P}') \xrightarrow{\pi^*} \mathrm{NS}(\mathbf{P})$, we have $\mathrm{NS}(\mathbf{P}) \cong \mathrm{NS}(\mathbf{P}') \oplus \mathbf{Z} \cdot \xi$. The class of D_j in $\mathrm{NS}(\mathbf{P})$ is $[D_j] = \pi^*[D'_j]$, and the class of E_i is $[E_i] = -\pi^*[E'_i] + \xi$.

Lemma 2.9. *For $\eta' \in \mathrm{NS}(\mathbf{P}')$ and $b \in \mathbf{Z}$, we set $\eta = \pi^*\eta' + b\xi \in \mathrm{NS}(\mathbf{P})$. Then η is in $A^{(l)}(\mathbf{P})$ if and only if $b \geq l$ and $\eta' + b[E'_i]$ is in $A^{(l)}(\mathbf{P}')$ for $0 \leq i \leq r$.*

Proof. Let (N, Σ) be a fan which defines $\mathbf{P}$, and let $\rho_j \in \Sigma(1)$ be the 1-dimensional cone corresponding to D_j, and let $\tau_i \in \Sigma(1)$ be corresponding to E_i. If σ is a $(n-1)$-dimensional cone in Σ, then $\sigma = \rho_{j_1} + \cdots + \rho_{j_{n-r-1}} + \tau_{i_1} + \cdots + \tau_{j_r}$ or $\sigma = \rho_{j_1} + \cdots + \rho_{j_{n-r}} + \tau_{i_1} + \cdots + \tau_{j_{r-1}}$. In the case when $\sigma = \rho_{j_1} + \cdots + \rho_{j_{n-r-1}} + \tau_{i_1} + \cdots + \tau_{j_r}$, $\sigma' = \rho'_{j_1} + \cdots + \rho'_{j_{n-r-1}}$ is $(n-r-1)$-dimensional cone in Σ', where ρ'_j is the 1-dimensional cone corresponding to D'_j. Let $V(\sigma)$ be the closure of the torus-orbit corresponding to σ. The intersection number $(\eta.V(\sigma))$ is equal to $(\eta' + b[E'_i].V(\sigma'))$, where $i \in [0, r]$ is the only integer different from $i_1, \cdots, i_r$. In the case when $\sigma = \rho_{j_1} + \cdots + \rho_{j_{n-r}} + \tau_{i_1} + \cdots + \tau_{j_{r-1}}$, the intersection number $(\eta.V(\sigma))$ is equal to b. Then the Lemma is checked by definition. □

Proof of Theorem 2.5. We use the homogeneous coordinate ring of a toric variety introduced in [4, Section 1]. The homogeneous coordinate ring is a k-algebra graded by the Néron-Severi group. Let $S' = k[z'_1, \cdots, z'_s]$ be the homogeneous coordinate ring of $\mathbf{P}'$, where the variable z'_j has a degree $[D'_j]$ in $\mathrm{NS}(\mathbf{P}')$, and let $S = k[z_1, \cdots, z_s, y_0, \cdots, y_r]$ be the homogeneous coordinate ring of $\mathbf{P}$ with $\deg(z_j) = [D_j]$ and $\deg(y_i) = [E_i]$. By [4, Proposition 1.1], $H^0(\mathbf{P}, \mathcal{L}_h)$ is identified with the degree $[\mathcal{L}_h]$-part $S_{[\mathcal{L}_h]}$ of S, and the multiplication

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

is identified with the multiplication of the polynomial ring S

$$S_{[\mathcal{L}_1]} \otimes S_{[\mathcal{L}_2]} \longrightarrow S_{[\mathcal{L}_1 \otimes \mathcal{L}_2]}.$$

For $h = 1, 2$, let $\mathcal{L}'_h$ be an invertible sheaf on $\mathbf{P}'$ and a_h be an integer with $[\mathcal{L}_h] = \pi^*[\mathcal{L}'_h] + a_h\xi$. By Lemma 2.9, we have $a_1 \geq l_1$ and $a_2 \geq l_2$, and $[\mathcal{L}'_1 \otimes \mathcal{O}_{\mathbf{P}'}(a_1 E'_i)] \in A^{(l_1)}(\mathbf{P}')$ and $[\mathcal{L}'_2 \otimes \mathcal{O}_{\mathbf{P}'}(a_2 E'_i)] \in A^{(l_2)}(\mathbf{P}')$ for $0 \leq i \leq r$.

Let $z_1^{d_1} \cdots z_s^{d_s} y_0^{e_0} \cdots y_r^{e_r}$ be a monomial in $S_{[\mathcal{L}_1 \otimes \mathcal{L}_2]}$. Then we have

$$\sum_{j=1}^{s} d_j [D'_j] - \sum_{i=0}^{r} e_i [E'_i] = [\mathcal{L}'_1 \otimes \mathcal{L}'_2]$$

and $\sum_{i=0}^r e_i = a_1 + a_2$. We can find nonnegative integers $e_{i,1}$ and $e_{i,2}$ which satisfy that $\sum_{i=0}^r e_{i,1} = a_1$, $\sum_{i=0}^r e_{i,2} = a_2$ and $e_i = e_{i,1} + e_{i,2}$.

Since $[\mathcal{L}'_h \otimes \mathcal{O}_{\mathbf{P}'}(a_h E'_i)]$ is in $A^{(l_h)}(\mathbf{P}')$ for $0 \leq i \leq r$,

$$\sum_{i=0}^{r} \frac{e_{i,h}}{a_h} ([\mathcal{L}'_h] + a_h [E'_i].V(\tau)) \geq \sum_{i=0}^{r} \frac{e_{i,h}}{a_h} \cdot l_h = l_h$$

for any $\tau \in \Sigma(\dim(\mathbf{P}') - 1)$, hence

$$[\mathcal{L}'_h \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^{r} e_{i,h} E'_i)] = \sum_{i=0}^{r} \frac{e_{i,h}}{a_h} ([\mathcal{L}'_h] + a_h [E'_i])$$

is in $A^{(l_h)}(\mathbf{P}')$. (If $a_h = 0$, then $[\mathcal{L}'_h] \in A^{(l_h)}(\mathbf{P}')$.) By the assumption $*$-2.4 for $\mathbf{P}'$, the multiplication

$$S'_{[\mathcal{L}'_1 \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^r e_{i,1} E'_i)]} \otimes S'_{[\mathcal{L}'_2 \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^r e_{i,2} E'_i)]} \longrightarrow S'_{[\mathcal{L}'_1 \otimes \mathcal{L}'_2 \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^r e_i E'_i)]}$$

is surjective, and ${z'}_1^{d_1} \cdots {z'}_s^{d_s}$ is contained in $S'_{[\mathcal{L}'_1 \otimes \mathcal{L}'_2 \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^r e_i E'_i)]}$. So there is a monomial ${z'}_1^{d_{1,h}} \cdots {z'}_s^{d_{s,h}} \in S_{[\mathcal{L}'_h \otimes \mathcal{O}_{\mathbf{P}'}(\sum_{i=0}^r e_{i,h} E'_i)]}$ for $h = 1, 2$, such that

$$({z'}_1^{d_{1,1}} \cdots {z'}_s^{d_{s,1}}) \cdot ({z'}_1^{d_{1,2}} \cdots {z'}_s^{d_{s,2}}) = {z'}_1^{d_1} \cdots {z'}_s^{d_s}.$$

The surjectivity of

$$S_{[\mathcal{L}_1]} \otimes S_{[\mathcal{L}_2]} \longrightarrow S_{[\mathcal{L}_1 \otimes \mathcal{L}_2]}.$$

is proved, because the monomial $z_1^{d_{1,h}} \cdots z_s^{d_{s,h}} y_0^{e_{0,h}} \cdots y_r^{e_{r,h}}$ is contained in $S_{[\mathcal{L}_h]}$, and

$$(z_1^{d_{1,1}} \cdots z_s^{d_{s,1}} y_0^{e_{0,1}} \cdots y_r^{e_{r,1}}) \cdot (z_1^{d_{1,2}} \cdots z_s^{d_{s,2}} y_0^{e_{0,2}} \cdots y_r^{e_{r,2}}) = z_1^{d_1} \cdots z_s^{d_s} y_0^{e_0} \cdots y_r^{e_r}.$$

□

Proposition 2.10. *We assume that* $\mathbf{P}$ *satisfies the condition* $*$*-2.4 for* (l_1, l_2)*. For* $\mathcal{L}_1 \in A^{(l_1)}(\mathbf{P})$ *and* $\mathcal{L}_2 \in A^{(l_2+1)}(\mathbf{P})$ *and for* $\sigma \in \Sigma$ *the multiplication map*

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2 \otimes \mathcal{I}_Z) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_Z)$$

is surjective, where $Z = V(\sigma)$ *is the closure of the torus-orbit corresponding to* σ*, and* $\mathcal{I}_Z$ *is the ideal sheaf of* Z *in* $\mathbf{P}$*.*

Proof. Let $\mathcal{F}_1$ be the locally free $\mathcal{O}_\mathbf{P}$-module defined by the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1) \otimes \mathcal{O}_\mathbf{P} \longrightarrow \mathcal{L}_1 \longrightarrow 0.$$

We want to show $H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_Z) = 0$.

Let $S = k[z_1, \cdots, z_s]$ be the homogeneous coordinate ring of $\mathbf{P}$. We may assume that σ is generated by the cones corresponding to $z_1, \cdots, z_l$. Let I_Z be the homogeneous ideal generated by $z_1, \cdots, z_l$. Then $\mathcal{I}_Z$ is the coherent $\mathcal{O}_\mathbf{P}$-module associated to I_Z. Since I_Z has the Koszul resolution

$$\begin{aligned} 0 \to S([-D_1 - \cdots - D_l]) \to \cdots \\ \cdots \to \bigoplus_{1 \le j_1 < j_2 \le l} S([-D_{j_1} - D_{j_2}]) \to \bigoplus_{1 \le j \le l} S([-D_j]) \to I_Z \to 0, \end{aligned}$$

we have a resolution of $\mathcal{I}_Z$

$$\begin{aligned} 0 \to \mathcal{O}_\mathbf{P}(-D_1 - \cdots - D_l) \to \cdots \\ \cdots \to \bigoplus_{1 \le j_1 < j_2 \le l} \mathcal{O}_\mathbf{P}(-D_{j_1} - D_{j_2}) \to \bigoplus_{1 \le j \le l} \mathcal{O}_\mathbf{P}(-D_j) \to \mathcal{I}_Z \to 0, \end{aligned}$$

by [4, Proposition 3.1]. The vanishing of $H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_Z)$ is shown by the next Lemma. □

Lemma 2.11. *We assume that* $\mathbf{P}$ *satisfies the condition* $*$*-2.4 for* (l_1, l_2)*, and let* $\mathcal{F}_1$ *and* $\mathcal{L}_2$ *be as above. If* $q \ge 2$ *or* $q = 1$*,* $0 \le p \le 1$*, then*

$$H^q(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2(-D_{j_1} - \cdots - D_{j_p})) = 0,$$

for $1 \le j_1 < \cdots < j_p \le s$*.*

Proof. First, we consider the case $q = 1$, $p = 1$. Since $\mathcal{L}_2(-D_j)$ is in $A^{(l_2)}(\mathbf{P})$ by [12, Proposition 4.3], we have $H^1(\mathbf{P}, \mathcal{L}_2(-D_j)) = 0$. By the assumption $*$-2.4 for (l_1, l_2), the multiplication

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2(-D_j)) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2(-D_j))$$

is surjective. So $H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2(-D_j)) = 0$. In the case $q = 1$, $p = 0$, the vanishing of $H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2)$ is proved by the same way.

We assume $q \ge 2$. By the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{F}_1 \otimes \mathcal{L}_2(-D_{j_1} - \cdots - D_{j_p}) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1) \otimes \mathcal{L}_2(-D_{j_1} - \cdots - D_{j_p}) \\ \longrightarrow \mathcal{L}_1 \otimes \mathcal{L}_2(-D_{j_1} - \cdots - D_{j_p}) \longrightarrow 0, \end{aligned}$$

$H^q(\mathcal{F}_1 \otimes \mathcal{L}_2(-D_{j_1} - \cdots - D_{j_p})) = 0$ is proved by the vanishing theorem of Mustaţă [12, Corollary 2.5]. □

Theorem 2.12. *Let $\mathbf{P}$ be a nonsingular projective toric variety satisfying the condition $*$-2.4 for (l_1, l_2). For $\mathcal{L}_1 \in A^{(l_1)}(\mathbf{P})$ and $\mathcal{L}_2 \in A^{(l_2+1)}(\mathbf{P})$ and for $p \in \mathbf{P}(k)$, the multiplication map*

$$H^0(\mathbf{P}, \mathcal{L}_1) \otimes H^0(\mathbf{P}, \mathcal{L}_2 \otimes \mathcal{I}_p) \longrightarrow H^0(\mathbf{P}, \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_p)$$

is surjective.

Proof. Let $\mathcal{I}_\Delta \subset \mathcal{O}_{\mathbf{P}\times\mathbf{P}}$ be the ideal sheaf of the diagonal subvariety $\mathbf{P} = \Delta \to \mathbf{P} \times \mathbf{P}$, and let $\mathbf{P}_p = \mathbf{P}$ be the fiber of the first projection $\mathbf{P} \times \mathbf{P} \xrightarrow{\pi_1} \mathbf{P}$ at p. Since the restriction of $\mathcal{O}_{\mathbf{P}\times\mathbf{P}} \to \mathcal{O}_\Delta$ to the fiber $\mathbf{P}_p$ is identified with $\mathcal{O}_{\mathbf{P}} \to k = \mathcal{O}_p$, we have a surjection $\mathcal{I}_\Delta|_{\mathbf{P}_p} \to \mathcal{I}_p$, whose kernel is supported on p. So the map

$$H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_\Delta|_{\mathbf{P}_p}) \longrightarrow H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_p)$$

is an isomorphism, where $\mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_\Delta|_{\mathbf{P}_p}$ is isomorphic to the restriction of $\pi_2^*(\mathcal{F}_1 \otimes \mathcal{L}_2) \otimes \mathcal{I}_\Delta$ to the fiber $\mathbf{P}_p$. Since $\pi_2^*(\mathcal{F}_1 \otimes \mathcal{L}_2) \otimes \mathcal{I}_\Delta$ is flat over $\mathbf{P}$, by the upper semicontinuity,

$$\{p \in \mathbf{P}(k) \mid H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_p) \neq 0\}$$

is a closed subset, and it is torus-invariant. If it is not empty, then it contains a torus-invariant point. But it contradicts Proposition 2.10, so $H^1(\mathbf{P}, \mathcal{F}_1 \otimes \mathcal{L}_2 \otimes \mathcal{I}_p)$ is vanish for any $p \in \mathbf{P}(k)$. □

3. Ample hypersurfaces on nonsingular projective toric varieties

Let $\mathbf{P}$ be a nonsingular projective toric variety of dimension n over an algebraically closed field k, and let $\{D_1, \cdots, D_s\}$ be the set of all torus-invariant prime divisors on $\mathbf{P}$. For an ample invertible sheaf $\mathcal{L}$, let

$$\mathcal{X}_\mathcal{L} \longrightarrow \mathcal{M}_\mathcal{L} = \mathrm{Proj}(\mathrm{Sym}_k H^0(\mathbf{P}, \mathcal{L})^*)$$

be the universal family of hypersurfaces in $|\mathcal{L}|$.

Remark 3.1. Let K be a field over k, and $f : \mathrm{Spec}(K) \to \mathcal{M}_\mathcal{L}$ be a k-morphism. The fiber of $\mathcal{X}_\mathcal{L} \to \mathcal{M}_\mathcal{L}$ at f is denoted by X_f. Then X_f is a hypersurface in $\mathbf{P}_K = \mathbf{P} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. Since $\mathbf{P}_K$ satisfies the same condition $*$-2.4 as $\mathbf{P}$, we don't note the definition field of $\mathbf{P}$.

First, we prove the infinitesimal Torelli problem under some assumption.

Theorem 3.2. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let $\mathcal{L}$ be an ample invertible sheaf on $\mathbf{P}$, and let X_f be a nonsingular fiber of $\mathcal{X}_\mathcal{L} \to \mathcal{M}_\mathcal{L}$. If $\mathcal{L} \in -K_{\mathbf{P}} + \mathrm{Ample}(\mathbf{P})$, then the infinitesimal period map*

$$H^1(X_f, T_{X_f}) \longrightarrow \bigoplus_{p=1}^{n-1} \mathrm{Hom}(H^{n-p-1}(X_f, \Omega^p_{X_f}), H^{n-p}(X_f, \Omega^{p-1}_{X_f}))$$

is injective.

Proof. We use the Jacobian ring of X_f introduced in [2, Section 10]. By the duality, we prove that the map

$$H^{n-p-1}(X_f, \Omega^p_{X_f}) \otimes H^{p-1}(X_f, \Omega^{n-p}_{X_f}) \longrightarrow H^{n-2}(X_f, \Omega^1_{X_f} \otimes \Omega^{n-1}_{X_f})$$

is surjective for some $1 \leq p \leq n-1$.

When $1 \le p \le n-2$, in the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \Omega_{\mathbf{P}}^{p+1}(\log X_f) \\ &\longrightarrow \Omega_{\mathbf{P}}^{p+1}(X_f) \longrightarrow \Omega_{\mathbf{P}}^{p+2}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes 2} \longrightarrow \Omega_{\mathbf{P}}^{p+3}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes 3} \longrightarrow \cdots \\ &\cdots \longrightarrow \Omega_{\mathbf{P}}^{n-1}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-p-1} \longrightarrow \Omega_{\mathbf{P}}^{n}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-p} \longrightarrow 0, \end{aligned}$$

the cohomology $H^j(\mathbf{P}, \Omega_{\mathbf{P}}^{p+1}(X_f))$ and $H^j(X_f, \Omega_{\mathbf{P}}^{p+i}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes i})$ is vanish for $i \ge 2$ and $j \ge 1$, because of the vanishing theorem of Bott-Steenbrink-Danilov [2, Theorem 7.1] and the exact sequence

$$0 \longrightarrow \Omega_{\mathbf{P}}^{p+i}((i-1)X_f) \longrightarrow \Omega_{\mathbf{P}}^{p+i}(iX_f) \longrightarrow \Omega_{\mathbf{P}}^{p+i}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes i} \longrightarrow 0.$$

So there is a surjection

$$H^0(X_f, \Omega_{\mathbf{P}}^{n}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-p}) \longrightarrow H^{n-p-1}(\mathbf{P}, \Omega_{\mathbf{P}}^{p+1}(\log X_f)).$$

By the residue map

$$H^{n-p-1}(\mathbf{P}, \Omega_{\mathbf{P}}^{p+1}(\log X_f)) \longrightarrow H^{n-p-1}(X_f, \Omega_{X_f}^{p}),$$

we have a natural map

$$H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n}((n-p)X_f)) \longrightarrow H^{n-p-1}(X_f, \Omega_{X_f}^{p}),$$

where the image of the map is called primitive cohomology in [2, Section 10]. In the case when $p = n-1$, this is an isomorphism. By the same way, we have a map

$$H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n}(pX_f)) \longrightarrow H^{p-1}(X_f, \Omega_{X_f}^{n-p}).$$

Since we assume $\Omega_{\mathbf{P}}^n \otimes \mathcal{L}$ is ample, in the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \Omega_{\mathbf{P}}^{2}(\log X_f) \otimes \Omega_{\mathbf{P}}^{n}(X_f) \\ &\longrightarrow \Omega_{\mathbf{P}}^{2}(X_f) \otimes \Omega_{\mathbf{P}}^{n}(X_f) \longrightarrow \Omega_{\mathbf{P}}^{3}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes 2} \otimes \Omega_{X_f}^{n-1} \longrightarrow \cdots \\ &\cdots \longrightarrow \Omega_{\mathbf{P}}^{n-1}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-2} \otimes \Omega_{X_f}^{n-1} \longrightarrow \Omega_{\mathbf{P}}^{n}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-1} \otimes \Omega_{X_f}^{n-1} \longrightarrow 0 \end{aligned}$$

the cohomology $H^j(\mathbf{P}, \Omega_{\mathbf{P}}^2(X_f) \otimes \Omega_{\mathbf{P}}^n(X_f))$ and $H^j(X_f, \Omega_{\mathbf{P}}^{i+1}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes i} \otimes \Omega_{X_f}^{n-1})$ is vanish for $i \ge 2$ and $j \ge 1$, so there is a surjection

$$H^0(X_f, \Omega_{\mathbf{P}}^{n}|_{X_f} \otimes N_{X_f/\mathbf{P}}^{\otimes n-1} \otimes \Omega_{X_f}^{n-1}) \longrightarrow H^{n-2}(\mathbf{P}, \Omega_{\mathbf{P}}^{2}(\log X_f) \otimes \Omega_{\mathbf{P}}^{n}(X_f)).$$

By the surjectivity of the residue map

$$H^{n-2}(\mathbf{P}, \Omega_{\mathbf{P}}^{2}(\log X_f) \otimes \Omega_{\mathbf{P}}^{n}(X_f)) \longrightarrow H^{n-2}(X_f, \Omega_{X_f}^{1} \otimes \Omega_{X_f}^{n-1}),$$

we have a natural surjection

$$H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n} \otimes \Omega_{\mathbf{P}}^{n}(nX_f)) \longrightarrow H^{n-2}(X_f, \Omega_{X_f}^{1} \otimes \Omega_{X_f}^{n-1}).$$

Since we assume $*$-2.2, in the commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n} \otimes \mathcal{L}^{n-p}) \otimes H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n} \otimes \mathcal{L}^{p}) & \longrightarrow & H^0(\mathbf{P}, \Omega_{\mathbf{P}}^{n} \otimes \Omega_{\mathbf{P}}^{n} \otimes \mathcal{L}^{n}) \\ \downarrow & & \downarrow \\ H^{n-p-1}(X_f, \Omega_{X_f}^{p}) \otimes H^{p-1}(X_f, \Omega_{X_f}^{n-p}) & \longrightarrow & H^{n-2}(X_f, \Omega_{X_f}^{1} \otimes \Omega_{X_f}^{n-1}), \end{array}$$

the top horizontal arrow is surjective, hence the bottom horizontal arrow is surjective. □

The next is a key proposition for our main results, and it is corresponding to [15, Proposition 1.1] in the case of a projective space. To state results, we use the following subset of $\mathrm{NS}(\mathbf{P})$,

$$B(\mathbf{P}) = \{\eta \in \mathrm{NS}(\mathbf{P}) \mid \eta \pm [D_i] \in \mathrm{Nef}(\mathbf{P})\ (1 \le i \le s)\}.$$

Proposition 3.3. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let $\mathcal{L}$ be an ample invertible sheaf on $\mathbf{P}$, and let $\mathcal{X} \to \mathcal{M}$ be a base change of $\mathcal{X}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{L}}$ by any étale morphism $\mathcal{M} \to \mathcal{M}_{\mathcal{L}}$. If X_f is a nonsingular fiber of $\mathcal{X} \to \mathcal{M}$, then $T_{\mathcal{X}}|_{X_f} \otimes \mathcal{L}_0|_{X_f}$ is generated by global sections, for any $[\mathcal{L}_0] \in B(\mathbf{P})$.*

Proof. Let $\mathcal{G}$ be defined by

$$0 \longrightarrow \mathcal{G} \longrightarrow H^0(X_f, \mathcal{L}|_{X_f}) \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{L}|_{X_f} \longrightarrow 0.$$

Since $T_{\mathcal{M},f} \simeq H^0(X_f, \mathcal{L}|_{X_f})$ and $N_{X_f/\mathbf{P}} \simeq \mathcal{L}|_{X_f}$, the sequence is

$$0 \longrightarrow \mathcal{G} \longrightarrow T_{\mathcal{M},f} \otimes \mathcal{O}_{X_f} \longrightarrow N_{X_f/\mathbf{P}} \longrightarrow 0.$$

By the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X_f} & \longrightarrow & T_{\mathcal{X}}|_{X_f} & \longrightarrow & T_{\mathcal{M},f} \otimes \mathcal{O}_{X_f} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{X_f} & \longrightarrow & T_{\mathbf{P}}|_{X_f} & \longrightarrow & N_{X_f/\mathbf{P}} & \longrightarrow & 0, \end{array}$$

we have an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow T_{\mathcal{X}}|_{X_f} \longrightarrow T_{\mathbf{P}}|_{X_f} \longrightarrow 0.$$

The global generation of $T_{\mathcal{X}}|_{X_f} \otimes \mathcal{L}_0|_{X_f}$ is proved by the followings.

(i) $H^1(X_f, \mathcal{G} \otimes \mathcal{L}_0|_{X_f}) \to H^1(X_f, T_{\mathcal{X}}|_{X_f} \otimes \mathcal{L}_0|_{X_f})$ is injective.
$(H^0(X_f, T_{\mathcal{X}}|_{X_f} \otimes \mathcal{L}_0|_{X_f}) \to H^0(X_f, T_{\mathbf{P}}|_{X_f} \otimes \mathcal{L}_0|_{X_f})$ is surjective.)
(ii) $\mathcal{G} \otimes \mathcal{L}_0|_{X_f}$ is generated by global sections.
(iii) $T_{\mathbf{P}}|_{X_f} \otimes \mathcal{L}_0|_{X_f}$ is generated by global sections.

In the commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}, \mathcal{L}) \otimes H^0(\mathbf{P}, \mathcal{L}_0) & \longrightarrow & H^0(\mathbf{P}, \mathcal{L} \otimes \mathcal{L}_0) \\ \downarrow & & \downarrow \\ T_{\mathcal{M},f} \otimes H^0(X_f, \mathcal{L}_0|_{X_f}) & \longrightarrow & H^0(X_f, N_{X_f/\mathbf{P}} \otimes \mathcal{L}_0|_{X_f}), \end{array}$$

by the assumption $*$-2.2,

$$H^0(\mathbf{P}, \mathcal{L}) \otimes H^0(\mathbf{P}, \mathcal{L}_0) \longrightarrow H^0(\mathbf{P}, \mathcal{L} \otimes \mathcal{L}_0)$$

is surjective, and by $H^1(\mathbf{P}, \mathcal{L}_0) = 0$,

$$H^0(\mathbf{P}, \mathcal{L} \otimes \mathcal{L}_0) \longrightarrow H^0(X_f, N_{X_f/\mathbf{P}} \otimes \mathcal{L}_0|_{X_f})$$

is surjective, hence

$$T_{\mathcal{M},f} \otimes H^0(X_f, \mathcal{L}_0|_{X_f}) \longrightarrow H^0(X_f, N_{X_f/\mathbf{P}} \otimes \mathcal{L}_0|_{X_f})$$

is surjective. So

$$H^1(X_f, \mathcal{G} \otimes \mathcal{L}_0|_{X_f}) \longrightarrow T_{\mathcal{M},f} \otimes H^1(X_f, \mathcal{L}_0|_{X_f})$$

is injective, and this implies the injectivity of (i).

In the commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P},\mathcal{L})\otimes H^0(\mathbf{P},\mathcal{L}_0\otimes\mathcal{I}_{x/\mathbf{P}}) & \longrightarrow & H^0(\mathbf{P},\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{I}_{x/\mathbf{P}}) \\ \downarrow & & \downarrow \\ T_{\mathcal{M},f}\otimes H^0(X_f,\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) & \longrightarrow & H^0(X_f,N_{X_f/\mathbf{P}}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}), \end{array}$$

by the assumption $*$-2.2 and Theorem 2.12,

$$H^0(\mathbf{P},\mathcal{L})\otimes H^0(\mathbf{P},\mathcal{L}_0\otimes\mathcal{I}_{x/\mathbf{P}}) \longrightarrow H^0(\mathbf{P},\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{I}_{x/\mathbf{P}})$$

is surjective, and by $H^1(\mathbf{P},\mathcal{L}_0)=0$,

$$H^0(\mathbf{P},\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{I}_{x/\mathbf{P}}) \longrightarrow H^0(X_f,N_{X_f/\mathbf{P}}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X})$$

is surjective, hence

$$T_{\mathcal{M},f}\otimes H^0(X_f,\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) \longrightarrow H^0(X_f,N_{X_f/\mathbf{P}}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X})$$

is surjective. So

$$H^1(X_f,\mathcal{G}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) \longrightarrow T_{\mathcal{M},f}\otimes H^1(X_f,\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X})$$

is injective. Since $\mathcal{L}$ is generated by global sections,

$$H^1(X_f,\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) \longrightarrow H^1(X_f,\mathcal{L}_0|_{X_f})$$

is injective, and by the commutative diagram

$$\begin{array}{ccc} H^1(X_f,\mathcal{G}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) & \longrightarrow & T_{\mathcal{M},f}\otimes H^1(X_f,\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) \\ \downarrow & & \downarrow \\ H^1(X_f,\mathcal{G}\otimes\mathcal{L}_0|_{X_f}) & \longrightarrow & T_{\mathcal{M},f}\otimes H^1(X_f,\mathcal{L}_0|_{X_f}), \end{array}$$

the map

$$H^1(X_f,\mathcal{G}\otimes\mathcal{L}_0|_{X_f}\otimes\mathcal{I}_{x/X}) \longrightarrow H^1(X_f,\mathcal{G}\otimes\mathcal{L}_0|_{X_f})$$

is injective. This means (ii).

By the Euler sequence [2, Theorem 12.1]

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}^{\oplus s-n} \longrightarrow \bigoplus_{i=1}^{s}\mathcal{O}_{\mathbf{P}}(D_i) \longrightarrow T_{\mathbf{P}} \longrightarrow 0,$$

$T_{\mathbf{P}}\otimes\mathcal{L}_0$ is generated by global sections, because $\mathcal{L}_0(D_i)$ is generated by global sections. (iii) is proved. □

Corollary 3.4. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let $\mathcal{L}$ be an ample invertible sheaf on $\mathbf{P}$, and let $\mathcal{X}\to\mathcal{M}$ be a base change of $\mathcal{X}_{\mathcal{L}}\to\mathcal{M}_{\mathcal{L}}$ by any étale morphism $\mathcal{M}\to\mathcal{M}_{\mathcal{L}}$. Let X_f be a nonsingular fiber of $\mathcal{X}\to\mathcal{M}$. If $[\mathcal{L}]\in -K_{\mathbf{P}}+(n-l-1)B(\mathbf{P})+A^{(j)}(\mathbf{P})$, then there is a $\mathcal{L}_1\in A^{(j)}(\mathbf{P})$ such that $\Omega_{\mathcal{X}}^{m+l}|_{X_f}\otimes\mathcal{L}_1^{-1}|_{X_f}$ is generated by global sections, where m is the dimension of $\mathcal{M}$.*

Proof. There is $\mathcal{L}_{0,i}\in B(\mathbf{P})$ for $1\le i\le n-l-1$ and $\mathcal{L}_1\in A^{(j)}(\mathbf{P})$ such that

$$\mathcal{L}\otimes\Omega_{\mathbf{P}}^n \simeq \mathcal{L}_{0,1}\otimes\cdots\otimes\mathcal{L}_{0,n-l-1}\otimes\mathcal{L}_1.$$

Since $T\mathcal{X}|_{X_f} \otimes \mathcal{L}_{0,i}|_{X_f}$ is generated by global sections,

$$\begin{aligned}\Omega_{\mathcal{X}}^{m+l}|_{X_f} \otimes \mathcal{L}_1^{-1}|_{X_f} &\simeq \Omega_{\mathcal{X}}^{m+n-1}|_{X_f} \otimes \bigwedge^{n-l-1} T\mathcal{X}|_{X_f} \otimes \mathcal{L}_1^{-1}|_{X_f} \\ &\simeq \Omega_{\mathcal{M},f}^{m} \otimes \Omega_{X_f}^{n-1} \otimes \bigwedge^{n-l-1} T\mathcal{X}|_{X_f} \otimes \mathcal{L}_1^{-1}|_{X_f} \\ &\simeq \Omega_{\mathcal{M},f}^{m} \otimes \mathcal{L}_{0,1}|_{X_f} \otimes \cdots \otimes \mathcal{L}_{0,n-l-1}|_{X_f} \otimes \bigwedge^{n-l-1} T\mathcal{X}|_{X_f}\end{aligned}$$

is generated by global sections. $\square$

Assume the base field k is of characteristic 0. We denote by $X_\xi \to \mathrm{Spec}(\overline{k(\mathcal{M}_\mathcal{L})})$ the geometric generic fiber of $\mathcal{X}_\mathcal{L} \to \mathcal{M}_\mathcal{L}$.

Theorem 3.5. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let C be a nonsingular projective curve over $\overline{k(\mathcal{M}_\mathcal{L})}$, and $\iota : C \to X_\xi$ be a $\overline{k(\mathcal{M}_\mathcal{L})}$-morphism of degree e. If $[\mathcal{L}] \in -K_\mathbf{P} + (n-2)B(\mathbf{P}) + A^{(l)}(\mathbf{P})$, then the genus satisfies $g(C) \geq \frac{1}{2} \cdot e \cdot l + 1$.*

Proof. There is an étale $\mathcal{M} \to \mathcal{M}_\mathcal{L}$, a proper smooth $\mathcal{C} \to \mathcal{M}$ and a $\mathcal{M}$-morphism $\mathcal{C} \to \mathcal{X}$ such that $\iota : C \to X_\xi$ is identified with the restriction of $\mathcal{C} \to \mathcal{X}$ to the fiber of geometric generic point of $\mathcal{M}$. By Corollary 3.4, there is a $\mathcal{L}_1 \in A^{(l)}(\mathbf{P})$ such that $\Omega_{\mathcal{X}}^{m+1}|_{X_\xi} \otimes \mathcal{L}_1^{-1}|_{X_\xi}$ is generated by global sections. Since $\iota^*(\Omega_{\mathcal{X}}^{m+1}|_{X_\xi}) \to \Omega_{\mathcal{C}}^{m+1}|_C$ is surjective, $\Omega_{\mathcal{C}}^{m+1}|_C \otimes \iota^*(\mathcal{L}_1^{-1}|_{X_\xi})$ is generated by global sections, hence $\Omega_C^1 \otimes \iota^*(\mathcal{L}_1^{-1}|_{X_\xi})$ has a positive degree;

$$2g(C) - 2 = \deg(\Omega_C^1) \geq \deg(\iota^*(\mathcal{L}_1|_{X_\xi})) = e \cdot (\mathcal{L}_1.\iota(C)) \geq e \cdot l.$$

$\square$

Corollary 3.6. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let C be a nonsingular projective curve over $\overline{k(\mathcal{M}_\mathcal{L})}$, and $C \to X_\xi$ be a $\overline{k(\mathcal{M}_\mathcal{L})}$-morphism which is birational to the image. If $[\mathcal{L}] \in -K_\mathbf{P} + (n-2)B(\mathbf{P}) + A^{(2g-1)}(\mathbf{P})$, then the genus satisfies $g(C) > g$.*

Theorem 3.7. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let Y be a nonsingular complete variety of dimension $l \geq 1$ over $\overline{k(\mathcal{M}_\mathcal{L})}$, and $Y \to X_\xi$ be a $\overline{k(\mathcal{M}_\mathcal{L})}$-morphism which is birational to the image.*

1. *If $[\mathcal{L}] \in -K_\mathbf{P} + (n-l-1)B(\mathbf{P})$, then the geometric genus $p_g(Y)$ is positive.*
2. *If $[\mathcal{L}] \in -K_\mathbf{P} + (n-l-1)B(\mathbf{P}) + \mathrm{Ample}(\mathbf{P})$, then Y is of general type.*

Proof. There is an étale $\mathcal{M} \to \mathcal{M}_\mathcal{L}$, a proper smooth $\mathcal{Y} \to \mathcal{M}$ and a $\mathcal{M}$-morphism $\mathcal{Y} \to \mathcal{X}$ such that $Y \to X_\xi$ is identified with the restriction of $\mathcal{Y} \to \mathcal{X}$ to the fiber of geometric generic point of $\mathcal{M}$. Let E be the exceptional set of $Y \to X_\xi$. Then

$$\Omega_{\mathcal{X}}^{m+l}|_{Y \setminus E} \longrightarrow \Omega_{\mathcal{Y}}^{m+l}|_{Y \setminus E}$$

is surjective. If we assume (1), then by Corollary 3.4,

$$H^0(X_\xi, \Omega_{\mathcal{X}}^{m+l}|_{X_\xi}) \otimes \mathcal{O}_{X_\xi} \longrightarrow \Omega_{\mathcal{X}}^{m+l}|_{X_\xi}$$

is surjective. By the commutative diagram

$$\begin{array}{ccc} H^0(X_\xi, \Omega_{\mathcal{X}}^{m+l}|_{X_\xi}) \otimes \mathcal{O}_{Y \setminus E} & \longrightarrow & \Omega_{\mathcal{X}}^{m+l}|_{Y \setminus E} \\ \downarrow & & \downarrow \\ H^0(Y, \Omega_{\mathcal{Y}}^{m+l}|_Y) \otimes \mathcal{O}_{Y \setminus E} & \longrightarrow & \Omega_{\mathcal{Y}}^{m+l}|_{Y \setminus E}, \end{array}$$

$H^0(Y, \Omega^{m+l}_{\mathcal{Y}}|_Y) \simeq \Omega^m_{\mathcal{M},\xi} \otimes H^0(Y, \Omega^l_Y)$ is not vanish.

If we assume (2), then there is an ample $\mathcal{L}_1$ on $\mathbf{P}$ such that

$$H^0(X_\xi, \Omega^{m+l}_{\mathcal{X}}|_{X_\xi} \otimes \mathcal{L}_1^{-1}|_{X_\xi}) \otimes \mathcal{O}_{X_\xi} \longrightarrow \Omega^{m+l}_{\mathcal{X}}|_{X_\xi} \otimes \mathcal{L}_1^{-1}|_{X_\xi}$$

is surjective. By the same argument using the commutative diagram

$$\begin{array}{ccc} H^0(X_\xi, \Omega^{m+l}_{\mathcal{X}}|_{X_\xi} \otimes \mathcal{L}_1^{-1}|_{X_\xi}) \otimes \mathcal{O}_{Y\setminus E} & \longrightarrow & \Omega^{m+l}_{\mathcal{X}}|_{Y\setminus E} \otimes \mathcal{L}_1^{-1}|_{Y\setminus E} \\ \downarrow & & \downarrow \\ H^0(Y, \Omega^{m+l}_{\mathcal{Y}}|_Y \otimes \mathcal{L}_1^{-1}|_Y) \otimes \mathcal{O}_{Y\setminus E} & \longrightarrow & \Omega^{m+l}_{\mathcal{Y}}|_{Y\setminus E} \otimes \mathcal{L}_1^{-1}|_{Y\setminus E}, \end{array}$$

$\Omega^l_Y|_{Y\setminus E} \otimes \mathcal{L}_1^{-1}|_{Y\setminus E}$ is generated by global sections. Since $\mathcal{L}_1|_{Y\setminus E}$ separates two points in $Y \setminus E$, the image of the canonical map of Y is l-dimensional. □

Theorem 3.8. *Assume the condition $*$-2.2 for $\mathbf{P}$. When $[\mathcal{L}] \in -K_{\mathbf{P}} + (n-1)B(\mathbf{P}) + A^{(2r-1)}(\mathbf{P})$, for any $2r$ distinct closed points $p_1, \cdots, p_r, q_1, \cdots, q_r \in X_\xi(\overline{k(\mathcal{M}_\mathcal{L})})$, $[p_1 + \cdots + p_r]$ and $[q_1 + \cdots + q_r]$ are not rationally equivalent as zero cycles on X_ξ.*

Proof. There is an étale $\mathcal{M} \to \mathcal{M}_\mathcal{L}$, and sections $\mathcal{M} \xrightarrow{\mathcal{P}_1} \mathcal{X}, \cdots, \mathcal{M} \xrightarrow{\mathcal{Q}_r} \mathcal{X}$ of $\mathcal{X} \to \mathcal{M}$ such that $\mathrm{Spec}(\overline{k(\mathcal{M}_\mathcal{L})}) \xrightarrow{p_1} X_\xi, \cdots, \mathrm{Spec}(\overline{k(\mathcal{M}_\mathcal{L})}) \xrightarrow{q_r} X_\xi$ is identified with the restriction of $\mathcal{M} \xrightarrow{\mathcal{P}_1} \mathcal{X}, \cdots, \mathcal{M} \xrightarrow{\mathcal{Q}_r} \mathcal{X}$ to the fiber of geometric generic point of $\mathcal{M}$. By Corollary 3.4, there is a $\mathcal{L}_1 \in A^{(2r-1)}(\mathbf{P})$ such that $\Omega^{m+1}_{\mathcal{X}}|_{X_\xi} \otimes \mathcal{L}_1^{-1}|_{X_\xi}$ is generated by global sections, and by [5, Theorem 4.2], $\mathcal{L}_1$ is separates $2r$ points. So

$$H^0(X_\xi, \Omega^m_{\mathcal{X}}|_{X_\xi}) \longrightarrow \Omega^m_{\mathcal{X}}|_{p_1} \oplus \cdots \oplus \Omega^m_{\mathcal{X}}|_{q_r}$$

is surjective. Using the duality, the cycle class in $H^{n-1}(X_\xi, \Omega^{n-1}_{\mathcal{X}}|_{X_\xi})$ of a section $\mathcal{M} \xrightarrow{\mathcal{P}} \mathcal{X}$ is the composition $\psi_\mathcal{P}$:

$$\bigwedge^m T_{\mathcal{M},\xi} \otimes H^0(X_\xi, \Omega^m_{\mathcal{X}}|_{X_\xi}) \longrightarrow \bigwedge^m T_{\mathcal{M},\xi} \otimes \Omega^m_{\mathcal{X}}|_p \longrightarrow \bigwedge^m T_{\mathcal{M},\xi} \otimes \Omega^m_{\mathcal{P}}|_p \simeq \overline{k(\mathcal{M}_\mathcal{L})}.$$

The cycle class of

$$[\mathcal{P}_1 + \cdots + \mathcal{P}_r - \mathcal{Q}_1 - \cdots - \mathcal{Q}_r] \in \mathrm{CH}^{n-1}(\mathcal{X})$$

in $H^{n-1}(X_\xi, \Omega^{n-1}_{\mathcal{X}}|_{X_\xi})$ is $\psi = \psi_{\mathcal{P}_1} + \cdots + \psi_{\mathcal{P}_r} - \psi_{\mathcal{Q}_1} - \cdots - \psi_{\mathcal{Q}_r}$. By the surjectivity of

$$H^0(X_\xi, \Omega^m_{\mathcal{X}}|_{X_\xi}) \longrightarrow \Omega^m_{\mathcal{P}_1}|_{p_1} \oplus \cdots \oplus \Omega^m_{\mathcal{Q}_r}|_{q_r},$$

there is a section $\omega \in \bigwedge^m T_{\mathcal{M},\xi} \otimes H^0(X_\xi, \Omega^m_{\mathcal{X}}|_{X_\xi})$ such that $\psi(\omega) \neq 0$. If $[p_1 + \cdots + p_r]$ and $[q_1 + \cdots + q_r]$ are rationally equivalent, then we can make $\mathcal{P}_1, \cdots, \mathcal{Q}_r$ that $[\mathcal{P}_1 + \cdots + \mathcal{P}_r]$ and $[\mathcal{Q}_1 + \cdots + \mathcal{Q}_r]$ are rationally equivalent. In this case ψ must be zero. □

Remark 3.9. Theorem 3.8 is a generalization of the result of C. Voisin [15, Theorem 1.11], and gives an explicit condition, in the toric case, to the result of N. Fakhruddin [9, Theorem 1]. As in [9, Corollary 2], we have the next corollary.

Corollary 3.10. *Assume the condition $*$-2.2 for $\mathbf{P}$. Let C be a nonsingular projective curve over $\overline{k(\mathcal{M}_\mathcal{L})}$, and $C \to X_\xi$ be a $\overline{k(\mathcal{M}_\mathcal{L})}$-morphism which is birational to the image. If $[\mathcal{L}] \in -K_{\mathbf{P}} + (n-1)B(\mathbf{P}) + A^{(2r-1)}(\mathbf{P})$, then the gonality satisfies $\mathrm{gon}(C) > r$.*

Proof. There are distinct $2 \cdot \mathrm{gon}(C)$ points $p_1, \cdots, p_{\mathrm{gon}(C)}, q_1, \cdots, q_{\mathrm{gon}(C)}$ on C, such that $[p_1 + \cdots + p_{\mathrm{gon}(C)}]$ and $[q_1 + \cdots + q_{\mathrm{gon}(C)}]$ are linearly equivalent on C, hence these are rationally equivalent on X_ξ. By Theorem 3.8, $\mathrm{gon}(C)$ must be larger than r. □

References

[1] V. Batyrev, *On the classification of smooth projective toric varieties,* Tohoku Math. J. **43** (1991), 569-585.

[2] V. Batyrev and D. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties,* Duke Math. J. **75** (1994), 293-338.

[3] H. Clemens, *Curves on generic hypersurfaces,* Ann. Sci. École Norm. Sup. **19** (1986), 626-636.

[4] D. Cox, *The homogeneous coordinate ring of a toric variety,* J. Algebraic Geom. **4** (1995), 17-50.

[5] S. Di Rocco, *Generation of k-jets on toric varieties,* Math. Z. **231** (1999), 169-188.

[6] L. Ein, *Subvarieties of generic complete intersections,* Invent. Math. **94** (1988), 163-169.

[7] L. Ein, *Subvarieties of generic complete intersections. II,* Math. Ann. **289** (1991), 465-471.

[8] N. Fakhruddin, *Multiplication maps of linear systems on smooth projective toric surfaces,* math. AG /0208178.

[9] N. Fakhruddin, *Zero cycles on generic hypersurfaces of large degree,* math. AG/ 0208179.

[10] P. Kleinschmidt, *A classification of toric varieties few generators,* Aequationes Math. **35** (1988), 254-266.

[11] A. Mavlyutov, *Semiample hypersurfaces in toric varieties,* Duke Math. J. **101** (2000), 85-116.

[12] M. Mustaţă, *Vanishing theorems on toric varieties,* Tohoku Math. J. **54** (2002), 451-470.

[13] T. Oda, *Convex Bodies and Algebraic Geometry — An Introduction to the Theory of Toric Varieties,* Ergebnisse der Math. (3) **15**, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.

[14] T. Oda, *Problems on Minkowski sums of convex lattice polytopes,* preprint (1997).

[15] C. Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces,* J. Differential Geom. **44** (1996), 200-214.

[16] C. Voisin, *A correction on "A conjecture of Clemens on rational curves on hypersurfaces",* J. Differential Geom. **49** (1998), 601-611.

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