<table>
<thead>
<tr>
<th>Title</th>
<th>Hodge Theory and Algebraic Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Matsushita, D.</td>
</tr>
<tr>
<td>Citation</td>
<td>Hokkaido University technical report series in mathematics, 75, 1</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2003-01-01</td>
</tr>
<tr>
<td>DOI</td>
<td>10.14943/633</td>
</tr>
<tr>
<td>Doc URL</td>
<td><a href="http://hdl.handle.net/2115/691">http://hdl.handle.net/2115/691</a>; <a href="http://eprints3.math.sci.hokudai.ac.jp/0278/">http://eprints3.math.sci.hokudai.ac.jp/0278/</a></td>
</tr>
<tr>
<td>Type</td>
<td>bulletin (article)</td>
</tr>
</tbody>
</table>

### Note
- Hodge Theory and Algebraic Geometry 2002/10/7-11 Department of Mathematics, Hokkaido University
- 石井志保子
- 内藤広嗣 (名大多元) 村上雅亮(京大理)
- 大野浩二(大阪大) On certain boundedness of fibred Calabi-Yau s threefolds
- 阿部健(京大理) 春井岳(大阪大) The gonarlity of curves on an elliptic ruled surface
- 山下剛(東大数理) 開多様体のp進etale cohomology と crystalline cohomology
- 中島幸喜(東京電機大) Theorie de Hodge III pour cohomologies p-adiques
- 皆川龍博 (東工大) On classification of weakened Fano 3-folds
- 斉藤夏男(東大数理) Fano threefold in positive characteristic
- 石井亮(京大工) Variation of the representation moduli of the McKay quiver
- 前野俊昭(京大理) 群のコホモロジーと量子変形
- 宮岡洋一(東大数理) 次数が低い有理曲線とファノ多様体
- 池田京司(大阪大) Subvarieties of generic hypersurfaces in a projective toric variety
- 竹田雄一郎(九大数理) Complexes of hermitian cubes and the Zagier conjecture
- 鈴木香織(東大数理) \( \rho(X) = 1, f \leq 2 \) のQ-Fano 3-fold Fanoの分類

---

### Additional Information
There are other files related to this item in HUSCAP. Check the above URL.

### File Information
- maeno.pdf
Quantum deformation and cohomology of groups

Toshiaki Maeno
Department of Mathematics
Kyoto University

Introduction

The notion of strongly homotopy associative algebra (or $A_\infty$-algebra) was considered by Stasheff [8] for the first time. The structure of $A_\infty$-algebra is given by multi-linear operations which satisfy the associativity conditions up to homotopy. In [4], Kontsevich proposed the homological mirror symmetry, which involves the notion of $A_\infty$-deformations. The extended moduli space introduced by Baramnikov and Kontsevich ([1], [2]) parametrizes the $A_\infty$-deformations of the derived category of coherent sheaves on a complex manifold. Here, we treat the $A_\infty$-deformations of the derived category of representations of groups. When a group $G$ acts on a complex manifold $X$, we can relate the $A_\infty$-deformations of the category of $G$-module with the extended moduli space of the derived category of coherent sheaves on a complex manifold. Here, we treat the $A_\infty$-deformations of the derived category of the representations of groups. On the other hand, group algebras are regarded as algebras of functions on noncommutative spaces. Hence, our object of consideration gives an example of the deformation theory in noncommutative geometry. This note contains some topics which was not mentioned in the talk at the workshop “Hodge Theory and Algebraic Geometry” at Hokkaido University. For details, see [6].

1 $A_\infty$-algebras and $A_\infty$-categories

In this section, we will work over a field $K$ of characteristic zero.

**Definition 1.1** Let $V$ be a $\mathbb{Z}$-graded $K$-vector space and $m_k : V^\otimes k \to V$ ($k \geq 1$) a family of $K$-linear maps of degree $2 - k$. Then, $(V, \{m_k\}_k)$ is called an $A_\infty$-algebra if the condition

\[
\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^j \sum_{i=1}^j [a_i] + j(l-1) + (k-1)\right \} m_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n) = 0
\]

is satisfied for $k, l \geq 1$.

**Remark 1.1** Let $W = V^*[1]$ The condition (*) can be interpreted in terms of a derivation on the tensor algebra

\[ T(W) = \bigoplus_{n=0}^{\infty} W^\otimes n. \]
A super-derivation on $T(W)$ is by definition a $K$-linear map $\hat{d} : T(W) \to T(W)$ satisfying

$$\hat{d}(a \otimes b) = \hat{d}(a) \otimes b + (-1)^{|a|}|\hat{d}|a \otimes \hat{d}(b).$$

So $\hat{d}$ is uniquely determined by its restriction $d = \hat{d}|_W$. Moreover, we have decompositions

$$d = \sum_k d_k, \quad d_k : W \to W^\otimes k,$$

and

$$\hat{d} = \sum_{k,l} d_{k,l}, \quad d_{k,l} : W^\otimes l \to W^\otimes k.$$  

Then, $\hat{d}^2 = 0$ is equivalent to $\sum_{k+l=n} d_{k,l}d_k = 0$ for all $n > 1$. This condition is nothing but the dual to $(\ast)$.

**Definition 1.2** An $A_\infty$-module structure on a graded vector space $M$ is an integrable connection

$$D : M^* \otimes T(W) \to M^* \otimes T(W)$$

of degree one with the integrability condition $[D, D] = 0$

**Remark 1.2** Since the $A_\infty$-algebra (resp. $A_\infty$-module) structure is given by a derivation $\hat{d}$ (resp. a connection $D$), its deformations are given by deformations of the corresponding derivation $\hat{d}$ (resp. $D$) keeping the integrability condition.

**Definition 1.3** An additive category $C$ over $K$ is called an $A_\infty$-category if

$$m_k(E_0, \ldots, E_k) : \operatorname{Hom}(E_0, E_1) \otimes \cdots \otimes \operatorname{Hom}(E_{k-1}, E_k) \to \operatorname{Hom}(E_0, E_k)$$

($\forall E_0, \ldots, E_k \in \operatorname{Ob}(C)$ and $k \geq 0$) of degree $2 - k$ subject to the condition $(\ast)$ in Definition 1.1 is defined.

2. $A_\infty$-deformations of the category of representations of groups

Let $G$ be a group. Denote by $\operatorname{Mod}(G)$ the category of $G$-bimodules over $C$ and $D(G) = D^\beta\operatorname{Mod}(G)$ its bounded derived category. For a $G$-bimodule $M$, we define the space of cochains

$$C^k(G, M) = \{\mu : C\langle G \rangle^\otimes k \to M, \text{ linear}\},$$

and the coboundary operators

$$d : C^k(G, M) \to C^{k+1}(G, M)$$

by

$$(d\mu)(g_0, \ldots, g_k) = g_0\mu(g_1, \ldots, g_k) + \sum_{i=1}^k (-1)^i\mu(g_0, \ldots, g_{i-1}g_i, \ldots, g_k) + (-1)^{k+1}\mu(g_0, \ldots, g_{k-1}g_k).$$
If $M = C\langle G \rangle$, we have a canonical bracket on $C^* = C^*\{G, C\langle G \rangle\}$ defined by

$$[\mu, \nu] = \mu \circ \nu - (-1)^{|\mu|-1}|\nu|-1) \nu \circ \mu,$$

where

$$(\mu \circ \nu)(g_1, \ldots, g_{|\nu|+|\nu|-1}) = \sum_{i=0}^{|\mu|-1} \mu(g_1, \ldots, \nu(g_{i+1}, \ldots, g_{i+|\nu|}), \ldots, g_{|\mu|+|\nu|-1}).$$

This bracket induces a structure of the Gerstenhaber algebra on the cohomology group $H^*(G, C\langle G \rangle)$ ([3]). The Hochschild complex $C^*$ with the bracket $[\ , \ ]$ has a structure of $L_\infty$-algebra ([9]).

Now we consider the Maurer-Cartan equation for the Hochschild complex $C^*$. We introduce a set of formal parameters $t = (t_g)_{g \in G}$. Let $R = C[[t]]$ be the ring of formal power series in $t$. For an element $\gamma(t) \in C^* \otimes R$, the equation

$$d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$$

is called the Maurer-Cartan equation. Let $\gamma(t)$ be a formal solution of the Maurer-Cartan equation. We define $(Q_{\gamma(t)}(\gamma'(t)) := d\gamma'(t) + [\gamma'(t), \gamma(t)])$ for $\gamma'(t) \in C^* \otimes R$. If $\epsilon$ is a parameter such that $\epsilon^2 = 0$ and $\epsilon Q_{\gamma(t)}(\gamma'(t))$ is odd, then $\gamma(t) + \epsilon Q_{\gamma(t)}(\gamma'(t))$ also satisfies the Maurer-Cartan equation. The transformation $\gamma(t) \mapsto \gamma(t) + \epsilon Q_{\gamma(t)}(\gamma'(t))$ is an analogue of the infinitesimal gauge transformation. We denote by $M_G$ the formal moduli space of the solutions of the Maurer-Cartan equation, i.e.

$$M_G = \{ \gamma(t) \in C^* \otimes R \mid d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0 \}/(\text{gauge equivalence}).$$

From the argument in [1], [2], [5], and Tamarkin and Tsygan’s formality theorem [9], [10], we can show the following.

**Theorem 2.1** (1) The moduli space $M_G$ is the formal $A_\infty$-deformation space of the category $D(G)$.

(2) The tangent space to $M_G$ at $t = 0$ is isomorphic to $H^*(G, C\langle G \rangle)$.

(3) The Kodaira-Spencer map $M_G \rightarrow H^*(G, C\langle G \rangle)$ is surjective.

**Corollary** If $G$ is finite, then there exists no nontrivial $A_\infty$-deformation of $D(G)$ in characteristic zero.

Let us consider the situation where $X$ is a contractible complex manifold, and a group $G$ acts on $X$ holomorphically, freely and discontinuously. Let $\pi : X \rightarrow Y := X/G$ be the natural projection. We assume that $Y$ is compact. Denote by $M(Y)$ the extended moduli space of $Y$, which parametrizes formal $A_\infty$-deformations of the derived category of the coherent sheaves on $Y$. As for the relationship between $M_G$ and $M(Y)$, we have the following.

**Theorem 2.2** Each section $s \in H^*(X, \pi^*(\wedge^n T_Y))$ gives a natural morphism

$$\alpha_s : M_G \rightarrow M(Y).$$

The morphism $\alpha_s$ is induced by the morphism

$$C^*(G, C\langle G \rangle) \rightarrow C^*(G, C\langle G \rangle) \otimes H^*(X, \pi^*(\wedge^n T_Y)) \rightarrow C^*(G, H^*(X, \pi^*(\wedge^n T_Y)))$$

$$\mu \mapsto \mu \otimes s \quad \mu \cdot s.$$
3 Example

Take a complex torus $T = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice. Fix a coordinate $(z_1, \ldots, z_g)$ on $\mathbb{C}^g$. In this case, we have a standard morphism $\alpha_0 : \mathcal{M}_\Lambda \to \mathcal{M}(T)$ such that the induced map between the tangent spaces at the origin coincides with the map

$$H^*(\Lambda, \mathcal{C}(\Lambda)) \to H^*(\Lambda, \mathcal{C}(\Lambda)) \otimes \Lambda \mathbb{C} \to H^*(T, \mathbb{C})$$

$$\mu \mapsto \mu \otimes 1.$$  

The embedding of the lattice $\Lambda \to \mathbb{C}^g$ defines a linear map $j : \mathcal{C}(\Lambda) \to H^0(\mathbb{C}^g, \mathcal{O}_\mathbb{C}^{\otimes g})$. Differentiation of $j$ by $\partial / \partial z_i$ ($i = 1, \ldots, g$) gives a homomorphism

$$H^1(\Lambda, \mathcal{C}(\Lambda)) \to H^1(T, \mathcal{O}_T^{\otimes g}).$$

Here, $H^1(T, \mathcal{O}_T^{\otimes g})$ is nothing but the tangent space of the classical deformation of $T$. The vector fields $\partial / \partial z_i$ determines the choice of $s$ in Theorem 2.2. On the other hand, the isomorphism

$$H^2(\Lambda, \mathcal{C}) \cong \wedge^2 \text{Hom}(\Lambda, \mathbb{C})$$

tells us that $H^2(\Lambda, \mathcal{C})$ is the tangent space to the locus of noncommutative deformations of $T$.

The Fourier transformation gives us another viewpoint to see the above situation. Denote by $\Lambda^\vee$ the space of irreducible representations of $\Lambda$. Consider the algebra homomorphism

$$\varphi : \mathcal{C}(\Lambda) \to C(\Lambda^\vee)$$

defined by the Fourier transformation. Then, $C(\Lambda^\vee)$ can be regarded as the space of functions on $\hat{T} = \text{Pic}^0(\mathcal{T})$, and it is easy to see that $\varphi$ induces a morphism $\phi : \mathcal{M}_\Lambda \to \mathcal{M}(\hat{T})$. We also have a morphism $\iota : \mathcal{M}(\hat{T}) \to \mathcal{M}(T)$ defined by the isomorphism $\hat{T} \cong T$. Then, it can be checked that $\iota \circ \phi = \alpha_0$. We can generalize this observation to the cases $G$ is noncommutative.

**Proposition 3.1** Under the assumption of Theorem 2.2, the Fourier transformation $\mathcal{C}(G) \to C(G^\vee)$ gives a morphism $\phi_{G,X} : \mathcal{M}_G \to \mathcal{M}(\text{Pic}^0(Y))$.

Let $\mathbf{H}$ be the upper half plane. We assume that a Fuchsian group $\Gamma$ of the first kind acts on $\mathbf{H}$ freely without cusps. By applying Proposition 3.1 to $Y = \mathbf{H}/\Gamma$, we obtain a morphism $\phi_{\Gamma, \mathbf{H}} : \mathcal{M}_\Gamma \to \mathcal{M}(\text{Pic}^0(Y))$. In this case, we can see that $\phi_{\Gamma, \mathbf{H}}$ is surjective. This result shows that the higher products on the spaces of automorphic forms induced by the $A_\infty$-deformations of $Y$ arises from the $A_\infty$-deformations of $\Gamma$-modules.

In general, the action of the Fuchsian group on $\mathbf{H}$ has fixed points and cusps, so the quotient space $Y$ is an orbifold. At present, the notion of $A_\infty$-deformations of the orbifolds is not clear. However, even in such a case, it is possible to define higher products group-theoretically on the space of automorphic forms.

**Remark 3.1** Polishchuk computed the higher products for the elliptic curves in [7].
References


