Quantum deformation and cohomology of groups

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Introduction

The notion of strongly homotopy associative algebra (or A_∞-algebra) was considered by Stasheff [8] for the first time. The structure of A_∞-algebra is given by multi-linear operations which satisfy the associativity conditions up to homotopy. In [4], Kontsevich proposed the homological mirror symmetry, which involves the notion of A_∞-deformations. The extended moduli space introduced by Barannikov and Kontsevich ([1], [2]) parametrizes the A_∞-deformations of the derived category of coherent sheaves on a complex manifold. Here, we treat the A_∞-deformations of the derived category of the representations of groups. When a group G acts on a complex manifold X, we can relate the A_∞-deformations of the category of G-module with the extended moduli space of the quotient space X/G under some assumptions on X and the action of G. On the other hand, group algebras are regarded as algebras of functions on *noncommutative spaces*. Hence, our object of consideration gives an example of the deformation theory in noncommutative geometry. This note contains some topics which was not mentioned in the talk at the workshop "Hodge Theory and Algebraic Geometry" at Hokkaido University. For details, see [6].

1 A_∞-algebras and A_∞-categories

In this section, we will work over a field K of characteristic zero.

Definition 1.1 *Let V be a $\mathbf{Z}$-graded K-vector space and $m_k : V^{\otimes k} \to V$ $(k \geq 1)$ a family of K-linear maps of degree $2-k$. Then, $(V, \{m_k\}_k)$ is called an A_∞-algebra if the condition*

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{l \sum_{i=1}^{j} |a_i| + j(l-1) + (k-1)l} m_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n) = 0$$

is satisfied for $k, l \geq 1$.

Remark 1.1 Let $W = V^*[1]$ The condition $(*)$ can be interpreted in terms of a derivation on the tensor algebra

$$T(W) = \bigoplus_{n=0}^{\infty} W^{\otimes n}.$$

A super-derivation on $T(W)$ is by definition a K-linear map $\hat{d} : T(W) \to T(W)$ satisfying

$$\hat{d}(a \otimes b) = \hat{d}(a) \otimes b + (-1)^{|a||\hat{d}|} a \otimes \hat{d}(b).$$

So $\hat{d}$ is uniquely determined by its restriction $d = \hat{d}|_W$. Moreover, we have decompositions

$$d = \sum_k d_k, \quad d_k : W \to W^{\otimes k},$$

and

$$\hat{d} = \sum_{k,l} d_{k,l}, \quad d_{k,l} : W^{\otimes l} \to W^{\otimes k}.$$

Then, $\hat{d}^2 = 0$ is equivalent to $\sum_{k+l=n} d_{k,l} d_k = 0$ for all $n > 1$. This condition is nothing but the dual to $(*)$.

Definition 1.2 *An A_∞-module structure on a graded vector space M is an integrable connection*

$$D : M^* \otimes T(W) \to M^* \otimes T(W)$$

of degree one with the integrability condition $[D, D] = 0$

Remark 1.2 Since the A_∞-algebra (resp. A_∞-module) structure is given by a derivation $\hat{d}$ (resp. a connection D), its deformations are given by deformations of the corresponding derivation $\hat{d}$ (resp. D) keeping the integrability condition.

Definition 1.3 *An additive category $\mathcal{C}$ over K is called an A_∞-category if* $\mathrm{Hom}(E, E')$ $(\forall E, E' \in Ob(\mathcal{C}))$ *is a* $\mathbf{Z}$*-graded vector space, and a family of linear maps*

$$m_k(E_0, \ldots, E_k) : \mathrm{Hom}(E_0, E_1) \otimes \cdots \otimes \mathrm{Hom}(E_{k-1}, E_k) \to \mathrm{Hom}(E_0, E_k)$$

$(\forall E_0, \ldots, E_k \in Ob(\mathcal{C})$ and $k \geq 0)$ *of degree* $2 - k$ *subject to the condition* $(*)$ *in Definition 1.1 is defined.*

2 A_∞-deformations of the category of representations of groups

Let G be a group. Denote by $\mathrm{Mod}(G)$ the category of G-bimodules over $\mathbf{C}$ and $D(G) = D^b\mathrm{Mod}(G)$ its bounded derived category. For a G-bimodule M, we define the space of cochains

$$C^k(G, M) = \{\mu : \mathbf{C}\langle G\rangle^{\otimes k} \to M, \text{ linear}\},$$

and the coboundary operators

$$d : C^k(G, M) \to C^{k+1}(G, M)$$

by

$$(d\mu)(g_0, \ldots, g_k) = g_0\mu(g_1, \ldots, g_k) + \sum_{i=1}^{k} (-1)^i \mu(g_0, \ldots, g_{i-1}g_i, \ldots, g_k) + (-1)^{k+1} \mu(g_0, \ldots, g_{k-1}) g_k.$$

If $M = \mathbf{C}\langle G\rangle$, we have a canonical bracket on $C^* = C^*(G, \mathbf{C}\langle G\rangle)$ defined by

$$[\mu, \nu] = \mu \circ \nu - (-1)^{(|\mu|-1)(|\nu|-1)} \nu \circ \mu,$$

where

$$(\mu \circ \nu)(g_1, \ldots, g_{|\mu|+|\nu|-1}) = \sum_{i=0}^{|\mu|-1} \mu(g_1, \ldots, \nu(g_{i+1}, \ldots, g_{i+|\nu|}), \ldots, g_{|\mu|+|\nu|-1}).$$

This bracket induces a structure of the Gerstenhaber algebra on the cohomology group $H^*(G, \mathbf{C}\langle G\rangle)$ ([3]). The Hochschild complex C^* with the bracket [,] has a structure of L_∞-algebra ([9]).

Now we consider the Maurer-Cartan equation for the Hochschild complex C^*. We introduce a set of formal parameters $t = (t_g)_{g \in G}$. Let $R = \mathbf{C}[[t]]$ be the ring of formal power series in t. For an element $\gamma(t) \in C^* \otimes R$, the equation

$$d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0$$

is called the Maurer-Cartan equation. Let $\gamma(t)$ be a formal solution of the Maurer-Cartan equation. We define $(Q_{\gamma(t)}(\gamma'(t)) := d\gamma'(t) + [\gamma'(t), \gamma(t)])$ for$\gamma'(t) \in C^* \otimes R$. If ϵ is a parameter such that $\epsilon^2 = 0$ and $\epsilon Q_{\gamma(t)}(\gamma'(t))$ is odd, then $\gamma(t) + \epsilon Q_{\gamma(t)}(\gamma'(t))$ also satisfies the Maurer-Cartan equation. The transformation $\gamma(t) \mapsto \gamma(t) + \epsilon Q_{\gamma(t)}(\gamma'(t))$ is an analogue of the infinitesimal gauge transformation. We denote by $\mathcal{M}_G$ the formal moduli space of the solutions of the Maurer-Cartan equation, i.e.

$$\mathcal{M}_G = \{\gamma(t) \in C^* \otimes R \mid d\gamma(t) + \frac{1}{2}[\gamma(t), \gamma(t)] = 0\}/(\text{gauge equivalence}).$$

From the argument in [1], [2], [5], and Tamarkin and Tsygan's formality theorem [9], [10], we can show the following.

Theorem 2.1 (1) *The moduli space $\mathcal{M}_G$ is the formal A_∞-deformation space of the category $D(G)$.*
(2) *The tangent space to $\mathcal{M}_G$ at $t = 0$ is isomorphic to $H^*(G, \mathbf{C}\langle G\rangle)$.*
(3) *The Kodaira-Spencer map $\mathcal{M}_G \to H^*(G, \mathbf{C}\langle G\rangle)$ is surjective.*

Corollary *If G is finite, then there exists no nontrivial A_∞-deformation of $D(G)$ in characteristic zero.*

Let us consider the situation where X is a contractible complex manifold, and a group G acts on X holomorphically, freely and discontinuously. Let $\pi : X \to Y := X/G$ be the natural projection. We assume that Y is compact. Denote by $\mathcal{M}(Y)$ the extended moduli space of Y, which parametrizes formal A_∞-deformations of the derived category of the coherent sheaves on Y. As for the relationship between $\mathcal{M}_G$ and $\mathcal{M}(Y)$, we have the following.

Theorem 2.2 *Each section $s \in H^*(X, \pi^*(\wedge^* T_Y))$ gives a natural morphism*

$$\alpha_s : \mathcal{M}_G \to \mathcal{M}(Y).$$

The morphism α_s is induced by the morphism

$$\begin{array}{ccccc} C^*(G, \mathbf{C}\langle G\rangle) & \to & C^*(G, \mathbf{C}\langle G\rangle) \otimes H^*(X, \pi^*(\wedge^* T_Y)) & \to & C^*(G, H^*(X, \pi^*(\wedge^* T_Y))) \\ \mu & \mapsto & \mu \otimes s & \mapsto & \mu \cdot s. \end{array}$$

3 Example

Take a complex torus $T = \mathbf{C}^g/\Lambda$, where $\Lambda \subset \mathbf{C}^g$ is a lattice. Fix a coodinate $(z_1, \ldots, z_g)$ on $\mathbf{C}^g$. In this case, we have a standard morphism $\alpha_0 : \mathcal{M}_\Lambda \to \mathcal{M}(T)$ such that the induced map between the tangent spaces at the origin coincides with the map

$$\begin{array}{ccccc} H^*(\Lambda, \mathbf{C}\langle\Lambda\rangle) & \to & H^*(\Lambda, \mathbf{C}\langle\Lambda\rangle) \otimes_\Lambda \mathbf{C} & \to & H^*(T, \mathbf{C}) \\ \mu & \mapsto & \mu \otimes 1. & & \end{array}$$

The embedding of the lattice $\Lambda \to \mathbf{C}^g$ defines a linear map

$$j : \mathbf{C}\langle\Lambda\rangle \to H^0(\mathbf{C}^g, \mathcal{O}_{\mathbf{C}^g}^\times).$$

Differentiation of j by $\partial/\partial z_i$ $(i = 1, \ldots, g)$ gives a homomorhism

$$H^1(\Lambda, \mathbf{C}\langle\Lambda\rangle) \to H^1(T, \mathcal{O}_T^{\oplus g}).$$

Here, $H^1(T, \mathcal{O}_T^{\oplus g})$ is nothing but the tangent space of the classical deformation of T. The vector fields $\partial/\partial z_i$ determines the choice of s in Theorem 2.2. On the other hand, the isomorphism

$$H^2(\Lambda, \mathbf{C}) \simeq \wedge^2 \mathrm{Hom}(\Lambda, \mathbf{C})$$

tells us that $H^2(\Lambda, \mathbf{C})$ is the tangent space to the locus of noncommutative deformations of T.

The Fourier transformation gives us another viewpoint to see the above situation. Denote by $\Lambda^\vee$ the space of irreducible representations of Λ. Consider the algebra homomorphism

$$\varphi : \mathbf{C}\langle\Lambda\rangle \to C(\Lambda^\vee)$$

defined by the Fourier transformation. Then, $C(\Lambda^\vee)$ can be regarded as the space of functions on $\hat{T} = \mathrm{Pic}^0(T)$, and it is easy to see that φ induces a morphism $\phi : \mathcal{M}_\Lambda \to \mathcal{M}(\hat{T})$. We also have a morphism $\iota : \mathcal{M}(\hat{T}) \to \mathcal{M}(T)$ defined by the isomorphism $\hat{\hat{T}} \simeq T$. Then, it can be checked that $\iota \circ \phi = \alpha_0$. We can generalize this observation to the cases G is noncommutative.

Proposition 3.1 *Under the assumption of Theorem 2.2, the Fourier transformation* $\mathbf{C}\langle G\rangle \to C(G^\vee)$ *gives a morphism* $\phi_{G,X} : \mathcal{M}_G \to \mathcal{M}(\mathrm{Pic}^0(Y))$.

Let $\mathbf{H}$ be the upper half plane. We assume that a Fuchsian group Γ of the first kind acts on $\mathbf{H}$ freely without cusps. By applying Proposition 3.1 to $Y = \mathbf{H}/\Gamma$, we obtain a morphism $\phi_{\Gamma,\mathbf{H}} : \mathcal{M}_\Gamma \to \mathcal{M}(\mathrm{Pic}^0(Y))$. In this case, we can see that $\phi_{\Gamma,\mathbf{H}}$ is surjective. This result shows that the higher products on the spaces of automorhic forms induced by the A_∞-deformations of Y arises from the A_∞-deformations of Γ-modules.

In general, the action of the Fuchsian group on $\mathbf{H}$ has fixed points and cusps, so the quotient space Y is an orbifold. At present, the notion of A_∞-deformations of the orbifolds is not clear. However, even in such a case, it is possible to define higher products group-theoretically on the space of automorphic forms.

Remark 3.1 Polishchuk computed the higher products for the elliptic curves in [7].

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