ON CLASSIFICATION OF WEAKENED FANO 3-FOLDS

TATSUHIRO MINAGAWA

1. INTRODUCTION

We will work over \( \mathbb{C} \) in this talk.

**Definition 1.1.** Let \( X \) be a 3-dimensional smooth projective variety and \((\Delta, 0)\) a germ of the 1-dimensional disk.

1. We call \( X \) a Fano 3-fold when its anti-canonical divisor \(-K_X\) is ample.
2. We call \( X \) a weak Fano 3-fold when its anti-canonical divisor \(-K_X\) is nef and big.

**Definition 1.2.** Let \( X \) be a smooth weak Fano 3-fold. We call \( X \) a weakened Fano 3-fold when

1. \( X \) is not a Fano 3-fold, and
2. there exists a small deformation \( f: X_s \rightarrow (\Delta, 0) \) of \( X \) such that the fiber \( X_s = f^{-1}(s) \) is a Fano 3-fold for any \( s \in (\Delta, 0) \setminus \{0\} \).

We will talk about the classification of weakened Fano 3-folds.

Let \( X \) be a weak Fano 3-fold. We remark that \( B_2(X) \geq 2 \) because \( X \) is a weak Fano which is not a Fano 3-fold. Fano 3-folds with \( B_2 \geq 2 \) are classified by Mori and Mukai (cf. [M-M 1], [M-M 2]). In particular, \( 2 \leq B_2 \leq 10 \).

**Example 1.3.** \( \mathbb{P} \times \mathbb{P}^2 \) is a weakened Fano 3-fold with \( B_2(X) = 3 \) and \((-K)^3 = 48\) which will deform to \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

Let \( S \) be a weak Del Pezzo surface but a Del Pezzo surface. Then it is a weakened Del Pezzo surface. Thus \( \mathbb{P}^1 \times S \) is a weakened Fano 3-fold. We call such weakened Fano 3-folds “product type”.

**Theorem 1.4.** (H. Sato [Sa]) There are exactly 15 toric weakened Fano 3-folds \( X \) up to isomorphism.

In this classification, 11 toric weakened Fano 3-folds are of product type. Moreover, we had the following on product type.

**Theorem 1.5.** Let \( X \) be a weakened Fano 3-fold which will deform to \( X_t \simeq \mathbb{P}^1 \times S_t \) where \( S_t \) is a Del Pezzo surface. Then \( X \) is of product type.

Our main result will be the following:

**Theorem 1.6.** Let \( X \) be a weakened Fano 3-fold. Set \( \delta_X := (-K_X)^3 \).

If \( B_2(X) = 2 \), then \( \delta_X = 12, 20, 28, 48 \)
If \( B_2(X) = 3 \), then \( \delta_X = 12, 18, 24, 26, 30, 36, 38, 44, 48, 52 \)
If \( B_2(X) = 4 \), then \( \delta_X = 24, 28, 30, 32, 34, 38, 42, 46 \)
If \( B_2(X) = 5 \), then \( \delta_X = 28, 36 \)
If $B_2(X) \geq 6$, then $\delta_X = 6(11 - B_2(X))$
Moreover, there exists an example in each case.

**Acknowledgement.** I would like to thank Doctor Hiroshi Sato for useful discussions and encouraging the author during the preparation of this contents. I also wishes to thank organizers for giving me the opportunity to talk about this contents.

**Notations.** (1) The $i$-th Betti number of a manifold $X$ is denoted by $B_i(X)$.
(2) The $\mathbb{P}^1$-bundle $\mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(n))$ over $\mathbb{P}^1$, a Hirzebruch surface of degree $n$, is denoted by $\mathbb{F}_n$.

2. **Examples of weakened Fano 3-folds**

We will study some typical examples of weakened Fano 3-folds in this section.

**Example 2.1.** Let $Y$ be a smooth quadric 3-fold and $E'$ be its hyperplane. Then $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We remark that $-K_Y|_{E'}$ is a divisor of bi-degree $(3,3)$. Let $\Gamma$ be the smooth curve of bi-degree $(3,1)$ on $E'$, $\psi : X \to Y$ be a blow-up of $Y$ along $\Gamma$, and $E$ the strict transform of $E'$. Then $X$ is a weakened Fano 3-fold with $B_2(X) = 2$ and $\delta_X = 28$. We remark that $-K_X|_E$ is a divisor of bi-degree $(0,2)((3,3) - (3,1))$ on $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

**Example 2.2.** Let $p \in Q$ where $Q$ is a smooth quadric 3-fold, $Y$ be a Fano 3-fold which is obtained by the blow-up of $Q$ along $p$, and $E' \simeq \mathbb{P}^2$ be its exceptional divisor. Let $\Sigma$ be the conic on $E'$ which parametrize lines on $Q$ which through $p$. Let $q \in E'$ which is not on $\Sigma$, $\psi : X \to Y$ the blow-up of $Y$ along $q$, and $E$ the strict transform of $E'$. Then $X$ is a weakened Fano 3-fold with $B_2(X) = 3$ and $\delta_X = 38$, and $E \simeq \mathbb{F}_1$. Let $h$ be a section with $h^2 = 1$, and $f$ a fiber with respect to the $\mathbb{P}^1$-bundle structure of $E$. We remark that $-K_X|_E = 2f$ on $E$.

**Example 2.3.** Let $l \subset \mathbb{P}^3$ be a line, $Y$ be a Fano 3-fold which is obtained by the blow-up of $\mathbb{P}^3$ along $l$, and $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be its exceptional divisor. We remark that $-K_Y|_{E'}$ is a divisor of bi-degree $(1,3)$. Let $\Gamma$ be a smooth curve of bi-degree $(1,1)$ on $E'$, $\psi : X \to Y$ the blow-up of $Y$ along $\Gamma$, and $E$ the strict transform of $E'$. Then $X$ is a weakened Fano 3-fold with $B_2(X) = 3$ and $\delta_X = 44$. We remark that $-K_X|_E$ is a divisor of bi-degree $(0,2)((1,3) - (1,1))$ on $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

**Example 2.4.** Let $Z = \mathbb{F}_2 \times \mathbb{P}^1 \times \mathbb{P}^1$, $p_i$ its projections, and $s$ section with $s^2 = 2$ on $\mathbb{F}_2$ with respect to its $\mathbb{P}^1$-bundle structure. Let $H_i$ be the pull back of a hyperplane by $p_i$ for $i = 2, 3$, and $X \in |p_1^*s + H_2 + H_3|$. Then $X$ is a weakened Fano 3-fold with $B_2(X) = 4$ and $\delta_X = 24$. We remark that $Z$ is a weakened Fano 4-fold.

**Example 2.5.** Let $Q$ be the blow-up of a smooth quadric 3-fold $Q$ with center a conic on it, $D'$ be its exceptional divisor, and $f_1, f_2 \subset D'$ be disjoint exceptional lines. Let $Y$ be the blow-up of $Q$ along $f_1$ and $f_2$, $E'$ its exceptional divisor over $f_1$, and $D$ the strict transform of $D'$. Set $\Gamma = D \cap E'$. Let $\psi : X \to Y$ be the blow-up of $Y$ along $\Gamma$. Then $X$ is a weakened Fano 3-fold with $B_2(X) = 5$ and $\delta_X = 28$. 

2
3. Ideas for classification

Key theorem for the classification is the following.

**Theorem 3.1.** (Cf. [Pa], [Mi1], [Mi2], [Sa]) Let $X$ be a weak Fano 3-fold which is not a Fano 3-fold. Then $X$ is weakened Fano 3-fold if and only if every primitive crepant contraction $\phi : X \to \overline{X}$ is a contraction which contracts a divisor $E$ to a curve $C \subset \overline{X}$ such that

1. $C \simeq \mathbb{P}^1$
2. $\phi|_E : E \to C$ is a $\mathbb{P}^1$-bundle structure
3. $(-K_X \cdot C) = 2$

Let $\phi_{ac} : X \to X_{ac}$ is multi-anti-canonical morphism. For simplicity, let $\phi_{ac}$ be primitive. Let $E \simeq \mathbb{P}_n$ be the exceptional divisor of $\phi_{ac}$, and $\phi_E : E \to C := \phi_{ac}$ its $\mathbb{P}^1$-bundle structure. Let $h$ be a section of $\phi_E$ with $h^2 = n$, and $f$ a fiber of $\phi_E$. Since $X$ is a weakened Fano 3-fold, there exist rational curves $l_i$ such that $(-K_X \cdot l_i) > 0$, and $\text{NE}(X) = \mathbb{R}_{\geq 0}[f] + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[l_i]$. Set $\mu_i = (-K_X \cdot l_i) > 0$, $c_i = (E \cdot l_i)$, and $\alpha = \max \left\{ \frac{c_i}{\mu_i} \mid i \right\}$. We have 2 lemmas by easy calculations.

**Lemma 3.2.** $L = -\alpha K_X - E$ is a nef $\mathbb{Q}$-divisor with $(L \cdot f) > 0$.

**Lemma 3.3.** $\alpha > 0$

Let $\psi : X \to Y$ be a contracton corresponding to $L$. If $\psi$ is not birational, then $L^3 = \delta_X \alpha^3 - 12 \alpha = 0$. By [M-M 2], we have that $\delta_X = 12$ or 48 since $\alpha \in \mathbb{Q}$.

The case that $\psi$ is birational, we have the following.

**Proposition 3.4.** If $\psi$ is birational, $\psi$ is a composition of blow-ups of smooth curves.

For the proof, for example, suppose $\psi : X \to Y$ be the blow-up of a smooth point $p$ of $Y$ and $D$ its exceptional divisor. We have that $D|_E = m(h - nf)$ for some $m > 0$ and $\alpha = \frac{n}{2}$. Moreover, we have that $n = m = 1$ by some calculations. Thus $L = -\frac{1}{2} K_X - E$. Let $E' = \psi(E)$ which is isomorphic to $\mathbb{P}^2$ since $n = 1$. Then $-\frac{1}{2} K_Y + E'$ is a nef divisor on $Y$, and $\psi' : Y \to Y'$ the contraction corresponding to $-\frac{1}{2} K_Y + E'$ contracts $E'$ to a smooth point. We remark that $\psi$ and $\psi'$ are contractions of same type. By [M-M 1], $Y'$ is smooth quadric 3-fold. But in this case, we have that $\alpha = 1$. It is contradiction.

By this proposition, [M-M 1], and [M-M 2], we can study the structures of weakened Fano 3-folds explicitly.

**Remark 3.5.** As above, we see that the nef cone of a deformed Fano 3-fold is “symmetric”. We know such phenomena by [W] and [Pa]. It is another key for the classification.
References


[Ha] Hartshorne R., Algebraic Geometry, GTM 52, Springer-Verlag (1977)


Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 153-8551, Japan

E-mail address: minagawa@math.titech.ac.jp