ON CLASSIFICATION OF WEAKENED FANO 3-FOLDS

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1. Introduction

We will work over $\mathbb{C}$ in this talk.

Definition 1.1. Let X be a 3-dimensional smooth projective variety and $(\Delta, 0)$ a germ of the 1-dimensional disk.

(1) We call X a Fano 3-fold when its anti-canonical divisor $-K_X$ is ample.
(2) We call X a weak Fano 3-fold when its anti-canonical divisor $-K_X$ is nef and big.

Definition 1.2. Let X be a smooth weak Fano 3-fold. We call X a weakened Fano 3-fold when

(i) X is not a Fano 3-fold, and
(ii) there exists a small deformation $\mathfrak{f}\colon \mathscr{X} \to (\Delta, 0)$ of X such that the fiber $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$ is a Fano 3-fold for any $s \in (\Delta, 0) \setminus \{0\}$.

We will talk about the classification of weakened Fano 3-folds.
Let X be a weak Fano 3-fold. We remark that $B_2(X) \geq 2$ because X is a weak Fano which is not a Fano 3-fold. Fano 3-folds with $B_2 \geq 2$ are classified by Mori and Mukai (cf. [M-M 1], [M-M 2]). In particular, $2 \leq B_2 \leq 10$.

Example 1.3. $\mathbb{P} \times \mathbb{F}_2$ is a weakened Fano 3-fold with $B_2(X) = 3$ and $(-K)^3 = 48$ which will deform to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let S be a weak Del Pezzo surface but a Del Pezzo surface. Then it is a weakened Del Pezzo surface. Thus $\mathbb{P}^1 \times S$ is a weakened Fano 3-fold. We call such weakened Fano 3-folds "product type".

Theorem 1.4. *(H.Sato* [Sa]*) There are exacty 15 toric weakened Fano 3-folds X up to isomorphism.*

In this classification, 11 toric weakened Fano 3-folds are of product type. Moreover, we had the following on product type.

Theorem 1.5. *Let X be a weakened Fano 3-fold which will deform to $X_t \simeq \mathbb{P}^1 \times S_t$ where S_t is a Del Pezzo surface. Then X is of product type.*

Our main result will be the following:

Theorem 1.6. *Let X be a weakened Fano 3-fold. Set $\delta_X := (-K_X)^3$.*
If $B_2(X) = 2$, then $\delta_X = 12, 20, 28, 48$
If $B_2(X) = 3$, then $\delta_X = 12, 18, 24, 26, 30, 36, 38, 44, 48, 52$
If $B_2(X) = 4$, then $\delta_X = 24, 28, 30, 32, 34, 38, 42, 46$
If $B_2(X) = 5$, then $\delta_X = 28, 36$

If $B_2(X) \geq 6$, then $\delta_X = 6(11 - B_2(X))$
Moreovere, there exists an example in each case.

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Notations. (1) The i-th Betti number of a manifold X is denoted by $B_i(X)$.
(2) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ over $\mathbb{P}^1$, a Hirzebruch surface of degree n, is denoted by $\mathbb{F}_n$.

2. Examples of weakened Fano 3-folds

We will study some typical examples of weakened Fano 3-folds in this section.

Example 2.1. Let Y be a smooth quadric 3-fold and E' be its hyperplane. Then $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We remark that $-K_Y|_{E'}$ is a divisor of bi-degree $(3, 3)$. Let Γ be the smooth curve of bi-degree $(3, 1)$ on E', $\psi : X \to Y$ be a blow-up of Y along Γ, and E the strict transform of E'. Then X is a weakened Fano 3-fold with $B_2(X) = 2$ and $\delta_X = 28$. We remark that $-K_X|_E$ is a divisor of bi-degree $(0, 2)(= (3, 3) - (3, 1))$ on $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.2. Let $p \in Q$ where Q is a smooth quadric 3-fold, Y be a Fano 3-fold which is obtained by the blow-up of Q along p, and $E' \simeq \mathbb{P}^2$ be its exceptional divisor. Let Σ be the conic on E' which parametrize lines on Q which through p. Let $q \in E'$ which is not on Σ, $\psi : X \to Y$ the blow-up of Y along q, and E the strict transform of E'. Then X is a weakened Fano 3-fold with $B_2(X) = 3$ and $\delta_X = 38$, and $E \simeq \mathbb{F}_1$. Let h be a section of with $h^2 = 1$, and f a fiber with respect to the $\mathbb{P}^1$-bundle structure of E. We remark that $-K_X|_E = 2f$ on E.

Example 2.3. Let $l \subset \mathbb{P}^3$ be a line, Y be a Fano 3-fold which is obtained by the blow-up of $\mathbb{P}^3$ along l, and $E' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be its exceptional divisor. We remark that $-K_Y|_{E'}$ is a divisor of bi-degree $(1, 3)$. Let Γ be a smooth curve of bi-degree $(1, 1)$ on E', $\psi : X \to Y$ the blow-up of Y along Γ, and E the strict transform of E'. Then X is a weakened Fano 3-fold with $B_2(X) = 3$ and $\delta_X = 44$. We remark that $-K_X|_E$ is a divisor of bi-degree $(0, 2)(= (1, 3) - (1, 1))$ on $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Example 2.4. Let $Z = \mathbb{F}_2 \times \mathbb{P}^1 \times \mathbb{P}^1$, p_i its projections, and s section with $s^2 = 2$ on $\mathbb{F}_2$ with respect to its $\mathbb{P}^1$-bundle structure. Let H_i be the pull back of a hyperplane by p_i for $i = 2, 3$, and $X \in |p_1^* s + H_2 + H_3|$. Then X is a weakened Fano 3-fold with $B_2(X) = 4$ and $\delta_X = 24$. We remark that Z is a weakened Fano 4-fold.

Example 2.5. Let $\tilde{Q}$ be the blow-up of a smooth quadric 3-fold Q with center a conic on it, D' be its exceptional divisor, and $f_1, f_2 \subset D'$ be disjoint exceptional lines. Let Y be the blow-up of $\tilde{Q}$ along f_1 and f_2, E' its exceptional divisor over f_1, and D the strict transform of D'. Set $\Gamma = D \cap E'$. Let $\psi : X \to Y$ be the blow-up of Y along Γ. Then X is a weakened Fano 3-fold with $B_2(X) = 5$ and $\delta_X = 28$.

3. Ideas for classification

Key theorem for the classification is the following.

Theorem 3.1. *(Cf.[Pa],[Mi1],[Mi2],[Sa]) Let X be a weak Fano 3-fold which is not a Fano 3-fold. Then X is weakened Fano 3-fold if and only if every primitive crepant contraction $\phi : X \to \bar{X}$ is a contraction which contracts a divisor E to a curve $C \subset \bar{X}$ such that*
(1) $C \simeq \mathbb{P}^1$
(2) $\phi|_E : E \to C$ is a $\mathbb{P}^1$-bundle structure
(3) $(-K_{\bar{X}} \cdot C) = 2$

Let $\phi_{ac} : X \to X_{ac}$ is multi-anti-canonical morphism. For simplicity, let ϕ_{ac} be primitive. Let $E \simeq \mathbb{F}_n$ be the exceptional divisor of ϕ_{ac} , and $\phi_E : E \to C := \phi_{ac}$ its $\mathbb{P}^1$-bundle structure . Let h be a section of ϕ_E with $h^2 = n$, and f a fiber of ϕ_E. Since X is a weakened Fano 3-fold, there exist rational curves l_i such that $(-K_X \cdot l_i) > 0$, and $\overline{NE}(X) = \mathbb{R}_{\geq 0}[f] + \sum_{finite} \mathbb{R}_{\geq 0}[l_i]$. Set $\mu_i = (-K_X \cdot l_i) > 0$, $c_i = (E \cdot l_i)$, and $\alpha = max\left\{ \frac{c_i}{\mu_i} \;\middle|\; i \right\}$. We have 2 lemmas by easy calculations.

Lemma 3.2. *$L = -\alpha K_X - E$ is a nef $\mathbb{Q}$-divisor with $(L \cdot f) > 0$.*

Lemma 3.3. $\alpha > 0$

Let $\psi : X \to Y$ be a contracton corresponding to L. If ψ is not birational, then $L^3 = \delta_X \alpha^3 - 12\alpha = 0$. By [M-M 2], we have that $\delta_X = 12$ or 48 since $\alpha \in \mathbb{Q}$.

The case that ψ is biratinal, we have the following.

Proposition 3.4. *If ψ is birational, ψ is a composition of blow-ups of smooth curves.*

For the proof, for example, suppose $\psi : X \to Y$ be the blow-up of a smooth point p of Y and D its exceptional divisor. We have that $D|_E = m(h - nf)$ for some $m > 0$ and $\alpha = \frac{n}{2}$. Moreover, we have that $n = m = 1$ by some calculations. Thus $L = -\frac{1}{2}K_X - E$. Let $E' = \psi(E)$ which is isomorphic to $\mathbb{P}^2$ since $n = 1$. Then $-\frac{1}{2}K_Y + E'$ is a nef divisor on Y, and $\psi' : Y \to Y'$ the contraction corresponding to $-\frac{1}{2}K_Y + E'$ contracts E' to a smooth point. We remark that ψ and ψ' are contractions of same type. By [M-M 1], Y' is smooth quadric 3-fold. But in this case, we have that $\alpha = 1$. It is contradiction.

By this proposition, [M-M 1], and [M-M 2], we can study the structures of weakened Fano 3-folds explicitly.

Remark 3.5. As above, we see that the nef cone of a deformed Fano 3-fold is "symmetric". We know such phenomena by [W] and [Pa]. It is another key for the classification.

References

[Ba-Hu] Barth W. and Hulek K., Monads and moduli of vector bundles, Manuscripta Math. 25 (1978)

[Be 1] Beauville A., Variété de Prym et jacobiennes intermédiaires, Ann. Sci. École Norm. Sup. 10 (1977)

[Be 2] Beauville A., Complex Algebraic Surfaces, Cambridge University Press (1983)

[Ha] Hartshone R., Algebraic Geometry, GTM 52, Springer-Verlag (1977)

[Is 1] Iskovskih V. A., Fano 3-fold I, English translation, Math. USSR Izv. 11 (1977)

[Is 2] Iskovskih V. A., Fano 3-fold II, English translation, Math. USSR Izv. 12 (1978)

[K-M-M] Kawamata Y., Matsuda, K., and Matsuki K., Introduction to the minimal model problem, Adv. Stud. Pure Math. 10, Kinokuniya-North Holland (1987)

[Mi1] Minagawa T., Global smoothing of singular weak Fano 3-folds, To appear in J. Math. Soc. Japan

[Mi2] Minagawa T., On classification of weakened Fano 3-folds with $B_2 = 2$, Proc. of alg. geom. symposium in Kinosaki (2000)

[Mo] Mori S., Threefolds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982)

[M-M 1] Mori S. and Mukai S., Classification of Fano 3-folds with $B_2 \geq 2$, Manuscripta Math. 36 (1981)

[M-M 2] Mori S. and Mukai S., On Fano 3-folds with $B_2 \geq 2$, Algebraic and Analytic Varieties, Adv. Stud. in Pure Math. 1 (1983)

[Pa] Paoletti, R., The Kähler cone in families of quasi-Fano threefolds, Math. Z. 227 (1998)

[Sa] The classification of smooth toric weakened Fano 3-folds, Manuscripta Math. 109 (2002)

[Shin] Shin K.H., 3-dimensional Fano varieties with canonical singularities, Tokyo J. Math. 12 (1989)

[Take] Takeuchi K., Some birational maps of Fano 3-folds, Compositio Math. 71 (1989)

[W] Wilson, P.M.H., The Kähler cone on Calabi-Yau threefolds, Invent. Math. 107 (1992)

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