Hodge Theory and Algebraic Geometry
On some invariant rings for the two dimensional additive group action

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(Joint work with S. Mukai)

Let \( S \) be a polynomial ring over \( \mathbb{C} \). We assume that an \( n \)-dimensional additive group \( G \) acts on \( S \) linearly. In [6], Weitzenböck proved that the invariant ring \( S^G \) is finitely generated for \( n = 1 \). For \( n \geq 3 \), in [1] and [2], Mukai proved that there exists an invariant ring \( S^G \) which is infinitely generated. In this article, we consider the case \( n = 2 \). There are two important examples for the case \( n = 2 \), namely, Nagata type and Sylvester type.

1 Nagata type

Let \( S = \mathbb{C}[a_0, \ldots, a_n, b_0, \ldots, b_n] \) be the polynomial ring of \( 2n + 2 \) variables. The group \( G = \mathbb{G}_a^2 \) acts as follows:

\[
\begin{align*}
    a_i & \mapsto a_i \\
    b_i & \mapsto b_i + sa_i + t\lambda_i a_i, \quad (s, t) \in G, \quad \lambda_i \in \mathbb{C}, \quad 0 \leq i \leq n.
\end{align*}
\]

This action was investigated by Nagata in [3]. The invariant ring \( S^G \) contains the minors of degree \( 2l + 1 \) of the matrix

\[
\begin{bmatrix}
    a_0 & \cdots & \cdots & \cdots & a_n \\
    \lambda_0 a_0 & \cdots & \cdots & \cdots & \lambda_n a_n \\
    \lambda_0^2 a_0 & \cdots & \cdots & \cdots & \lambda_n^2 a_n \\
    \vdots & \ddots & & \cdots & \vdots \\
    \lambda_0^l a_0 & \cdots & \cdots & \cdots & \lambda_n^l a_n \\
    b_0 & \cdots & \cdots & \cdots & b_n \\
    \vdots & \ddots & & \cdots & \vdots \\
    \lambda_0 b_0 & \cdots & \cdots & \cdots & \lambda_n b_n \\
    \vdots & \ddots & & \cdots & \vdots \\
    \lambda_0^{l-1} b_0 & \cdots & \cdots & \cdots & \lambda_n^{l-1} b_n
\end{bmatrix}
(0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor).
\]

**Problem 1.** Is the invariant ring \( S^G \) generated by the above invariants?

This answer is unknown, however we can prove that the invariant ring \( S^G \) is finitely generated.
Let $X \rightarrow \mathbb{P}^{n-2}$ be the blowing up of $\mathbb{P}^{n-2}$ at $n+1$ points. These $n+1$ points are determined by the values of $\lambda_i$. We consider the total coordinate ring of $X$

$$TC(X) := \bigoplus_{\alpha, \beta_1, \ldots, \beta_{n+1} \in \mathbb{Z}} H^0(X, \mathcal{O}(\alpha h - \beta_1 e_1 - \cdots - \beta_{n+1} e_{n+1})),$$

where $h$ is pull back of $\mathcal{O}(1)$ and $e_i$ are the exceptional divisors. Mukai proved the next Theorem in [1] and [2].

**Theorem 1.** The invariant ring $S^G$ is isomorphic to the total coordinate ring $TC(X)$.

By an argument similar to the proof of Corollary 2, we have the next result.

**Corollary 1.** The invariant ring $S^G$ is finitely generated.

## 2 Sylvester type

Let $S = \mathbb{C}[a_0, \ldots, a_n, b_0, \ldots, b_{n+1}]$ be the polynomial ring of $2n + 3$ variables. The group $G = \mathbb{G}_a^2$ acts as follows:

$$\begin{cases}
    a_i &\mapsto a_i & 0 \leq i \leq n, \\
    b_j &\mapsto b_j + sa_j + ta_{j-1} & (s, t) \in G, \ 0 \leq j \leq n + 1,
\end{cases}$$

where we put $a_{-1} = a_{n+1} = 0$. This action is due to some moduli space. Let

$$f = a_0 x^n + \cdots + a_n y^n,$$

$$g = b_0 x^{n+1} + \cdots + b_{n+1} y^{n+1}$$

be homogeneous elements of $\mathbb{C}[x, y]$ of degree $n$ and $n + 1$. The ideal $I$ generated by $f$ and $g$ is invariant of the above action. The invariant ring $S^G$ contains the minors of degree $2l + 1$ of the matrix

$$M_l := \begin{bmatrix}
    a_0 & a_1 & \cdots & \cdots & \cdots & a_n & 0 & \cdots & \cdots & 0 \\
    0 & a_0 & a_1 & \cdots & \cdots & \cdots & a_n & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \ddots & \ddots & \cdots & \vdots \\
    \vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \cdots & \vdots \\
    0 & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & a_n & 0 \\
    0 & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & a_n & 0 \\
    b_0 & b_1 & \cdots & \cdots & \cdots & b_{n+1} & 0 & \cdots & 0 & 0 \\
    0 & b_0 & b_1 & \cdots & \cdots & \cdots & b_{n+1} & 0 & \cdots & 0 \\
    \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
    \vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
    0 & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & b_{n+1} & 0 \\
    0 & \cdots & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & b_{n+1}
\end{bmatrix} \quad (0 \leq l \leq n),$$

whose size is $(2l + 1, n + l + 1)$. 

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Problem 2. Is the invariant ring generated by the above invariants?

In case $l = 0$, the matrix $M_0$ is $[a_0, \cdots, a_n]$. In case $l = 1$, we put $R(i, j, k)$ ($1 \leq i < j < k \leq n + 2$) the determinant of the matrix made from the $i, j$ and $k$th columns of the matrix $M_1$. In case $l = n$, the determinant of the matrix $M_n$ is Sylvester resultant $R$. In case $l = n - 1$, we put $R_i$ ($1 \leq i \leq 2n$) the determinant of the matrix omitted the $i$th column from the matrix $M_{n-1}$. In case $l = n - 2$, we put $i$, ($1 \leq i < j \leq 2n - 1$) the determinant of the matrix omitted the $i$ and $j$th columns from the matrix $M_{n-2}$. Then, there are following relations among the invariants.

$$R^{n-1}a_i = (-1)^i \theta_{i+1}^{n+1} \theta_i^{R_1} \theta_{i+1}^{R_2} \cdots \theta_{i+1}^{R_{n+1}}$$ \hspace{1cm} (1)

$$R^{n-2}R(i, j, k) = (-1)^{n-1} \theta_{k+1}^{R_1} \theta_{k+1}^{R_2} \cdots \theta_{k+1}^{R_{n+1}}$$ \hspace{1cm} (2)

$$RR_{i, j} = \begin{vmatrix} R_i & R_j \\ R_i+1 & R_j+1 \end{vmatrix}$$ \hspace{1cm} (3)

We consider $\mathbb{P}^{2n-1}$ with the homogeneous coordinate $(R_1, \cdots, R_{2n})$. Common zeros of

$$\begin{vmatrix} R_i & R_j \\ R_i+1 & R_j+1 \end{vmatrix}$$ \hspace{1cm} (4)

are the normal rational curve $C_{2n-1} \subset \mathbb{P}^{2n-1}$. Let $X \rightarrow \mathbb{P}^{2n-1}$ be the blowing up of $\mathbb{P}^{2n-1}$ along $C_{2n-1}$.

Theorem 2.

$$S^G \cong TC(X) := \bigoplus_{\alpha, \beta \in \mathbb{Z}} H^0(X, \mathcal{O}(\alpha h - \beta e)),$$

where $h$ is pull back of $\mathcal{O}(1)$ and $e$ is the exceptional divisor.

Proof. We put $A := \mathbb{C}[R_1, \cdots, R_{2n}, R]$. At first, we prove that $\text{spec} G \cong A[R^{-1}] \cap S$. The ring $A$ is a subring of the polynomial ring $S$, therefore we can consider the morphism $\phi : \text{spec} S \rightarrow \text{spec} A$. The group $G$ acts $\text{spec} S$ naturally. Let $O_P$ be the $G$-orbit of $P \in \text{spec} S$. Since $R_1, \cdots, R_{2n}, R$ are invariants, $\phi(O_P)$ is one point of $\text{spec} A$. We can verify that if $\phi(O_P) = \phi(O_P')$, then $O_P = O_{P'}$ for $P, P' \in \text{spec} S \setminus (R = 0 \cap a_0 = \cdots = a_n = 0)$, by using the relations (1) and (2). Thus there is a non empty open set $U \subset \text{spec} A$ such that $\phi^{-1}(Q)$ contains a dense orbit for any $Q \in U$. We put $V_1 := (R \neq 0) \subset \text{spec} A$ and
Let $V_2 := \text{Spec} A \setminus \text{Spec}(A/I)$, where ideal $I \subset A$ is generated by the determinants of the matrix omitted $i$th ($1 \leq i \leq n + 1$) column from the matrix

$$\begin{bmatrix}
R_1 & \cdots & R_i & \cdots & R_{n+1} \\
\vdots & \vdots & \vdots & & \vdots \\
R_n & \cdots & R_{i+n-1} & \cdots & R_{2n}
\end{bmatrix}.$$

Then, we can prove that $\text{Image}(\phi) \supset (V_1 \cap V_2)$, by using the relations (1) and (2). Therefore, for the localized morphism $\phi_{R^{-1}} : \text{Spec} S[R^{-1}] \longrightarrow \text{Spec} A[R^{-1}]$, we have

$$\text{codim}(\text{Spec} A[R^{-1}] \setminus \text{Image}(\phi_{R^{-1}})) \geq 2.$$

By Igusa’s lemma in [4], $\phi_{R^{-1}} : A[R^{-1}] \longrightarrow S^G[R^{-1}]$ is isomorphism. Thus, the invariant ring $S^G$ is isomorphic to the ring $A[R^{-1}] \cap S$.

Next we prove $A[R^{-1}] \cap S \cong TC(X)$. We can consider the total coordinate ring $TC(X)$ is subring of $A[R^{-1}]$ as follows.

$$
\begin{align*}
TC(X) & \longrightarrow A[R^{-1}] \\
H^0(X, \mathcal{O}(e)) \ni 1 & \longmapsto R \\
H^0(X, \mathcal{O}(h)) & \longrightarrow <R_1, \cdots, R_{2n}>.
\end{align*}
$$

Then, the subring

$$\bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{\leq 0}} H^0(X, \mathcal{O}(\alpha h - \beta e))$$

of the total coordinate ring $TC(X)$ is isomorphic to the polynomial ring $A$. Hence, we consider the case $\beta \geq 1$.

At first, we prove

$$\bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{>0}} H^0(X, \mathcal{O}(\alpha h - \beta e)) = \bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{>0}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}_{C_{2n-1}}^\beta(\alpha)) \subset A[R^{-1}] \cap S. \quad (5)$$

We consider the case $\beta = 1$. The defining ideal of $C_{2n-1}$

$$I := \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}_{C_{2n-1}}^1(\alpha)) \subset \mathbb{C}[R_1, \cdots, R_{2n}]$$

is generated by (4). By the relations (3), if $F \in I$, then $R|F$. Thus, (5) is satisfied.

We consider the case $\beta \geq 2$. We put the ideal

$$I_\beta := \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}_{C_{2n-1}}^\beta(\alpha)) \subset \mathbb{C}[R_1, \cdots, R_{2n}].$$

For $F \in I_\beta$, there is an integer $n_F \in \mathbb{Z}_{>0}$ such that $FR_1^{n_F} \in I^\beta$. This implies $F \in I^\beta$, thus we have $R^\beta|F$. Therefore, (5) is satisfied. Hence, we have $TC(X) \subset A[R^{-1}] \cap S$.  

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Finally, we prove $A[R^{-1}] \cap S \subset TC(X)$. We consider the case $\beta = 1$. We put
\[
a_0 = p^n, a_1 = 2p^{n-1}q, \ldots, a_{n-1} = 2pq^{n-1}, a_n = q^n,
\]
\[
b_0 = p^{n+1}, b_1 = 2p^nq, \ldots, b_n = 2pq^n, b_{n+1} = q^{n+1}.
\]
Then, we have
\[
R = 0 \quad \text{and} \quad R_i = (pq)^{n^2-p^i-1}q^{2n-i}.
\]
Therefore, for $F \in C[R_1, \cdots, R_{2n}]$, if $R|F$ then $F$ vanishes on the curve $C_{2n-1}$.

Next, we consider the case $\beta \geq 2$. We prove that for $F \in C[R_1, \cdots, R_{2n}]$ if $R^3|F$ then
\[
F \in \bigoplus_{\alpha \in \Z} H^0(\P^{2n-1}, \mathcal{I}_{C_{2n-1}}(\alpha)).
\]

We use induction about $\beta$. For $P \in A$, we have
\[
\frac{\partial P}{\partial a_0} = \frac{\partial P}{\partial R_1} \frac{\partial R_1}{\partial a_0} + \cdots + \frac{\partial P}{\partial R_{2n}} \frac{\partial R_{2n}}{\partial a_0} + \frac{\partial P}{\partial R} \frac{\partial R}{\partial a_0},
\]
\[
\vdots
\]
\[
\frac{\partial P}{\partial a_n} = \frac{\partial P}{\partial R_1} \frac{\partial R_1}{\partial a_n} + \cdots + \frac{\partial P}{\partial R_{2n}} \frac{\partial R_{2n}}{\partial a_n} + \frac{\partial P}{\partial R} \frac{\partial R}{\partial a_n},
\]
\[
\vdots
\]
\[
\frac{\partial P}{\partial b_{n-1}} = \frac{\partial P}{\partial R_1} \frac{\partial R_1}{\partial b_{n-1}} + \cdots + \frac{\partial P}{\partial R_{2n}} \frac{\partial R_{2n}}{\partial b_{n-1}} + \frac{\partial P}{\partial R} \frac{\partial R}{\partial b_{n-1}}.
\]

By Cramer’s rule, we have the identical equation
\[
\frac{\partial P}{\partial R_i} = \frac{1}{J} \begin{vmatrix}
\frac{\partial R_1}{\partial a_0} & \cdots & \frac{\partial R_{n-1}}{\partial a_0} & \frac{\partial P}{\partial a_0} & \frac{\partial R_{n+1}}{\partial a_0} & \cdots & \frac{\partial R}{\partial a_0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial R_1}{\partial b_{n-1}} & \cdots & \frac{\partial R_{n-1}}{\partial b_{n-1}} & \frac{\partial P}{\partial b_{n-1}} & \frac{\partial R_{n+1}}{\partial b_{n-1}} & \cdots & \frac{\partial R}{\partial b_{n-1}}
\end{vmatrix} \quad (1 \leq i \leq 2n),
\]
(6)

where $J$ is Jacobian. By applying the relations (1) and (2) to the identical equation (6), we have
\[
R_j J_{i,j} \in S \quad (1 \leq i \leq n+1, 1 \leq j \leq 2n),
\]
(7)
\[
R_j J_{i,j} a_n^2 \in S \quad (n+2 \leq i \leq 2n+1, 1 \leq j \leq 2n),
\]
(8)
where $J_{i,j}$ is the cofactor. Since $R^3|F$, we can express $F = R^3 F'$. By the identical equation (6), we have
\[
\frac{\partial F}{\partial R_i} = \sum_{j=1}^{n+1} R_j^2 \frac{\partial F'}{\partial a_{j-1}} J_{i,j} + \sum_{j=n+2}^{2n+1} R_j^3 \frac{\partial F'}{\partial b_{j-n-2}} J_{i,j} \quad (1 \leq i \leq 2n).
\]

By using (7) and (8), we have $R^3|a_n^2 \frac{\partial F}{\partial R_i}$, this implies $R^3|\frac{\partial F}{\partial R_i}$. By induction assumption, we have
\[
\frac{\partial F}{\partial R_i} \in \bigoplus_{\alpha \in \Z} H^0(\P^{2n-1}, \mathcal{I}_{C_{2n-1}}^{3-1}(\alpha)).
\]
Therefore, we have
\[ F \in \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, T^{\beta}_{C_{2n-1}}(\alpha)). \]

\[ \square \]

**Corollary 2.** The invariant ring $S^G$ is finitely generated.

**Proof.** In [5], Thaddeus proved that
\[ h^0(X, O(\alpha h - \beta e)) = \left[ \frac{(1 - t^{\alpha - \beta + 1})^{2n\alpha - 2n\beta - \alpha - \beta + 2n + 2}}{(1 - t^{\alpha - \beta + 2})^{2n\alpha - 2n\beta - \alpha - \beta + 1}(1 - t^{2n})} \right]_\alpha, \]
where \([ \ ]_\alpha\) means the coefficient of $t^\alpha$. The support of $TC(X)$ is the semi-group
\[ \text{Eff}(X) := \{ L \in \text{Pic}(X) | H^0(X, L) \neq 0 \}. \]

The Picard group $\text{Pic}(X)$ is generated by $h$ and $e$. By using (9), $\text{Eff}(X)$ is contained in the semi-group
\[ (e\mathbb{Z}_{\geq 0} \oplus h\mathbb{Z}_{\geq 0}) \cup (h\mathbb{Z}_{\geq 0} \oplus (2h - e)\mathbb{Z}_{\geq 0}) \cup \cdots \cup ((n - 1)h - (n - 2)e)\mathbb{Z}_{\geq 0} \oplus (nh - (n - 1)e)\mathbb{Z}_{\geq 0}. \]

We put $\psi_1 : X_1 := X \longrightarrow Y_0 := \mathbb{P}^{2n-1} = \text{Proj}\mathbb{C}[R_i]$ the blowing up along $C_{2n-1}$. We consider the morphism $\varphi_1 : X_1 := X \longrightarrow Y_1$ which is defined by blowing down at the strict transform of 2-secant line of $C_{2n-1}$. Then, $Y_1$ is isomorphic to $\text{Proj}\mathbb{C}[R_{i,j}]$. We consider the morphism $\psi_2 : X_2 \longrightarrow Y_1$ which is defined by blowing up along the image of 2-secant line in another direction of $\varphi_1$. We consider the morphism $\varphi_2 : X_2 \longrightarrow Y_2$ which is defined by blowing down at the strict transform of 3-secant plane of $C_{2n-1}$. Then, $Y_2$ is isomorphic to $\text{Proj}\mathbb{C}[R_{i,j,k}]$, where $R_{i,j,k}$ are the minors of the matrix $M_{n-3}$. By similar argument, we can define the morphism $\psi_l : X_l \longrightarrow Y_{l-1}$ and $\varphi_l : X_l \longrightarrow Y_l$ ($1 \leq l \leq n - 1$). Then, $X_l$ is embedded in $Y_{l-1} \times Y_l$ ($1 \leq l \leq n - 1$). There is a birational morphism $X_1 \leftarrow X_l$ ($2 \leq l \leq n - 1$) which is isomorphism except on sets of codimension $\geq 2$, therefore $TC(X_1) \cong TC(X_l)$. Hence, the subring of $TC(X)$
\[ TC(X)\big|_{(lh - (l-1)e)\mathbb{Z}_{\geq 0} \oplus (l+1)h - le)\mathbb{Z}_{\geq 0}} \]
is finitely generated. Thus, $TC(X)$ is finitely generated. \[ \square \]

**References**


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