On some invariant rings for the two dimensional additive group action

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(Joint work with S. Mukai)

Let S be a polynomial ring over $\mathbb{C}$. We assume that a n-dimensional additive group G acts on S linearly. In [6], Weitzenböck proved that the invariant ring S^G is finitely generated for $n = 1$. For $n \geq 3$, in [1] and [2], Mukai proved that there exists an invariant ring S^G which is infinitely generated. In this article, we consider the case $n = 2$. There are two important examples for the case $n = 2$, namely, Nagata type and Sylvester type.

1 Nagata type

Let $S = \mathbb{C}[a_0, \cdots, a_n, b_o, \cdots, b_n]$ be the polynomial ring of $2n + 2$ variables. The group $G = \mathbb{G}_a^2$ acts as follows:

$$\begin{cases} a_i & \longmapsto & a_i \\ b_i & \longmapsto & b_i + sa_i + t\lambda_i a_i \quad (s,t) \in G, \quad \lambda_i \in \mathbb{C}, \end{cases} \qquad 0 \leq i \leq n.$$

This action was investigated by Nagata in [3]. The invariant ring S^G contains the minors of degree $2l + 1$ of the matrix

$$\begin{bmatrix} a_0 & \cdots & \cdots & \cdots & a_n \\ \lambda_0 a_0 & \cdots & \cdots & \cdots & \lambda_n a_n \\ {\lambda_0}^2 a_0 & \cdots & \cdots & \cdots & {\lambda_n}^2 a_n \\ \vdots & & & & \vdots \\ {\lambda_0}^l a_0 & \cdots & \cdots & \cdots & {\lambda_n}^l a_n \\ b_0 & \cdots & \cdots & \cdots & b_n \\ \lambda_0 b_0 & \cdots & \cdots & \cdots & \lambda_n b_n \\ \vdots & & & & \vdots \\ {\lambda_0}^{l-1} b_0 & \cdots & \cdots & \cdots & {\lambda_n}^{l-1} b_n \end{bmatrix} \qquad (0 \leq l \leq [\frac{n}{2}]).$$

Problem 1. *Is the invariant ring S^G generated by the above invariants?*

This answer is unknown, however we can prove that the invariant ring S^G is finitely generated.

Let $X \longrightarrow \mathbb{P}^{n-2}$ be the blowing up of $\mathbb{P}^{n-2}$ at $n+1$ points. These $n+1$ points are determined by the values of λ_i. We consider the total coordinate ring of X

$$TC(X) := \bigoplus_{\alpha,\beta_1,\cdots,\beta_{n+1}\in\mathbb{Z}} H^0(X, \mathcal{O}(\alpha h - \beta_1 e_1 - \cdots - \beta_{n+1} e_{n+1})),$$

where h is pull back of $\mathcal{O}(1)$ and e_i are the exceptional divisors. Mukai proved the next Theorem in [1] and [2].

Theorem 1. *The invariant ring S^G is isomorphic to the total coordinate ring $TC(X)$.*

By an argument similar to the proof of Corollary 2, we have the next result.

Corollary 1. *The invariant ring S^G is finitely generated.*

2 Sylvester type

Let $S = \mathbb{C}[a_0, \cdots, a_n, b_0, \cdots, b_{n+1}]$ be the polynomial ring of $2n+3$ variables. The group $G = \mathbb{G}_a^2$ acts as follows:

$$\begin{cases} a_i & \longmapsto & a_i & & 0 \le i \le n, \\ b_j & \longmapsto & b_j + s a_j + t a_{j-1} & (s,t) \in G, & 0 \le j \le n+1, \end{cases}$$

where we put $a_{-1} = a_{n+1} = 0$. This action is due to some moduli space. Let

$$f = a_0 x^n + \cdots + a_n y^n,$$

$$g = b_0 x^{n+1} + \cdots + b_{n+1} y^{n+1}$$

be homogeneous elements of $\mathbb{C}[x,y]$ of degree n and $n+1$. The ideal I generated by f and g is invariant of the above action. The invariant ring S^G contains the minors of degree $2l+1$ of the matrix

$$M_l := \begin{bmatrix} a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_n & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_n & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_n & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_{n+1} & 0 & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_{n+1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_{n+1} & 0 \\ 0 & \cdots & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_{n+1} \end{bmatrix} \quad (0 \le l \le n),$$

whose size is $(2l+1, n+l+1)$.

Problem 2. *Is the invariant ring generated by the above invariants?*

In case $l = 0$, the matrix M_0 is $[a_0, \cdots, a_n]$. In case $l = 1$, we put $R(i, j, k)$ $(1 \le i < j < k \le n + 2)$ the determinant of the matrix made from the i, j and kth columns of the matrix M_1. In case $l = n$, the determinant of the matrix M_n is Sylvester resultant R. In case $l = n-1$, we put R_i $(1 \le i \le 2n)$ the determinant of the matrix omitted the ith column from the matrix M_{n-1}. In case $l = n-2$, we put $R_{i,j}$ $(1 \le i < j \le 2n-1)$ the determinant of the matrix omitted the i and jth columns from the matrix M_{n-2}. Then, there are following relations among the invariants.

$$R^{n-1}a_i = (-1)^{[\frac{n}{2}]} \begin{vmatrix} R_1 & \cdots & \check{R}_i & \cdots & R_{n+1} \\ R_2 & \cdots & \check{R}_{i+1} & \cdots & R_{n+2} \\ \vdots & & \vdots & & \vdots \\ R_n & \cdots & \check{R}_{i+n-1} & \cdots & R_{2n} \end{vmatrix}, \tag{1}$$

$$R^{n-2}R(i,j,k) = (-1)^{[\frac{n-1}{2}]} \begin{vmatrix} R_1 & \cdots & \check{R}_i & \cdots & \check{R}_j & \cdots & \check{R}_k & \cdots & R_{n+2} \\ R_2 & \cdots & \check{R}_{i+1} & \cdots & \check{R}_{j+1} & \cdots & \check{R}_{k+1} & \cdots & R_{n+3} \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ R_{n-1} & \cdots & \check{R}_{i+n-2} & \cdots & \check{R}_{j+n-2} & \cdots & \check{R}_{k+n-2} & \cdots & R_{2n} \end{vmatrix}, \tag{2}$$

$$RR_{i,j} = \begin{vmatrix} R_i & R_j \\ R_{i+1} & R_{j+1} \end{vmatrix}. \tag{3}$$

We consider $\mathbb{P}^{2n-1}$ with the homogeneous coordinate $(R_1, \cdots, R_{2n})$. Common zeros of

$$\begin{vmatrix} R_i & R_j \\ R_{i+1} & R_{j+1} \end{vmatrix} \tag{4}$$

are the normal rational curve $C_{2n-1} \subset \mathbb{P}^{2n-1}$. Let $X \longrightarrow \mathbb{P}^{2n-1}$ be the blowing up of $\mathbb{P}^{2n-1}$ along C_{2n-1}.

Theorem 2.
$$S^G \cong TC(X) := \bigoplus_{\alpha,\beta \in \mathbb{Z}} H^0(X, \mathcal{O}(\alpha h - \beta e)),$$
where h is pull back of $\mathcal{O}(1)$ and e is the exceptional divisor.

Proof. We put $A := \mathbb{C}[R_1, \cdots, R_{2n}, R]$. At first, we prove that $S^G \cong A[R^{-1}] \cap S$. The ring A is a subring of the polynomial ring S, therefore we can consider the morphism $\phi :$ Spec$S \longrightarrow$ Spec$A$. The group G acts Spec$S$ naturally. Let O_P be the $G-$orbit of $P \in$ Spec$S$. Since $R_1, \cdots, R_{2n}, R$ are invariants, $\phi(O_P)$ is one point of Spec$A$. We can verify that if $\phi(O_P) = \phi(O_{P'})$, then $O_P = O_{P'}$ for $P, P' \in$ Spec$S \setminus (R = 0 \cap a_0 = \cdots = a_n = 0)$, by using the relations (1) and (2). Thus there is a non empty open set $U \subset$ Spec$A$ such that $\phi^{-1}(Q)$ contains a dense orbit for any $Q \in U$. We put $V_1 := (R \neq 0) \subset$ Spec$A$ and

$V_2 := \mathrm{Spec}A \setminus \mathrm{Spec}(A/I)$, where ideal $I \subset A$ is generated by the determinants of the matrix omitted ith $(1 \leq i \leq n+1)$ column from the matrix

$$\begin{bmatrix} R_1 & \cdots & R_i & \cdots & R_{n+1} \\ \vdots & & \vdots & & \vdots \\ R_n & \cdots & R_{i+n-1} & \cdots & R_{2n} \end{bmatrix}.$$

Then, we can prove that $\mathrm{Image}(\phi) \supset (V_1 \cap V_2)$, by using the relations (1) and (2). Therefore, for the localized morphism $\phi_{R^{-1}} : \mathrm{Spec}S[R^{-1}] \longrightarrow \mathrm{Spec}A[R^{-1}]$, we have

$$\mathrm{codim}(\mathrm{Spec}A[R^{-1}] \setminus \mathrm{Image}(\phi_{R^{-1}})) \geq 2.$$

By Igusa's lemma in [4], ${\phi_{R^{-1}}}^* : A[R^{-1}] \longrightarrow S^G[R^{-1}]$ is isomorphism. Thus, the invariant ring S^G is isomorphic to the ring $A[R^{-1}] \cap S$.

Next we prove $A[R^{-1}] \cap S \cong TC(X)$. We can consider the total coordinate ring $TC(X)$ is subring of $A[R^{-1}]$ as follows.

$$\begin{array}{lcc} TC(X) & \longrightarrow & A[R^{-1}] \\ H^0(X, \mathcal{O}(e)) \ni 1 & \longmapsto & R \\ H^0(X, \mathcal{O}(h)) & \longrightarrow & < R_1, \cdots, R_{2n} > . \end{array}$$

Then, the subring

$$\bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{\leq 0}} H^0(X, \mathcal{O}(\alpha h - \beta e))$$

of the total coordinate ring $TC(X)$ is isomorphic to the polynomial ring A. Hence, we consider the case $\beta \geq 1$.

At first, we prove

$$\bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{>0}} H^0(X, \mathcal{O}(\alpha h - \beta e)) = \bigoplus_{\alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_{>0}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}^{\beta}_{C_{2n-1}}(\alpha)) \subset A[R^{-1}] \cap S. \tag{5}$$

We consider the case $\beta = 1$. The defining ideal of C_{2n-1}

$$I := \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}_{C_{2n-1}}(\alpha)) \subset \mathbb{C}[R_1, \cdots, R_{2n}]$$

is generated by (4). By the relations (3), if $F \in I$, then $R | F$. Thus, (5) is satisfied.

We consider the case $\beta \geq 2$. We put the ideal

$$I_\beta := \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}^{\beta}_{C_{2n-1}}(\alpha)) \subset \mathbb{C}[R_1, \cdots, R_{2n}].$$

For $F \in I_\beta$, there is a integer $n_F \in \mathbb{Z}_{>0}$ such that $F R_1^{n_F} \in I^\beta$. This implies $F \in I^\beta$, thus we have $R^\beta | F$. Therefore, (5) is satisfied. Hence, we have $TC(X) \subset A[R^{-1}] \cap S$.

Finally, we prove $A[R^{-1}] \cap S \subset TC(X)$. We consider the case $\beta = 1$. We put

$$a_0 = p^n, a_1 = 2p^{n-1}q, \cdots, a_{n-1} = 2pq^{n-1}, a_n = q^n,$$

$$b_0 = p^{n+1}, b_1 = 2p^n q, \cdots, b_n = 2pq^n, b_{n+1} = q^{n+1}.$$

Then, we have

$$R = 0 \quad \text{and} \quad R_i = (pq)^{n^2-n} p^{i-1} q^{2n-i}.$$

Therefore, for $F \in \mathbb{C}[R_1, \cdots, R_{2n}]$, if $R|F$ then F vanishes on the curve C_{2n-1}.

Next, we consider the case $\beta \geq 2$. We prove that for $F \in \mathbb{C}[R_1, \cdots, R_{2n}]$ if $R^\beta | F$ then

$$F \in \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}^\beta_{C_{2n-1}}(\alpha)).$$

We use induction about β. For $P \in A$, we have

$$\begin{array}{ccccccccc}
\frac{\partial P}{\partial a_0} & = & \frac{\partial P}{\partial R_1}\frac{\partial R_1}{a_0} & + & \cdots & + & \frac{\partial P}{\partial R_{2n}}\frac{\partial R_{2n}}{\partial a_0} & + & \frac{\partial P}{\partial R}\frac{\partial R}{\partial a_0}, \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\frac{\partial P}{\partial a_n} & = & \frac{\partial P}{\partial R_1}\frac{\partial R_1}{a_n} & + & \cdots & + & \frac{\partial P}{\partial R_{2n}}\frac{\partial R_{2n}}{\partial a_n} & + & \frac{\partial P}{\partial R}\frac{\partial R}{\partial a_n}, \\
\frac{\partial P}{\partial b_0} & = & \frac{\partial P}{\partial R_1}\frac{\partial R_1}{b_0} & + & \cdots & + & \frac{\partial P}{\partial R_{2n}}\frac{\partial R_{2n}}{\partial b_0} & + & \frac{\partial P}{\partial R}\frac{\partial R}{\partial b_0}, \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\frac{\partial P}{\partial b_{n-1}} & = & \frac{\partial P}{\partial R_1}\frac{\partial R_1}{b_{n-1}} & + & \cdots & + & \frac{\partial P}{\partial R_{2n}}\frac{\partial R_{2n}}{\partial b_{n-1}} & + & \frac{\partial P}{\partial R}\frac{\partial R}{\partial b_{n-1}}.
\end{array}$$

By Cramer's rule, we have the identical equation

$$\frac{\partial P}{\partial R_i} = \frac{1}{J}\begin{vmatrix} \frac{\partial R_1}{\partial a_0} & \cdots & \frac{\partial R_{i-1}}{\partial a_0} & \frac{\partial P}{\partial a_0} & \frac{\partial R_{i+1}}{\partial a_0} & \cdots & \frac{\partial R}{\partial a_0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial R_1}{\partial b_{n-1}} & \cdots & \frac{\partial R_{i-1}}{\partial b_{n-1}} & \frac{\partial P}{\partial b_{n-1}} & \frac{\partial R_{i+1}}{\partial b_{n-1}} & \cdots & \frac{\partial R}{\partial b_{n-1}} \end{vmatrix} \quad (1 \leq i \leq 2n), \tag{6}$$

where J is Jacobian. By applying the relations (1) and (2) to the identical equation (6), we have

$$\frac{R}{J} J_{i,j} \in S \quad (1 \leq i \leq n+1, 1 \leq j \leq 2n), \tag{7}$$

$$\frac{R}{J} J_{i,j} {a_n}^2 \in S \quad (n+2 \leq i \leq 2n+1, 1 \leq j \leq 2n), \tag{8}$$

where $J_{i,j}$ is the cofactor. Since $R^\beta | F$, we can express $F = R^\beta F'$. By the identical equation (6), we have

$$\frac{\partial F}{\partial R_i} = \sum_{j=1}^{n+1} \frac{R^\beta}{J} \frac{\partial F'}{\partial a_{j-1}} J_{j,i} + \sum_{j=n+2}^{2n+1} \frac{R^\beta}{J} \frac{\partial F'}{\partial b_{j-n-2}} J_{j,i} \quad (1 \leq i \leq 2n).$$

By using (7) and (8), we have $R^{\beta-1} | {a_n}^2 \frac{\partial F}{\partial R_i}$, this implies $R^{\beta-1} | \frac{\partial F}{\partial R_i}$. By induction assumption, we have

$$\frac{\partial F}{\partial R_i} \in \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}^{\beta-1}_{C_{2n-1}}(\alpha)).$$

Therefore, we have

$$F \in \bigoplus_{\alpha \in \mathbb{Z}} H^0(\mathbb{P}^{2n-1}, \mathcal{I}^{\beta}_{C_{2n-1}}(\alpha)).$$

□

Corollary 2. *The invariant ring S^G is finitely generated.*

Proof. In [5], Thaddeus proved that

$$h^0(X, \mathcal{O}(\alpha h - \beta e)) = \left[\frac{(1 - t^{\alpha-\beta+1})^{2n\alpha-2n\beta-\alpha-\beta+2n+2}}{(1 - t^{\alpha-\beta+2})^{2n\alpha-2n\beta-\alpha-\beta+1}(1-t)^{2n}} \right]_{\alpha}, \tag{9}$$

where $[\quad]_\alpha$ means the coefficient of t^α. The support of $TC(X)$ is the semi-group

$$\mathrm{Eff}(X) := \{L \in \mathrm{Pic}(X) | H^0(X, L) \neq 0\}.$$

The Picard group $\mathrm{Pic}(X)$ is generated by h and e. By using (9), $\mathrm{Eff}(X)$ is contained in the semi-group

$$(e\mathbb{Z}_{\geq 0} \oplus h\mathbb{Z}_{\geq 0}) \cup (h\mathbb{Z}_{\geq 0} \oplus (2h-e)\mathbb{Z}_{\geq 0}) \cup \cdots \cup (((n-1)h-(n-2)e)\mathbb{Z}_{\geq 0} \oplus (nh-(n-1)e)\mathbb{Z}_{\geq 0}).$$

We put $\psi_1 : X_1 := X \longrightarrow Y_0 := \mathbb{P}^{2n-1} = \mathrm{Proj}\mathbb{C}[R_i]$ the blowing up along C_{2n-1}. We consider the morphism $\varphi_1 : X_1 := X \longrightarrow Y_1$ which is defined by blowing down at the strict transform of 2-secant line of C_{2n-1}. Then, Y_1 is isomorphic to $\mathrm{Proj}\mathbb{C}[R_{i,j}]$. We consider the morphism $\psi_2 : X_2 \longrightarrow Y_1$ which is defined by blowing up along the image of 2-secant line in another direction of φ_1. We consider the morphism $\varphi_2 : X_2 \longrightarrow Y_2$ which is defined by blowing down at the strict transform of 3-secant plane of C_{2n-1}. Then, Y_2 is isomorphic to $\mathrm{Proj}\mathbb{C}[R_{i,j,k}]$, where $R_{i,j,k}$ are the minors of the matrix M_{n-3}. By similar argument, we can define the morphism $\psi_l : X_l \longrightarrow Y_{l-1}$ and $\varphi_l : X_l \longrightarrow Y_l$ $(1 \leq l \leq n-1)$. Then, X_l is embedded in $Y_{l-1} \times Y_l$ $(1 \leq l \leq n-1)$. There is a birational morphism $X_1 \longleftrightarrow X_l$ $(2 \leq l \leq n-1)$ which is isomorphism except on sets of codimension ≥ 2, therefore $TC(X_1) \cong TC(X_l)$. Hence, the subring of $TC(X)$

$$TC(X)|_{(lh-(l-1)e)\mathbb{Z}_{\geq 0} \oplus ((l+1)h-le)\mathbb{Z}_{\geq 0}}$$

is finitely generated. Thus, $TC(X)$ is finitely generated. □

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