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Note	Hodge Theory and Algebraic Geometry 2002/10/7-11 Department of Mathematics, Hokkaido University 石井志保子（東工大）Nash problem on arc families for singularities 内藤広嗣　（名大多元） 村上雅亮（京大理）Surfaces with c^2_1= 3 and \kai(O) = 2, which have non-trivial 3-torsion divisors 大野浩二（大阪大）On certain boundedness of fibred Calabi-Yau s threefolds 阿部健（京大理） 春井岳（大阪大）The gonarlity of curves on an elliptic ruled surface 山下剛（東大数理）開多様体のp進etale cohomology と crystalline cohomology 中島幸喜（東京電機大）Theorie de Hodge III pour cohomologies p-adiques 皆川龍博　（東工大）On classification of weakened Fano 3-folds 斉藤夏男（東大数理）Fano threefold in positive characteristic 石井亮（京大工）Variation of the representation moduli of the McKay quiver 前野俊昭（京大理）群のコホモロジーと量子変形 宮岡洋一（東大数理）　次数が低い有理曲線とファノ多様体 池田京司（大阪大）Subvarieties of generic hypersurfaces in a projective toric variety 竹田雄一郎（九大数理）Complexes of hermitian cubes and the Zagier conjecture 臼井三平（大阪大）SL(2)-orbit theorem and log Hodge structures (Joint work with Kazuya Kato) 鈴木香織（東大数理) \rho(X) = 1, f \le 2 のQ-Fano 3-fold Fanoの分類
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THE GONALITY AND THE CLIFFORD INDEX OF CURVES ON AN ELLIPTIC RULED SURFACE

TAKESHI HARUI

Abstract. Under certain numerical conditions, the gonality of curves on an elliptic ruled surface is twice the degree of the bundle map of the ruled surface and the Clifford index of such curves is computed by pencils of minimal degree.

0. Introduction

Throughout this paper, a curve or a surface always means a projective, reduced and irreducible one over $\mathbb{C}$. For a Cartier divisor D on a curve X, $r(D) := h^0(X, \mathcal{O}_X(D)) - 1$ is the projective dimension of the complete linear system $|D|$. A linear system on X of degree d and projective dimension r is denoted by g^r_d. For D, D', two divisors on a surface, we will write $D \sim D'$ (resp. $D \equiv D'$) if D, D' are linearly equivalent (resp. numerically equivalent).

Let X be a smooth curve of genus $g \geq 1$. The minimal degree of surjective morphisms from X to $\mathbb{P}^1$ is called the *gonality* of X, and denoted by $\mathrm{gon}(X)$:

$$\begin{aligned}\mathrm{gon}(X) :&= \min\{\deg\varphi \mid \varphi : X \to \mathbb{P}^1, \text{ a surjective morphism}\} \\ &= \min\{d \mid \exists\, g^1_d \text{ on } X\}\end{aligned}$$

A curve of gonality k is called *k-gonal.* On a k-gonal curve any g^1_k is complete and free from base points.

The gonality is an old invariant. H. Martens introduced another one which is called the Clifford index of X. Recall the following classical result:

Theorem 0.1 (Clifford's Theorem)**.** *Let D be an effective divisor on X. If D is special, i.e. $h^1(D) \neq 0$, then $2r(D) \leq \deg D$.*

Motivated by this theorem, we define the Clifford index. Let D be an effective divisor on X. The *Clifford index* of D is defined by

$$\mathrm{Cliff}(D) := \deg D - 2r(D).$$

D is said to *contribute to the Clifford index* if both $h^0(D) \geq 2$ and $h^1(D) \geq 2$ hold. Finally, the *Clifford index* of X is defined as follows:

$$\mathrm{Cliff}(X) := \min\{\mathrm{Cliff}(D) \mid D \text{ contributes to the Clifford index}\}.$$

This is a non-negative integer by Clifford's Theorem. A divisor D is said to *compute the Clifford index* of X if D contributes to the Clifford index of X and satisfies that

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$\mathrm{Cliff}(D) = \mathrm{Cliff}(X)$.

Remark 0.2. (1) It is known from Existence Theorem (c.f. [ACGH]) that there exists a divisor on X which contributes to the Clifford index if $g \geq 4$. On the other hand, if $g \leq 3$, then we define formally

$$\mathrm{Cliff}(X) := \begin{cases} 0, & \text{if } X \text{ has a } g_2^1 \\ 1, & \text{otherwise.} \end{cases}$$

(2) A divisor computing the Clifford index is always complete and free from base points.

In this paper we consider a problem for these invariants of curves:

Problem. Determine the gonality and the Clifford index of a curve X when X lies on a "well-known" surface.

This seems to be natural and basic, but only few results are known so far. The case of $\mathbb{P}^2$ is classical.

Lemma 0.3. *Let X be a smooth plane curve of degree $d \geq 5$. Then The gonality and the Clifford index of X is determined only by the degree of X:*

$$\mathrm{gon}(X) = d - 1, \quad \mathrm{Cliff}(X) = d - 4.$$

Several years ago G. Martens studied curves on a Hirzebruch surface. Let Σ_e be a Hirzebruch surface of invariant $e \geq 0$. Recall that $\mathrm{Pic}(\Sigma_e)$ is generated by the classes of two rational curves Δ, Γ where Δ is a minimal section ($\Delta^2 = -e$) and Γ is a fiber of the bundle map $\Sigma_e \to \mathbb{P}^1$.

Proposition 0.4 ([M3]). *Let X be a smooth curve on Σ_e linearly equivalent to $a\Delta + b\Gamma$ ($a, b \in \mathbb{Z}$). Assume that $a \geq 2$, $b \geq ae$, and $a \neq b$ for $e = 1$, $b \geq a$ for $e = 0$. Then*

$$\mathrm{gon}(X) = a, \quad \mathrm{Cliff}(X) = a - 2.$$

The main result of this paper is the following theorem:

Main theorem. *Let S be a geometrically ruled surface over an elliptic curve E. Let C_0,f denote a minimal section, a fiber of the bundle map $\pi : S \to E$, respectively. Let X be a smooth curve on S numerically equivalent to $aC_0 + bf$ ($a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq ae$). If either*

(a) $b > \frac{1}{2}a(e + 4)$, *or*

(b) *e is even and* $b > \frac{1}{2}a(e+2)+1$,

then

$$\mathrm{gon}(X) = 2a \quad \textit{and} \quad \mathrm{Cliff}(X) = 2a-2.$$

1. Preliminary results

Let X be a smooth curve of genus $g \geq 1$. We denote $\mathrm{gon}(X)$, $\mathrm{Cliff}(X)$ by k, c, respectively.

Proposition 1.1 ([ACGH]). *Both the gonality and the Clifford index of X are bounded upwards by its genus:*

$$k \leq \left[\frac{g+3}{2}\right], \quad c \leq \left[\frac{g-1}{2}\right],$$

where $[x]$ is the integer part of x. Furthermore if X is a general curve, then equalities hold.

Example 1.2 ([ELMS]). Curves with small Clifford index are classified:

$c = 0 \Longleftrightarrow X$ is elliptic or hyperelliptic ($k = 2$).
$c = 1 \Longleftrightarrow X$ is trigonal ($k = 3$) or plane quintic ($k = 4$).
$c = 2 \Longleftrightarrow X$ is tetragonal ($k = 4$) or plane sextic ($k = 5$).
$c = 3 \Longleftrightarrow X$ is pentagonal ($k = 5$) or plane septic ($k = 6$),
or a complete intersection of type (3,3) in $\mathbb{P}^3$ ($k = 6$).

Such a classification is known for $c \leq 33$.

This example indicates a close relation between the gonality and the Clifford index. In fact, it is always true:

Theorem 1.3 ([CM]). *The gonality and the Clifford index of X satisfies the following inequality:*

$$k-3 \leq c \leq k-2.$$

We define the *Clifford dimension* to know the difference between the gonality and the Clifford index.

Definition 1.4. *The Clifford dimension* of X is defined as

$$\mathrm{Cliffdim}(X) := \min\{r(D) \mid D \text{ computes the Clifford index of } X\}.$$

A divisor D, which achieves the minimum (and computes the Clifford index), is said to *compute* the Clifford dimension of X.

Clearly a k-gonal curve has Clifford index $k-2$ if and only if its Clifford dimension is equal to 1.

Lemma 1.5 ([ELMS]). *Put* $r := \mathrm{Cliffdim}(X)$ *and let* D *be a divisor which computes the Clifford dimension of* X*.*

(1) *If* $r \geq 2$*, then* D *is very ample. In particular* $r = 2$ *holds if and only if* X *is isomorphic to a smooth plane curve.*

(2) *If* $r \geq 3$*, then* $c \geq 2r - 3$*.*

Lemma 1.6 ([ELMS]). *Let* X *be a smooth curve of genus* g*, Clifford index* c*, and Clifford dimension* r*. If* $r \geq 3$*, then either*

(1) $(g, c) = (4r - 2, 2r - 3)$*, or*

(2) $g \geq 8r - 7$ *and* $c \geq 4r - 6$*.*

Moreover (2) *never occurs for* $3 \leq r \leq 9$*.*

Definition 1.7. In this paper we will name the equation in (1) the ELMS-condition and call a curve satisfying it an ELMS-curve. For such a curve $g = 2c + 4 = 2k - 2$ holds, where k is its gonality.

Lemma 1.8 ([M1]). *The curves of Clifford dimension* 3 *are exactly the complete intersections of two cubics in* $\mathbb{P}^3$*.*

Theorem 1.9 (Castelnuovo bound [ACGH]). *Let* X *be a smooth curve of genus* g*. Assume that* X *admits a base point free linear system* g^r_d *and the morphism induced by the* g^r_d *is birational onto its image. Then* $g \leq \pi(d, r)$*, where*

$$\pi(d, r) := m(d-1) - \frac{m(m+1)}{2}(r-1), \quad m := \left[\frac{d-1}{r-1}\right].$$

Lemma 1.10 ([M2]). *Let* X *be a smooth plane curve of degree* $d \geq 5$ *and let* g_1, g_2 *be different pencils of degree* $d-1$*. Then they are independent. Here, "independent" means that the following situation does not occur: there exist another curve* X' *and a covering* $p : X \to X'$ *with* $1 < \deg p < d-1$ *and two pencils* g'_1, g'_2 *on* X' *such that* $p^*(g'_i) = g_i$ *for* $i = 1, 2$*.*

2. Main theorem

From now on, let S be a geometrically ruled surface over an elliptic curve E unless otherwise mentioned. Let C_0, f denote a minimal section, a fiber of the bundle map $\pi : S \to E$, respectively. Put $e := -C_0^2$.

Recall some elementary facts for geometrically ruled surfaces (see [H]). The Picard group and the Néron-Severi group of S are as follows:

$$\mathrm{Pic}(S) \simeq \mathbb{Z}[C_0] \oplus \pi^*\mathrm{Pic}(E),$$

and

$$\mathrm{NS}(S) \simeq \mathbb{Z}[C_0] \oplus \mathbb{Z}[f].$$

Note that

$$C_0.f = 1, \quad f^2 = 0 \quad \text{and} \quad e = -C_0^2 \geq -1.$$

The canonical bundle K_S is numerically equivalent to $-2C_0 - ef$.

If $\mathfrak{a}$ is a divisor on E, $\mathfrak{a}f$ will denote the pullback of $\mathfrak{a}$ to S by the projection π. When $\deg \mathfrak{a} = 1$ we will write, by abuse of notation, f instead of $\mathfrak{a}f$.

Let X be a smooth curve of genus g on S numerically equivalent to $aC_0 + bf$ ($a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq ae$). The adjunction formula says

$$\begin{aligned} 2g - 2 &= X.(X + K_S) \\ &= (aC_0 + bf).((a-2)C_0 + (b-e)f) \\ &= -a(a-2)e + a(b-e) + b(a-2) \\ &= 2ab - 2b - a(a-1)e \\ &= (a-1)(2b - ae). \end{aligned}$$

It follows that

$$g = (a-1)\left(b - \frac{1}{2}ae\right) + 1.$$

We remark a simple fact about the gonality of X. Since $\pi_{|X} : X \to E$ is a surjective morphism of degree a and E is a double covering of $\mathbb{P}^1$, we have a surjective morphism from X to $\mathbb{P}^1$ of degree $2a$. Hence the inequality $\mathrm{gon}(X) \leq 2a$ always holds.

The following result is a key of the proof of Main Theorem.

Theorem 2.1. *Assume that $e = -C_0^2 \geq 1$ or $S = E \times \mathbb{P}^1$ ($e = 0$). Let X be a smooth curve on S numerically equivalent to $aC_0 + bf$ ($a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq ae$).*

(1) *If* $\mathrm{Cliffdim}(X) \geq 3$*, then X is an ELMS-curve.*

(2) *If $e \geq 2$ or $S = E \times \mathbb{P}^1$, then* $\mathrm{Cliffdim}(X) \leq 2$ *unless $e = 3$ and $(a, b) = (3, 9)$ ($g = 10$). Furthermore if $k = 2a$ then* $\mathrm{Cliffdim}(X) = 1$.

Proof. Put $k := \mathrm{gon}(X)$, $c := \mathrm{Cliff}(X)$ and $r := \mathrm{Cliffdim}(X)$. Note that $k \leq 2a$, and by Theorem 1.3 it follows that $c = k - 3 \leq 2a - 3 \quad \text{if } r \geq 2$.

We shall assume that $e \geq 1$. The following proof applies to the case that $S = E \times \mathbb{P}^1$ with no significant changes. Note that $k \leq \min\{2a, b\}$ in the case.

(1) Suppose that X does not satisfy the ELMS-condition. Then by Lemma 1.6 we have

$$r \geq 10, \quad 4r - 6 \leq c \leq 2a - 3.$$

In particular $a \geq 19$. From Lemma 1.5 (1) and Castelnuovo bound we have

$$g \leq \pi(c + 2r, r),$$

where

$$\pi(c + 2r, r) = m(c + 2r - 1) - \frac{m(m+1)}{2}(r - 1), \quad m := \left[\frac{c + 2r - 1}{r - 1}\right]$$

Note that $m \geq 5$ since $c \geq 4r - 6$. Then

$$\begin{aligned}\pi(c + 2r, r) &= m(c + 2r - 1) - \frac{m(m+1)}{2}(r - 1) \\ &= m(c + 1) - \frac{m(m-3)}{2}(r - 1) \\ &< m(c + 1) - 4m(m - 3) \\ &= m(c + 13 - 4m) \\ &\leq \frac{1}{16}(c + 13)^2 \\ &\leq \frac{1}{16}(2a + 10)^2 \\ &= \frac{1}{4}(a + 5)^2.\end{aligned}$$

On the other hand

$$g = (a - 1)\left(b - \frac{1}{2}ae\right) + 1 \geq \frac{1}{2}a(a - 1) + 1.$$

Hence

$$\frac{1}{2}a(a - 1) + 1 \leq \frac{1}{4}(a + 5)^2,$$

which implies $a \leq 13$, a contradiction.

(2) Suppose that $r \geq 3$. Then X is an ELMS-curve by (1). By Lemma 1.5 it follows that $2r - 3 = c \leq 2a - 3$. So we have $a \geq r \geq 3$. We also have $g = 2k - 2 \leq 4a - 2$. On the other hand

$$g = (a - 1)\left(b - \frac{1}{2}ae\right) + 1 \geq \frac{1}{2}ae(a - 1) + 1.$$

Hence

$$\frac{1}{2}ae(a - 1) + 1 \leq 4a - 2,$$

which implies $e \leq 3$.

If $e = 3$ then we have $a = 3$ and $b = 3a = 9$.

If $e = 2$ then we have $a = 3$ or $a = 4$. But $a = 3$ implies $r = 3$ and $g = 2(b-3)+1 = 10$, which is absurd. So suppose that $a = 4$. Then

$$g = 3(b-4)+1 = \begin{cases} 10 & \text{if } r = 3 \\ 14 & \text{if } r = 4 \end{cases}$$

The latter case is impossible. The former one implies $b = 7$, which contradicts to the assumption that $b \geq 2a$. So $r \leq 2$.

Finally, assume that $k = 2a$. Since X is a covering of an elliptic curve E of degree a, X has infinitely many dependent pencils of degree $k = 2a$. Then from Lemma 1.10, X cannot be isomorphic to a smooth plane curve, which means that $r = 1$. □

Remark 2.2. In the next section we will prove that the exceptinal case $e = 3$ and $(a, b) = (3, 9)$ gives a curve of Clifford dimension 3.

Theorem 2.3. *Assume that $e \geq 1$. Let X be a smooth curve of genus g on S numerically equivalent to $aC_0 + bf$ $(a, b \in \mathbb{Z},\ a \geq 2,\ b \geq ae)$. If either*

(a) $b > \frac{1}{2}a(e+4)$, *or*

(b) *e is even and $b > \frac{1}{2}a(e+2)+1$,*

then

$$\mathrm{gon}(X) = 2a \quad \textit{and} \quad \mathrm{Cliff}(X) = 2a - 2.$$

Remark 2.4. For simplicity we prove Main Theorem assuming that $e \geq 1$ as in Theorem 2.3. But our proof needs no essential changes in the other case.

The main tool for the proof of the theorem above is the folloing result.

Theorem 2.5 (Serrano's theorem [S])**.** *Let X be a smooth curve of genus g on a smooth surface S. Let $\varphi : X \to \mathbb{P}^1$ be a surjective morphism of degree d. Suppose that either*

(a) $X^2 > (d+1)^2$, *or*

(b) $X^2 > \frac{1}{2}(d+2)^2$ *and K_S is numerically even (i.e. $K_S.D$ is even for any divisor D on S).*

Then there exists a morphism $\psi : S \to \mathbb{P}^1$ such that $\psi_{|X} = \varphi$.

Remark 2.6. For a geometrically elliptic ruled surface S we have $K_S \equiv -2C_0 - ef$. So K_S is numerically even if and only if e is an even integer.

Lemma 2.7. *Assume that $e \geq 1$. Let X be a smooth curve of genus g on S numerically equivalent to $aC_0 + bf$ ($a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq ae$) and let $\varphi : X \to \mathbb{P}^1$ be a morphism which is the restriction to X of a morphism $\psi : S \to \mathbb{P}^1$. Then $\deg \varphi \geq 2a$.*

Proof. Let $D \equiv \alpha C_0 + \beta f$ be a fiber of ψ. Then we have

$$f.D \geq 0, \quad C_0.D \geq 0 \quad \text{and} \quad D^2 = 0.$$

It follows that

$$\alpha \geq 0, \quad \beta \geq \alpha e \quad \text{and} \quad \alpha(2\beta - \alpha e) = 0.$$

Thus

$$0 = \alpha(2\beta - \alpha e) \geq \alpha^2 e.$$

Since $e > 0$, we have $\alpha = 0$, $D \equiv \beta f$. Then we can write $D \sim \mathfrak{b} f$, with an effective divisor $\mathfrak{b}$ on E of degree β. Since $H^0(E, \mathcal{O}(\mathfrak{b})) \simeq H^0(S, \mathcal{O}(D))$, we obtain by Riemann-Roch theorem

$$\beta = h^0(E, \mathcal{O}(\mathfrak{b})) = h^0(S, \mathcal{O}(D)) \geq 2.$$

Thus we have

$$\deg \varphi = X.D = a\beta \geq 2a.$$

Hence the conclusion.

□

Proof of Main Theorem. Put $k := \mathrm{gon}(X)$ and $c := \mathrm{Cliff}(X)$. Note that $k \leq 2a$.

First we will show that $k = 2a$. Suppose, to the contrary, that $k < 2a$ and let $\varphi : X \to \mathbb{P}^1$ be a morphism of degree k. Then the lemma above implies that k is not computed by a morphism from S to $\mathbb{P}^1$. Note that

$$\begin{aligned} X^2 &= (aC_0 + bf)(aC_0 + bf) \\ &= a(2b - ae). \end{aligned}$$

From Serrano's theorem we have

$$X^2 \leq (k+1)^2 \leq 4a^2.$$

So $a(2b - ae) \leq 4a^2$, thus we have

$$b \leq \frac{1}{2}a(e+4).$$

Furthermore if e is an even integer, then K_S is numerically even. So again by Serrano's theorem, we obtain

$$X^2 \leq \frac{1}{2}(k+2)^2 \leq \frac{1}{2}(2a+1)^2 = 2a^2 + 2a + \frac{1}{2}.$$

Since X^2 is an integer

$$a(2b - ae) = X^2 \leq 2a^2 + 2a,$$

hence we have

$$b \leq \frac{1}{2}a(e+2) + 1.$$

Thus we obtain the assertion for the gonality.

It remains to show that $\mathrm{Cliffdim}(X) = 1$ yet. Suppose that $r := \mathrm{Cliffdim}(X) \geq 2$. By Theorem 2.1 and by Lemma 1.10 we have only to consider the case that $e = 1$ and X is an ELMS-curve. We then have $g = 2k - 2 = 4a - 2$. On the other hand, our assumption $b > \frac{5}{2}a$ implies

$$g = (a-1)\left(b - \frac{1}{2}a\right) + 1 > 2a(a-1) + 1.$$

Hence

$$2a(a-1) + 1 < 4a - 2,$$

which implies $a \leq 2$, a contradiction.

□

3. The case that $e = 3$

In the following sections we consider the cases not covered by Main Theorem, with fixed e. First of all we obtain an useful fact from the proof of the Serrano's theorem (see [S]).

Lemma 3.1. *Let S, X, φ, d be as in Serrano's theorem. Assume that $X^2 > 4d$. If φ cannot be extended to a morphism from S to $\mathbb{P}^1$, then there exists an effective divisor V on S satisfying the following condition:*
Let

$$s := V^2 \quad \text{and} \quad t := V.(X - V).$$

Then

$$0 < s < t \leq d \quad \text{and} \quad X^2 \leq \frac{(d+s)^2}{s}.$$

Theorem 3.2. *Assume that $e := -C_0^2 = 3$. Let X be a smooth curve on S numerically equivalent to $aC_0 + bf$ ($a, b \in \mathbb{Z}$, $a \geq 2$, $b \geq 3a$).*

(1) *If $b > 3a$, then* $\mathrm{gon}(X) = 2a$, $\mathrm{Cliff}(X) = 2a - 2$.

(2) *If $b = 3a$, then X is isomorphic to a complete intersection of type $(3, a)$ in $\mathbb{P}^3$ and*

$$\mathrm{gon}(X) = 2a \text{ or } 2a - 1, \quad \text{and}$$
$$\mathrm{Cliff}(X) = \mathrm{gon}(X) - 2 \quad \text{unless } (a, b) = (3, 9).$$

In the exceptional case $(a, b) = (3, 9)$, $\mathrm{Cliffdim}(X) = 3$, $\mathrm{gon}(X) = 6$ *and* $\mathrm{Cliff}(X) = 3$.

Proof. Put $k := \mathrm{gon}(X)$ and $c := \mathrm{Cliff}(X)$. Note that $k \leq 2a$. First we will prove the claim for the gonality and the Clifford index. We can disregard the case that $(a, b) = (2, 6)$ since then $g = 4$ and we have $k = 3$ and $c = 1$. So assume that $(a, b) \neq (2, 6)$. Suppose that $k < 2a$. Then $X^2 > 4k$, since otherwise

$$3a^2 \leq a(2b - 3a) = X^2 \leq 4k \leq 4(2a - 1),$$

therefore

$$a(3a-8) \le -4,$$

which implies $(a,b)=(2,6)$. Hence by Lemma 2.7 and the lemma above we obtain an effective divisor V on S such that

$$s := V^2 > 0, \quad t := V.(X-V) \le k \quad \text{and} \quad X^2 \le \frac{(k+s)^2}{s}$$

Let

$$V \equiv \alpha C_0 + \beta f \quad (\alpha, \beta \in \mathbb{Z}).$$

Then

$$s = V^2 = \alpha(2\beta - 3\alpha),$$

and $\alpha = V.f \ge 0$. If $s \ge 3$

$$3a^2 \le X^2 \le \frac{1}{3}(k+3)^2 \le \frac{1}{3}(2a+2)^2,$$

which implies $(a,b)=(2,6)$. So $s \le 2$. In addition $s = \alpha(2\beta - 3\alpha)$ cannot be equal to 2, so we have

$$s = \alpha(2\beta - 3\alpha) = 1.$$

Hence $\alpha = 2\beta - 3\alpha = 1$. So $\alpha = 1, \beta = 2$ and $V \equiv C_0 + 2f$. Then $X - V \equiv (a-1)C_0 + (b-2)f$ and

$$\begin{aligned} t = V.(X-V) &= -3(a-1) + 2(a-1) + (b-2) \\ &= -a + b - 1 \end{aligned}$$

So $-a+b-1 \le k \le 2a-1$ and we obtain $b = 3a$ and $k = 2a-1$. Hence the assertion for the gonality holds.

Next we will show that $c = k-2$, i.e. $\mathrm{Cliffdim}(X) = 1$ unless $(a,b) = (3,9)$. By Theorem 2.1 it suffices to exclude the case that $k = 2a-1$ and X is isomorphic to a smooth plane curve of degree $2a$. If such a case occurs, we have $b = 3a$, $a \ge 4$ and

$$\begin{aligned} \frac{3}{2}a(a-1) + 1 = g &= \frac{1}{2}(2a-1)(2a-2) \\ &= (a-1)(2a-1). \end{aligned}$$

We then have $a = 3$, which contradicts the assumption.

It remains to show the first part and the last one of (2). We can write $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-\mathfrak{e}))$, where $\mathfrak{e}$ is an effective divisor of degree 3 on E. Let $D \in |C_0 + \mathfrak{e}f|$.

Claim. $h^0(S, \mathcal{O}_S(D)) = 4$ and $|D|$ is free from base points.

Since $\pi_* \mathcal{O}_S(C_0) \simeq \mathcal{O}_E \oplus \mathcal{O}_E(-\mathfrak{e})$ we have

$$\begin{aligned} \pi_* \mathcal{O}_S(D) &\simeq \pi_*(\mathcal{O}_S(C_0) \otimes \pi^* \mathcal{O}_E(\mathfrak{e})) \\ &\simeq \pi_* \mathcal{O}_S(C_0) \otimes \mathcal{O}_E(\mathfrak{e}) \\ &\simeq \mathcal{O}_E \oplus \mathcal{O}_E(\mathfrak{e}). \end{aligned}$$

Therefore

$$\begin{aligned} H^0(S,\mathcal{O}_S(D)) &\simeq H^0(E,\pi_*\mathcal{O}_S(D)) \\ &\simeq H^0(E,\mathcal{O}_E)\oplus H^0(E,\mathcal{O}_E(\mathfrak{e})). \end{aligned}$$

Thus we obtain $h^0(S,\mathcal{O}_S(D)) = 1+3 = 4$.

To prove that $|D|$ is free from base points, it suffices to show the surjectivity of the restriction map $H^0(S,\mathcal{O}_S(D)) \to H^0(x,\mathcal{O}_S(D)|_x)$ for any $x \in S$. Let $F := \pi^{-1}(\pi(x))$ and consider the following exact sequence:

$$\begin{aligned} H^0(S,\mathcal{O}_S(D-F)) &\to H^0(S,\mathcal{O}_S(D)) \to H^0(F,\mathcal{O}_S(D)|_F) \\ \to H^1(S,\mathcal{O}_S(D-F)) &\to H^1(S,\mathcal{O}_S(D)) \to 0. \end{aligned}$$

All we have to do is to show that $\rho : H^0(S,\mathcal{O}_S(D)) \to H^0(F,\mathcal{O}_S(D)|_F)$ is surjective. Now

$$H^1(S,\mathcal{O}_S(D-F)) \simeq H^1(E,\mathcal{O}_E(-\pi(x))) \oplus H^1(E,\mathcal{O}_E(\mathfrak{e}-\pi(x))) \simeq \mathbb{C}.$$

$$H^1(S,\mathcal{O}_S(D)) \simeq H^1(E,\mathcal{O}_E) \oplus H^1(E,\mathcal{O}_E(\mathfrak{e})) \simeq \mathbb{C}.$$

Hence ρ is surjective, as desired.

Thus $|D|$ defines a morphism $\Phi_D : S \to \mathbb{P}^3$. $S' := \Phi_D(S)$ is a non-degenerate surface because $D^2 > 0$. Since

$$\deg \Phi_D \cdot \deg S' = D^2 = 3,$$

we have $\deg \Phi = 1$, $\deg S' = 3$. Note that S' is a cone over a plane cubic curve and Φ_D contracts C_0 to the vertex.

Now $X.C_0 = 0$ since $b = 3a$, hence X is embedded into S' isomorphically by Φ_D. Let $X \sim aC_0 + \mathfrak{b}f$, where $\mathfrak{b}$ is an effective divisor of degree $3a$ on X. Take $z_0 \in H^0(S,\mathcal{O}_S(C_0))$, $z_1 \in H^0(S,\mathcal{O}_S(C_0+\mathfrak{e}f))$. Then $\{z_0,z_1\}$ is a system of homogeneous fiber coordinates of Φ_D and $X \subset S$ is defined by the equation

$$\sum_{i=0}^{a} \alpha_i z_0^{a-i} z_1^i = 0$$

where $\alpha_i \in H^0(S,\mathcal{O}_S((\mathfrak{b}-i\mathfrak{e})f))$. If $\mathfrak{b} \nsim a\mathfrak{e}$ then $\alpha_a = 0$, which contradicts the irreducibility of X. So $\mathfrak{b} \sim a\mathfrak{e}$, i.e. $X \in |aD|$. In the exact sequence

$$H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(a)) \to H^0(S',\mathcal{O}_S(a)) \to H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(a-3))$$

the first map is surjective since $H^1(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(a-3)) = 0$. Hence X is an irreducible component of the intersection of S' and a surface of degree a in $\mathbb{P}^3$, say T. Since

$$3a = X.D = \deg X \le \deg(S' \cap T) = 3a,$$

we obtain that X is the only component of $S' \cap T$. So we have $X = S' \cap T$, the first part of (2).

The last part of (2) is nothing but Lemma 1.8. Hence the conclusion.

□

4. The case that $e = 2$

Theorem 4.1. *Assume that $e := -C_0^2 = 2$. Let X be a smooth curve on S numerically equivalent to $aC_0 + bf$ $(a, b \in \mathbb{Z},\ a \geq 2,\ b \geq 2a)$.*

(1) $\mathrm{gon}(X) = 2a$ *except for the following cases:*

- $b = 2a + 1$ *and X is birationalty equivalent to a singular plane curve of degree $2a + 1$ with ordinary nodes or cusps as its only singularities. Then* $\mathrm{gon}(X) = 2a - 1$.
- $b = 2a$ *and X is birationalty equivalent to a plane curve of degree $2a$ with ordinary nodes or cusps as its only singularities. Then* $\mathrm{gon}(X) = 2a - 2$.
- $(a, b) = (2, 4)$ *and X is isomorphic to a plane curve of degree 4. Then* $\mathrm{gon}(X) = 3$.
- $b = 2a$ *and X is a double covering of a smooth plane curve of degree a. Then* $\mathrm{gon}(X) = 2a - 2$.

(2) $\mathrm{Cliff}(X) = \mathrm{gon}(X) - 2$.

Proof. Put $k := \mathrm{gon}(X)$ and $c := \mathrm{Cliff}(X)$. Note that $k \leq 2a$. If $b > 2a+1$, there is nothing to prove since then we can apply Main Theorem. So assume that $b \leq 2a+1$.
(1) First we will show that $k \geq 2a - 2$. Suppose, to the contrary, that $k \leq 2a - 3$. Then k is not computed by a morphism from S to $\mathbb{P}^1$ by Lemma 2.7, and we have

$$X^2 = 2a(b - a) \geq 2a^2 > 4k.$$

Then by Lemma 3.1 there exists an effective divisor V on S such that

$$s := V^2 > 0, \quad t := V.(X - V) \leq k \quad \text{and} \quad X^2 \leq \frac{(k + s)^2}{s}.$$

Let

$$V \equiv \alpha C_0 + \beta f \quad (\alpha, \beta \in \mathbb{Z}).$$

Then

$$s = V^2 = 2\alpha(\beta - \alpha)$$

is an even integer, and $\alpha = V.f \geq 0$. If $s \geq 4$ then

$$2a^2 \leq X^2 \leq \frac{1}{4}(k + 4)^2 \leq \frac{1}{4}(2a + 1)^2,$$

which implies $a \leq 1$, contradicting our assumption. So we have

$$s = 2\alpha(\beta - \alpha) = 2.$$

Hence $\alpha = \beta - \alpha = 1$. So $\alpha = 1, \beta = 2$ and $V \equiv C_0 + 2f$. Then $X - V \equiv (a - 1)C_0 + (b - 2)f$ and we have

$$k \geq V.(X - V) = b - 2 \geq 2a - 2,$$

as desired.

Next we write $S = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-\mathfrak{e}))$, where $\mathfrak{e}$ is an effective divisor of degree 2 on E, and let $D \in |C_0 + \mathfrak{e}f|$. Then, similarly as in the proof of Theorem 3.2, we can show that $|D|$ defines a morphism $\Phi_D : S \to \mathbb{P}^2$, which is a double covering of

$\mathbb{P}^2$. Since Φ_D contracts the minimal section C_0 to a point, say P, it is lifted to a morphism $\Psi : S \to \Sigma_1$, where Σ_1 is the blow-up of $\mathbb{P}^2$ at P.

Let X' be the image of X by Ψ. It is a curve since $X.D = b > 0$. Then we can write $X' \equiv a'\Delta + b'\Gamma$ ($a', b' \in \mathbb{Z}$), where Δ, Γ is a minimal section, a fiber of Σ_1, respectively. Note that X' is birationally equivalent to $Y := \Phi_D(X)$, a plane curve of degree b'. Let $d := \deg \Psi_{|X}$. Then $d = 1$ or 2. By the projection formula we have

$$X.C_0 = dX'.\Delta \quad \text{i.e. } b - 2a = d(b' - a'),$$
$$X.D = dX'.(\Delta + \Gamma) \quad \text{i.e. } b = db'.$$

Hence we obtain

$$\text{if } d = 1 \quad \text{then } (a', b') = (2a, b),$$
$$\text{if } d = 2 \quad \text{then } b = 2a \text{ and } (a', b') = (a, a).$$

Thus we have the assertion except for the claim for the singularity.

If Y has a singular point with multiplicity n, then the degree of the projection from the point to $\mathbb{P}^1$ equals $b' - n$, which implies that $k \le d(b' - n)$. Hence, by the inequality $k \ge b - 2$, we obtain

$$\text{if } d = 1 \text{ (and } b' = b) \quad \text{then } n \le 2,$$
$$\text{if } d = 2 \text{ (and } b' = a,\ b = 2a) \quad \text{then } n = 1.$$

Moreover, in the case that $d = 1$(and $b' = b$), the arithmetic genus of Y is equals to $\frac{1}{2}(b-1)(b-2)$ and the geometric genus of Y is g. A straightforward calculation shows that $g < \frac{1}{2}(b-1)(b-2)$, i.e. Y is singular, unless $(a, b) = (2, 4)$. If $(a, b) = (2, 4)$ and Y is smooth, then X is isomorphic to Y since X is also smooth. Thus we completes the proof.

(2) By Theorem 2.1 it suffices to exclude the case that $k < 2a$ and Cliffdim$(X) = 2$. Suppose that such a case occurs. Since then X is isomorphic to a smooth plane curve of degree $k + 1$ we obtain

$$(a-1)(b-a) + 1 = g = \frac{1}{2}k(k-1)$$
$$= \begin{cases} (a-1)(2a-1) & \text{if } \quad k = 2a - 1, \\ (a-1)(2a-3) & \text{if } \quad k = 2a - 2. \end{cases}$$

It follows that $(a, b) = (2, 4)$, in which case we have $g = 3$ and $c = k - 2$. Hence the conclusion.

□

5. The case of the trivial bundle

Theorem 5.1. *Let $S = E \times \mathbb{P}^1$ and let X be a smooth curve on S numerically equivalent to $aC_0 + bf$ $(a, b \in \mathbb{Z},\ a \ge 2,\ b \ge 2)$. Then*

$$\text{gon}(X) = \min\{2a, b\}, \quad \text{Cliff}(X) = \text{gon}(X) - 2.$$

Proof. Put $k := \mathrm{gon}(X)$ and $c := \mathrm{Cliff}(X)$. Note that $k \leq \min\{2a, b\}$. We show that the following claim:

Claim. Let $\varphi : X \to \mathbb{P}^1$ be a morphism which is the restriction of a morphism $\psi : S \to \mathbb{P}^1$. Then $\deg \varphi \geq \min\{2a, b\}$.

To see this, let $D \equiv \alpha C_0 + \beta f$ be a fiber of ψ. Then we have

$$f.D \geq 0, \quad C_0.D \geq 0 \quad \text{and} \quad D^2 = 0.$$

Since $e = 0$ we have

$$\alpha \geq 0, \quad \beta \geq 0 \quad \text{and} \quad 2\alpha\beta = 0.$$

Thus $\alpha = 0$ or $\beta = 0$. If $\alpha = 0$, then we can write $D \sim \mathfrak{b} f$, with an effective divisor $\mathfrak{b}$ on E of degree β. Since

$$H^0(E, \mathcal{O}_E(\mathfrak{b})) \simeq H^0(S, \mathcal{O}_S(D)),$$

we obtain by Riemann-Roch theorem

$$\beta = h^0(E, \mathcal{O}_E(\mathfrak{b})) = h^0(S, \mathcal{O}_S(D)) \geq 2.$$

Thus we have

$$\deg \varphi = X.D = a\beta \geq 2a.$$

If $\beta = 0$, then $\alpha \geq 1$ since $D \not\sim 0$. So we have

$$\deg \varphi = X.D = b\alpha \geq b.$$

Hence we have the assertion of the claim.

Let $\varphi : X \to \mathbb{P}^1$ be a morphism of degree k. Suppose that $k < \min\{2a, b\}$. Then, by the claim above, φ is not extended to a morphism from S to $\mathbb{P}^1$. Hence from Serrano's theorem we obtain

$$X^2 \leq \frac{1}{2}(k + 2)^2.$$

On the other hand, since $e = 0$, we have

$$X^2 = 2ab \geq (k + 1)^2.$$

Thus $2(k + 1)^2 \leq (k + 2)^2$, which implies $k = 1$, a contradiction. Hence $k = \min\{2a, b\}$.

It remains to show that $\mathrm{Cliffdim}(X) = 1$. By Theorem 2.1 it suffices to exclude the case that $k = b < 2a$ and X is isomorphic to a smooth plane curve of degree $b + 1$. In such a case we have

$$(a - 1)b + 1 = g = \frac{1}{2}b(b - 1),$$

which implies $(a, b) = (1, 2)$, contradicting our assumption. Hence the conclusion. □

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