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Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field

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Abstract

This paper presents a rigorous derivation of an N-body Schrödinger Hamiltonian from taking a weak coupling limit of a renormalized Hamiltonian describing an interaction of N-spineless particles and a massive scalar field *at the same time* as an ultraviolet cut-off is removed. In particular, in the case where the space dimension equals three, the Yukawa potential appears in the N-body Schrödinger Hamiltonian.

1 INTRODUCTION

In this paper we are concerned with a model of a quantum N-particle system interacting with a quantutized massive scalar field. The main result is a mathematically rigorous derivation, from the point of view of the operator theory, of an N-body Schrödinger Hamiltonian by taking a weak coupling (scaling) limit of the Hamiltonian of the model with an ultraviolet cut-off ([1]) *at the same time* as the ultraviolet cut-off is removed ([2]). Such kind of scaling limits of Hamiltonians, but with ultraviolet cut-off fixed, have been discussed in [3,4,5] and references therein. A new feature in our approach is in taking a scaling limit and a removal of the ultraviolet cut-off simultaneously. For mathematical generality, we consider the case where the N-particles move in d-dimensional space, d=1,2,3. The total Hamiltonian H we analyze in this paper acts in the tensor product of Hilbert spaces $\underbrace{L^2(\mathbb{R}^d)\otimes ... \otimes L^2(\mathbb{R}^d)}_{N} \cong L^2(\mathbb{R}^{dN})$ and the Boson Fock space $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^d))$ over $L^2(\mathbb{R}^d)$, and has the following form

$$H = -\frac{1}{2m}\Delta \otimes I + I \otimes H_b - g\phi(\Lambda),$$

where H_b denotes the free Hamiltonian in $\mathcal{F}$, Δ the Laplacian in $L^2(\mathbb{R}^{dN})$, $\phi(\Lambda)$ a scalar field with an ultraviolet cut-off parameterized by $\Lambda > 0$, and $g \in \mathbb{R}$, $m > 0$ a coupling constant, the mass of particles, respectively. Let

$$H(\Lambda) = -\frac{1}{2m}\Delta \otimes I + \Lambda^2 I \otimes H_b - g\Lambda\phi(\Lambda^\alpha), \quad \alpha > 0.$$

We are concerned with the asymptotic behavior of $H(\Lambda)$ with a renormalization term subtracted as $\Lambda \to \infty$ in the strong resolvent sense, and determine a range of the parameter α, in which $H(\Lambda)$ converges to an N-body Schrödinger Hamiltonian. In the case where we take Λ's in the coefficients of $I \otimes H_b$ and ϕ infinity with Λ^α in $\phi(\Lambda^\alpha)$ replaced by a fixed parameter, this limit is called a "weak coupling limit". Conversely, in the case where Λ^α in $\phi(\Lambda^\alpha)$ tends to infinity with the other Λ's replaced by fixed parameters, the limit is called a "removal of the ultraviolet cut-off". It is well known that both of these limits exist in some sense ([1,2]). In this paper we take the weak coupling limit and the removal of the ultraviolet cut-off simultaneously. The basic idea to analyze $H(\Lambda)$ is to construct a dressing transformation ([2,6]). By using the dressing transformation and an abstract theory discussed in [7], we prove our main result in Theorem 3.10; assume that $0 < \alpha < \frac{2}{d+1}$, then

$$s - \lim_{\Lambda \to \infty} (H(\Lambda) - g^2 N E(\Lambda^\alpha) - z)^{-1} = \left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\infty) - z\right)^{-1} \otimes P_\Omega, \qquad (1.\ 1)$$

where $E(\Lambda^\alpha)$ is a renormalization term which goes to minus infinity as $\Lambda \to \infty$, $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large, P_Ω a projection operator on $\mathcal{F}$ and $V_{eff}(\infty)$ a multiplication operator in $L^2(\mathbb{R}^{dN})$. In particular, in the case $d = 3$, $V_{eff}(\infty)$ coincides with the Yukawa potential with the mass of the scalar field $\mu > 0$:

$$V_{eff}(x^1, ..., x^N; \infty) = -\frac{1}{4\pi} \sum_{i<j}^{N} \frac{e^{-\mu|x^i - x^j|}}{|x^i - x^j|}.$$

Moreover in the case $d = 1$, $V_{eff}(\infty)$ is as follows:

$$V_{eff}(x^1, ..., x^N; \infty) = -\frac{1}{2} \sum_{i<j}^{N} \frac{e^{-\mu|x^i - x^j|}}{\mu}.$$

Consequently the classical Schrödinger Hamiltonian with the Yukawa potential may be interpreted as the approximation in the sense proposed in (1.1).

The article is organized as follows. In section 2, for mathematical maximum generality, we specify our model and consider its asymptotic behavior in an abstract form. In section 3, we construct a dressing transformation which implements a unitary equivalence between the Hamiltonian $H(\Lambda)$ and a special version of an operator discussed in section 2. Using the dressing transformation, we present an asymptotic behavior of $H(\Lambda)$ with renormalization term $E(\Lambda^\alpha)$ subtracted in the strong resolvent sense. In section 4, we consider an asymptotic behavior of the ground state of $H(\Lambda)$ and spectral concentration. In section 5, we give concluding remarks.

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2 AN ABSTRACT SCALING LIMIT

In this section, for the succeeding discussion and for the sake of self-consistency of this paper, we review a theory of scaling limit of self-adjoint operators acting in the tensor product of two Hilbert spaces discussed in [7] with a little modification, which is a mathematical generalization that can be applied to our model. To begin with, let us introduce preliminary notations. The scalar product and the associated norm of a Hilbert space $\mathcal{L}$ are denoted by $(*,\ \cdot)_{\mathcal{L}}$ and $||\cdot||_{\mathcal{L}}$, respectively; the scalar product is linear in $\cdot$, antilinear in $*$. The domain and the spectrum of an operator T are denoted by $D(T)$ and $\sigma(T)$, respectively. Moreover the uniform operator norm of bounded operators on $\mathcal{L}$ is denoted by $||\cdot||_{B(\mathcal{L})}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, respectively. Put $\mathcal{H}\otimes\mathcal{K}=\mathcal{X}$. Suppose that an operator A (resp.B) is a non-negative self-adjoint operator in $\mathcal{H}$ (resp.$\mathcal{K}$), and Ker $B\neq\{0\}$. We denote the projection from $\mathcal{K}$ onto Ker B by P_0. We suppose that a family of symmetric operators $\{C_\Lambda\}_{\Lambda>0}$ in $\mathcal{X}$ admits the following conditions:

(i) *For any $\epsilon>0$ there exists Λ' such that for all $\Lambda>\Lambda'$, $D(C_\Lambda)\subset D(A\otimes I)\cap D(I\otimes B)=D(A\otimes I+I\otimes B)\equiv D_{AB}$ with*

$$||C_\Lambda\Phi||_{\mathcal{X}}\leq\epsilon||(A\otimes I+I\otimes B)\Phi||_{\mathcal{X}}+b(\epsilon)||\Phi||_{\mathcal{X}},$$

where $b(\epsilon)>0$ is a constant independent of $\Lambda>\Lambda'$.

(ii) *There exists a symmetric operator C in $\mathcal{X}$ such that for $z\in\mathbb{C}\setminus[0,\infty)$,*

$$s-\lim_{\Lambda\to\infty}C_\Lambda(A\otimes I+\Lambda I\otimes B-z)^{-1}=C(A-z)^{-1}\otimes P_0.$$

Proposition 2.1 **([7,Theorem 2.1])** *Let operators A, B, C_Λ and C be given as above. Then the following* $(1)\sim(4)$ *hold.*

(1) *For $\Lambda>\Lambda_0$ with some Λ_0, $K_\Lambda\equiv A\otimes I+\Lambda I\otimes B+C_\Lambda$ is self-adjoint on D_{AB} and* *uniformly bounded from below. Moreover it is essentially self-adjoint on any core of* $A \otimes I + I \otimes B$.

(2) $K_\infty \equiv A \otimes I + (I \otimes P_0)C(I \otimes P_0)$ *is self-adjoint on* $D(A \otimes I)$ *and bounded from below. Moreover it is essentially self-adjoint on any core of* $A \otimes I$.

(3) *For* $z \in \cap_{\Lambda > \Lambda_0} \rho(K_\Lambda) \bigcap \rho(K_\infty)$ *(*$\rho(T)$ *denotes the resolvent set of an operator* T*),*

$$s - \lim_{\Lambda \to \infty} (K_\Lambda - z)^{-1} = (K_\infty - z)^{-1}(I \otimes P_0).$$

(4) K_∞ *is reduced by* $\mathcal{H} \otimes \mathrm{Ker} B$. *Moreover the following holds:*

$$\overline{\lim}_{\Lambda \to \infty} \inf \sigma(K_\Lambda) \leq \inf \sigma(K_\infty|_{\mathcal{H} \otimes \mathrm{Ker} B}). \tag{2. 1}$$

3 DEFINITION OF A HAMILTONIAN AND A DRESSING TRANSFORMATION

In this section we define a Hamiltonian $H(\Lambda)$ describing an interaction of N-particles and a scalar field with an ultraviolet cut-off. To analyze an asymptotic behavior of the Hamiltonian, we construct a dressing transformation. The Boson Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^d)$ is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \left[\otimes_s^n L^2(\mathbb{R}^d) \right],$$

where $\otimes_s^0 L^2(\mathbb{R}^d) = \mathbb{C}$ and $\otimes_s^n L^2(\mathbb{R}^d)$ $(n \geq 1)$ is the n-fold symmetric tensor product of $L^2(\mathbb{R}^d)$. The vacuum vector is defined by $\Omega = \{1, 0, 0, 0..\}$ and the projection from $\mathcal{F}$ onto the subspace $\{\alpha\Omega | \alpha \in \mathbb{C}\}$ is denoted by P_Ω. Set

$$\mathcal{F}^\infty(\mathcal{W}) = \{\{\Psi_n\}_{n=0}^\infty \in \mathcal{F} | \Psi_n = 0 \text{ for all but finitely many } n'\text{s}\}.$$

The creation operator and the annihilation operator are denoted by $a^\dagger(f)$ and $a(g)$, $f, g \in L^2(\mathbb{R}^d)$, respectively. Formally we write

$$a^\sharp(f) = \int a^\sharp(k) f(k) dk, \quad f \in L^2(\mathbb{R}^d).$$

The operators $a^\sharp(f)$ leave $\mathcal{F}^\infty(\mathcal{W})$ invariant and satisfy

$$
\begin{aligned}
[a(f), a^\dagger(g)]\Phi &= (\bar{f}, g)_{L^2(\mathbb{R}^d)}\Phi \\
[a^\sharp(f), a^\sharp(g)]\Phi &= 0 \\
(a(f)\Phi, \Psi)_{\mathcal{F}} &= \left(\Phi, a^\dagger(\bar{f})\Psi\right)_{\mathcal{F}}, f, g \in L^2(\mathbb{R}^d), \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).
\end{aligned}
$$

The scalar field $\phi(x, f), (x \in \mathbb{R}^d)$ and the momentum conjugate $\Pi(x, g), (x \in \mathbb{R}^d)$ are defined by

$$
\begin{aligned}
\phi(x, f) &= \frac{1}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) \frac{f(-k)e^{-ikx}}{\sqrt{\omega(k)}} + a(k) \frac{f(k)e^{ikx}}{\sqrt{\omega(k)}} \right\} dk, f/\sqrt{\omega} \in L^2(\mathbb{R}^d), \\
\Pi(x, g) &= \frac{i}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) f(-k)\sqrt{\omega(k)} e^{-ikx} - a(k) f(k) \sqrt{\omega(k)} e^{ikx} \right\} dk, f\sqrt{\omega} \in L^2(\mathbb{R}^d),
\end{aligned}
$$

where $\omega(k) = \sqrt{k^2 + \mu^2}$, $\mu > 0$. In particular, for an indicator function Ξ_Λ:

$$
\Xi_\Lambda(k) = \begin{cases} 1, & |k| < \Lambda, \\ 0, & |k| \geq \Lambda, \end{cases}
$$

we set

$$
\phi(x, \Xi_\Lambda) = \phi(x, \Lambda).
$$

Consider the semi-group generated by the multiplication operator ω in $L^2(\mathbb{R}^d)$. Its second quantization $\Gamma(e^{-t\omega}), t \geq 0$, is a strongly continuous contraction semi-group on $\mathcal{F}$ (cf.[8]). The free Hamiltonian H_b is defined to be its generator:

$$
\Gamma(e^{-t\omega}) = e^{-tH_b}.
$$

Formally we may write

$$
H_b = \int \omega(k) a^\dagger(k) a(k) dk.
$$

Thus we have the following commutation relations:

$$
[\phi(x, f), \phi(y, g)] = 0, \quad f/\sqrt{\omega}, g/\sqrt{\omega} \in L^2(\mathbb{R}^d),
$$

$$\begin{aligned} [\Pi(x,f),\Pi(y,g)] &= 0, \quad \sqrt{\omega}f, \sqrt{\omega}g \in L^2(\mathbb{R}^d), \\ [\phi(x,f),\Pi(y,g)] &= \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d} f(-k)g(k)e^{-ik(x-y)}dk, \quad f/\sqrt{\omega}, \sqrt{\omega}g \in L^2(\mathbb{R}^d), \end{aligned}$$

on $\mathcal{F}^\infty(\mathcal{W})$ and on $D(H_b^{\frac{3}{2}})$

$$\begin{aligned} [H_b,\phi(x,f)] &= -i\Pi(x,f), \quad f/\sqrt{\omega}, \sqrt{\omega}f \in L^2(\mathbb{R}^d), \\ [H_b,\Pi(x,f)] &= i\phi(x,\omega^2 f), \quad \sqrt{\omega}f, \omega\sqrt{\omega}f \in L^2(\mathbb{R}^d). \end{aligned}$$

The total Hamiltonian $H(\Lambda)$ of our model acts in

$$\mathcal{L} = \underbrace{L^2(\mathbb{R}^d)\otimes ... \otimes L^2(\mathbb{R}^d)}_{N} \bigotimes \mathcal{F} \cong \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F}dx^1...dx^N$$

and is defined as

$$H(\Lambda) = -\frac{1}{2m}\Delta\otimes I + \Lambda^2 I\otimes H_b - g\Lambda\phi(\Lambda^\alpha), \quad \Lambda > 0, \quad \alpha > 0,$$

where

$$\phi(\Lambda^\alpha) = \int_{\mathbb{R}^{dN}}^{\oplus} \sum_{i=1}^{N} \phi(x^i,\Lambda^\alpha)dx^1...dx^N.$$

For later use, we have introduced the scaling parameters Λ and α; a range of α is determined in the rest of this section. Put

$$H_0 = -\frac{1}{2m}\Delta\otimes I + I\otimes H_b.$$

Proposition 3.1 ([2]) *The operator $H(\Lambda)$ is self-adjoint on $D(H_0)$ and bounded from below. Moreover it is essentially self-adjoint on any core of H_0.*

It is well known that for each $(x^1,...,x^N) \in \mathbb{R}^{dN}$, $\sum_{j=1}^{N}\Pi\left(x^j, \frac{\Xi_{\Lambda^\alpha}}{\omega^2}\right)$ is essentially self-adjoint on $\mathcal{F}^\infty(\mathcal{W})$. We denote its self-adjoint extension by the same symbol. Let

$$U_\Lambda(g) = \int_{\mathbb{R}^{dN}}^{\oplus} \exp\left(-i\frac{g}{\Lambda}\sum_{j=1}^{N}\Pi\left(x^j, \frac{\Xi_{\Lambda^\alpha}}{\omega^2}\right)\right) dx^1...dx^N.$$

We define a renormalization term $E(\Lambda^\alpha)$ by

$$E(\Lambda^\alpha) = -\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\Xi_{\Lambda^\alpha}(k)}{k^2+\mu^2}dk.$$

Set

$$\mathcal{L}^\infty \equiv \left[C_0^\infty(\mathbb{R}^{dN})\hat{\otimes}\mathcal{F}^\infty(\mathcal{W})\right]\bigcap D(I\otimes H_b),$$

where $\hat{\otimes}$ denotes the algebraic tensor product. The following lemma is the key lemma in this paper.

Lemma 3.2 *The unitary operator $U_\Lambda(g)$ maps $\mathcal{L}^\infty$ into $D(H(\Lambda))$ with*

$$\begin{aligned} U_\Lambda(g)^{-1}(H(\Lambda)-g^2NE(\Lambda^\alpha))U_\Lambda(g) &= \frac{1}{2m}\sum_{j=1}^N\sum_{\mu=1}^d\left(p_\mu^j\otimes I-\frac{g}{\Lambda}\phi_\mu^j(\Lambda^\alpha)-\frac{g^2}{\Lambda^2}V_\mu^j(\Lambda^\alpha)\otimes I\right)^2 \\ &\quad +\Lambda^2 I\otimes H_b+g^2V_{eff}(\Lambda^\alpha)\otimes I, \end{aligned} \tag{3. 1}$$

where $p_\mu^j = -iD_{x_\mu^j}, \mu=1,...,d, j=1,...,N$, $(x^j=(x_1^j,...,x_d^j)\in\mathbb{R}^d)$ denotes the generalized L^2-derivative.

$$\begin{aligned} \phi_\mu^j(\Lambda^\alpha) &= \int_{\mathbb{R}^{dN}}^\oplus \phi_\mu(x^j,\Lambda^\alpha)dx^1...dx^N, j=1,...,N,\mu=1,...,d, \\ \phi_\mu(x^j,\Lambda^\alpha) &= \frac{1}{\sqrt{2(2\pi)^d}}\int\left\{a^\dagger(k)\frac{\Xi_{\Lambda^\alpha}(k)k_\mu e^{-ikx^j}}{\sqrt{\omega(k)^3}}+a(k)\frac{\Xi_{\Lambda^\alpha}(k)k_\mu e^{ikx^j}}{\sqrt{\omega(k)^3}}\right\}dk, \end{aligned}$$

$V_{eff}(\Lambda^\alpha)$ and $V_\mu^j(\Lambda^\alpha)$ are multiplication operators in $L^2(\mathbb{R}^{dN})$:

$$\begin{aligned} V_{eff}(x^1,...,x^N;\Lambda^\alpha) &= -\frac{1}{2}\sum_{i=1,j=1,i\neq j}^N\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\Xi_{\Lambda^\alpha}(k)e^{-ik(x^i-x^j)}}{\omega(k)^2}dk, \\ V_\mu^j(x^1,...,x^N;\Lambda^\alpha) &= \sum_{i=1,i\neq j}^N\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{k_\mu\Xi_{\Lambda^\alpha}(k)e^{-ik(x^j-x^i)}}{\omega(k)^3}dk, \\ & \quad j=1,...,N,\mu=1,...,d. \end{aligned}$$

Proof: Note that $V_{eff}(\Lambda^\alpha)$ is bounded operator on $L^2(\mathbb{R}^{dN})$. By [6,Theorem 3.3], $U_\Lambda(g)$ maps $D(I\otimes H_b)$ onto itself with

$$U_\Lambda(g)^{-1}(-g\Lambda\phi(\Lambda^\alpha)+\Lambda^2I\otimes H_b)U_\Lambda(g) = \Lambda^2I\otimes H_b+g^2V_{eff}(\Lambda^\alpha)\otimes I+g^2NE(\Lambda^\alpha). \tag{3. 2}$$

On the other hand, it can be easily seen that on $\mathcal{L}^\infty$

$$U_\Lambda(g)^{-1}(p^j_\mu \otimes I)U_\Lambda(g) = p^j_\mu \otimes I - \frac{g}{\Lambda}\phi^j_\mu(\Lambda^\alpha) - \frac{g^2}{\Lambda^2}V^j_\mu(\Lambda^\alpha) \otimes I.$$

We have thus proved the lemma. □

Remark 3.3 *The renormalization term $E(\Lambda^\alpha)$ coincides with that of [2]. If $d = 2,3$, $\lim_{\Lambda\to\infty} E(\Lambda^\alpha) = -\infty$, while, if $d = 1$, $\lim_{\Lambda\to\infty} E(\Lambda^\alpha) > -\infty$. However, to proceed our argument systematically, even in the case where $d = 1$, we subtract the renormalization term from the total Hamiltonian, and do not distinguish between them.*

Let the right hand side of (3.1) be $H^D(\Lambda)$ and define a symmetric operator $C(\Lambda^\alpha)$ by

$$H^D(\Lambda) = -\frac{1}{2m}\Delta \otimes I + \Lambda^2 I \otimes H_b + C(\Lambda^\alpha).$$

Moreover, for notational simplicity, we introduce the following notations:

$$\begin{aligned} Q_1(\Lambda^\alpha) &= \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d}\left((p^j_\mu \otimes I)q^j_\mu(\Lambda^\alpha) + q^j_\mu(\Lambda^\alpha)(p^j_\mu \otimes I)\right), \\ Q_2(\Lambda^\alpha) &= \frac{1}{2m}\sum_{j=1}^{N}\sum_{\mu=1}^{d} q^j_\mu(\Lambda^\alpha)^2, \\ q^j_\mu(\Lambda^\alpha) &= -\phi^j_\mu(\Lambda^\alpha) - \frac{g}{\Lambda}V^j_\mu(\Lambda^\alpha) \otimes I. \end{aligned}$$

Thus in terms of the operators $Q_1(\Lambda^\alpha), Q_2(\Lambda^\alpha)$ and $V_{eff}(\Lambda^\alpha)$, we may write

$$C(\Lambda^\alpha) = g^2 V_{eff}(\Lambda^\alpha) \otimes I + \frac{g}{\Lambda}Q_1(\Lambda^\alpha) + \frac{g^2}{\Lambda^2}Q_2(\Lambda^\alpha).$$

We write $f(\Lambda) \sim \Lambda^k$ if $\frac{f(\Lambda)}{\Lambda^k}$ is bounded as $\Lambda \to \infty$.

Lemma 3.4 *It holds that $D(Q_1(\Lambda^\alpha)) \supset D(H_0)$ and $D(Q_2(\Lambda^\alpha)) \supset D(H_0)$. Moreover the following inequalities hold for $\Phi \in D(H_0)$*

$$||Q_1(\Lambda^\alpha)\Phi||_{\mathcal{L}} \leq (\epsilon_1(\Lambda^\alpha) + \epsilon_2(\Lambda^\alpha))||(H_0 + I)\Phi||_{\mathcal{L}}, \tag{3. 3}$$

$$||Q_2(\Lambda^\alpha)\Phi||_{\mathcal{L}} \leq (\eta_1(\Lambda^\alpha) + \eta_2(\Lambda^\alpha) + \eta_3(\Lambda^\alpha))||(H_0 + I)\Phi||_{\mathcal{L}}, \tag{3. 4}$$

where $\epsilon_1(\Lambda^\alpha), \epsilon_2(\Lambda^\alpha), \eta_1(\Lambda^\alpha), \eta_2(\Lambda^\alpha)$ *and* $\eta_3(\Lambda^\alpha)$ *are positive numbers depending on* Λ *such that*

$$\epsilon_1(\Lambda^\alpha) \sim \Lambda^{\frac{\alpha(d+1)}{2}}, \epsilon_2(\Lambda^\alpha) \sim \Lambda^{\alpha(d-1)-1},$$
$$\eta_1(\Lambda^\alpha) \sim \Lambda^{\frac{\alpha(2d-1)}{2}}, \eta_2(\Lambda^\alpha) \sim \Lambda^{\frac{\alpha(3d-5)}{2}-1}, \eta_3(\Lambda^\alpha) \sim \Lambda^{2\alpha(d-2)-2}.$$

Proof: In the proof we write operators of the form $A \otimes I$ in $\mathcal{L}$ as A and set for $k = (k_1, ..., k_d) \in \mathbb{R}^d$

$$\begin{aligned} f_\mu(k) &= \frac{\Xi_{\Lambda^\alpha}(k) k_\mu}{\omega(k)}, \\ f_{\mu\mu}(k) &= \frac{\Xi_{\Lambda^\alpha}(k) k_\mu^2}{\omega(k)}, \mu = 1, ..., d. \end{aligned}$$

Moreover

$$\begin{aligned} \pi_{\mu\mu}^j(\Lambda^\alpha) &= i \int_{\mathbb{R}^{dN}}^{\oplus} \pi_{\mu\mu}(x^j, \Lambda^\alpha) dx^1 ... dx^N, \\ \pi_{\mu\mu}(x^j, \Lambda^\alpha) &= \frac{i}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) \frac{\Xi_{\Lambda^\alpha}(k) k_\mu^2 e^{-ikx^j}}{\sqrt{\omega(k)^3}} - a(k) \frac{\Xi_{\Lambda^\alpha}(k) k_\mu^2 e^{ikx^j}}{\sqrt{\omega(k)^3}} \right\} dk, \\ V_{\mu\mu}^j(x^1, ..., x^N; \Lambda^\alpha) &= \sum_{i=1, i\neq j}^{N} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{k_\mu^2 \Xi_{\Lambda^\alpha}(k) e^{-ik(x^j - x^i)}}{\omega(k)^3} dk, \\ & \quad j = 1, ..., N, \mu = 1, ..., d. \end{aligned}$$

Set $||\sqrt{\omega}^n g||_{L^2(\mathbb{R}^d)} \equiv ||g||_n$. Then note that

$$\begin{aligned} &||f_\mu||_n \sim \Lambda^{\frac{\alpha(n+d)}{2}}, (n+d \neq 0), \\ &||f_\mu||_n \sim \sqrt{\alpha \log \Lambda}, (n+d = 0), \\ &||f_{\mu\mu}||_n \sim \Lambda^{\frac{\alpha(n+d+2)}{2}}, (n+d+2 \neq 0), \\ &||f_{\mu\mu}||_n \sim \sqrt{\alpha \log \Lambda}, (n+d+2 = 0). \end{aligned} \tag{3. 5}$$

The following inequalities are well known ([9,Lemmas 2.1,2.4])

$$||\phi(x, f)\Phi||_{\mathcal{F}} \leq A \left(2||f||_{-2} ||H_b^{\frac{1}{2}} \Phi||_{\mathcal{F}} + ||f||_{-1} ||\Phi||_{\mathcal{F}} \right),$$

$$\begin{aligned}||\pi(x,f)\Phi||_{\mathcal{F}} &\leq A\left(2||f||_0||H_b^{\frac{1}{2}}\Phi||_{\mathcal{F}}+||f||_1||\Phi||_{\mathcal{F}}\right), \quad \Phi\in D(H_b^{\frac{1}{2}})\\ ||[(H_b+I)^{\frac{1}{2}},\phi(x,f)]\Psi||_{\mathcal{F}} &\leq B\left(||f||_0||H_b^{\frac{1}{2}}\Psi||_{\mathcal{F}}+||f||_1||\Psi||_{\mathcal{F}}\right),\\ ||[(H_b+I)^{\frac{1}{2}},\pi(x,f)]\Psi||_{\mathcal{F}} &\leq B\left(||f||_2||H_b^{\frac{1}{2}}\Psi||_{\mathcal{F}}+||f||_3||\Psi||_{\mathcal{F}}\right), \quad \Psi\in D(H_b), \quad (3.\ 6)\end{aligned}$$

where

$$\begin{aligned}A &= \frac{1}{\sqrt{2(2\pi)^d}},\\ B &= \frac{1}{\sqrt{2(2\pi)^d}}\left(\frac{1}{2\pi}\int_0^\infty\frac{\sqrt{\lambda}}{(\lambda+1)^2}d\lambda\right).\end{aligned}$$

It can be directly seen that there exists $\alpha(\Lambda^\alpha)$ and $\beta(\Lambda^\alpha)$ so that

$$\begin{aligned}|V_\mu^j(x^1,...,x^N;\Lambda^\alpha)| &\leq \alpha(\Lambda^\alpha),\\ |V_{\mu\mu}^j(x^1,...,x^N;\Lambda^\alpha)| &\leq \beta(\Lambda^\alpha), j=1,...,N,\mu=1,...,d,\end{aligned}$$

with $\alpha(\Lambda^\alpha)\sim\Lambda^{\alpha(d-2)}$ and $\beta(\Lambda^\alpha)\sim\Lambda^{\alpha(d-1)}$. We first prove (3.3). Since, on $D(H_0)$,

$$[p_\mu^j,q_\mu^j(\Lambda^\alpha)] = -\pi_{\mu\mu}^j(\Lambda^\alpha)-\frac{g}{\Lambda}V_{\mu\mu}^j(\Lambda^\alpha),$$

in terms of (3.6), we have that for $\Phi\in D(H_0)$,

$$\begin{aligned}&||Q_1(\Lambda^\alpha)\Phi||_{\mathcal{L}}\\ &\leq \frac{1}{2m}\sum_{j=1}^N\sum_{\mu=1}^d\left(2||\phi_\mu^j(\Lambda^\alpha)p_\mu^j\Phi||_{\mathcal{L}}+\frac{2g}{\Lambda}||V_\mu^j(\Lambda^\alpha)p_\mu^j\Phi||_{\mathcal{L}}+||\pi_{\mu\mu}^j(\Lambda^\alpha)||_{\mathcal{L}}+\frac{g}{\Lambda}||V_{\mu\mu}^j(\Lambda^\alpha)\Phi||_{\mathcal{L}}\right)\\ &\leq \sum_{\mu=1}^d\left\{\left(||f_\mu||_{-2}+||f_\mu||_{-1}+||f_{\mu\mu}||_{-2}+||f_{\mu\mu}||_{-1}\right)+\left(\frac{\alpha(\Lambda^\alpha)}{\Lambda}+\frac{\beta(\Lambda^\alpha)}{\Lambda}\right)\right\}\\ &\times constant\times||(H_0+I)\Phi||_{\mathcal{L}}. \qquad (3.\ 7)\end{aligned}$$

Due to (3.5), in the first term on the right hand side of (3.7), the term $||f_{\mu\mu}||_{-1}\sim\Lambda^{\frac{\alpha(d+1)}{2}}$ is the highest order with respect to Λ as $\Lambda\to\infty$, and so is $\frac{\beta(\Lambda^\alpha)}{\Lambda}\sim\Lambda^{\alpha(d-1)-1}$ in the second term. Then (3.3) follows. Next we prove (3.4). As is seen in the proof of (3.3), one can easily see that

$$\begin{aligned}
&||Q_2(\Lambda^\alpha)\Phi||_{\mathcal{L}} \\
&\leq \frac{1}{2m}\sum_{j=1}^N\sum_{\mu=1}^d \left(||\phi_\mu^j(\Lambda^\alpha)^2\Phi||_{\mathcal{L}} + \frac{2g}{\Lambda}||\phi_\mu^j(\Lambda^\alpha)V_\mu^j(\Lambda^\alpha)\Phi||_{\mathcal{L}} + \frac{g^2}{\Lambda^2}||V_\mu^j(\Lambda^\alpha)^2\Phi||_{\mathcal{L}}\right), \\
&\leq \sum_{\mu=1}^d \left\{\left(||f_\mu||_{-2}\sum_{n=-2}^{1}||f_\mu||_n + ||f_\mu||_{-1}\sum_{n=-2}^{-1}||f_\mu||_n\right) + \frac{\alpha(\Lambda^\alpha)}{\Lambda}\left(||f_\mu||_{-2} + ||f_\mu||_{-1}\right) + \frac{\alpha(\Lambda^\alpha)^2}{\Lambda^2}\right\} \\
&\times constant \times ||(H_0 + I)\Phi||_{\mathcal{L}}. \qquad (3.\ 8)
\end{aligned}$$

Due to (3.5), in the first term on the right hand side of (3.8), the term $||f_\mu||_{-2}||f_\mu||_1 \sim \Lambda^{\frac{\alpha(2d-1)}{2}}$ is the highest order, and so is $\frac{\alpha(\Lambda^\alpha)}{\Lambda}||f_\mu||_{-1} \sim \Lambda^{\frac{\alpha(3d-5)}{2}-1}$ in the second term. Moreover in the third term, $\frac{\alpha(\Lambda^\alpha)^2}{\Lambda^2} \sim \Lambda^{2\alpha(d-2)-2}$. Thus (3.4) follows. This completes the proof. □

Lemma 3.5 *It holds that $D(V_{eff}(\Lambda^\alpha)) \supset D(-\Delta)$ and, for any $\epsilon > 0$, there exists $b(\epsilon) > 0$ such that for $\Phi \in D(-\Delta)$ and for any $\Lambda > \Lambda_0$ with some $\Lambda_0 > 0$,*

$$||V_{eff}(\Lambda^\alpha)\Phi||_{L^2(\mathbb{R}^{dN})} \leq \epsilon||-\Delta\Phi||_{L^2(\mathbb{R}^{dN})} + b(\epsilon)||\Phi||_{L^2(\mathbb{R}^{dN})}. \qquad (3.\ 9)$$

Proof: For each $(x^1, ..., x^N) \in \mathbb{R}^{dN}$, due to the spherically symmetry of $\frac{\Xi_{\Lambda^\alpha}}{\omega^2}$, one can see that

$$V_{eff}(x^1; ..., x^N; \Lambda^\alpha) = -\frac{1}{\sqrt{(2\pi)^d}}\sum_{i<j}^N \frac{1}{|x^i - x^j|^{\frac{d-1}{2}}}\int_0^{\Lambda^\alpha}\frac{r^{\frac{d-1}{2}}}{r^2+\mu^2}\sqrt{r|x^i - x^j|}J_{\frac{d-2}{2}}(r|x^i - x^j|)dr, \qquad (3.\ 10)$$

where J_ν is the Bessel function:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n}.$$

Let

$$G(r) = \frac{r^{\frac{d-1}{2}}}{r^2+\mu^2}.$$

At first, we prove (3.9) in the cases where $d = 1, 2$. Then, since $G \in L^1(\mathbb{R}^d)$, by a fundamental estimate of the Bessel functions (see [10,§ 7.21])

$$\sup_{u\in[0,\infty)} \left|\sqrt{u}J_\nu(u)\right| \equiv \tilde{J}_\nu < \infty,$$

we see that

$$|V_{eff}(x^1, ..., x^N; \Lambda^\alpha)| \leq \frac{1}{\sqrt{(2\pi)^d}}\tilde{J}_{\frac{d-2}{2}}\sum_{i<j}^{N}\frac{1}{|x^i - x^j|^{\frac{d-1}{2}}}\int_0^\infty G(r)dr. \qquad (3.\ 11)$$

Since, with the help of a direct extension of [8.Theorem X.16], one can easily see that the right hand side of (3.11) is infinitesimally small with respect to the Laplacian Δ in $L^2(\mathbb{R}^{dN})$, $d = 1, 2$. Thus (3.9) follows. Next we consider the case where $d = 3$. We note that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin x}{\sqrt{x}}.$$

Hence, changing the variables in the integrand (3.10) and using the contour integrals along with closed curves γ_{ij}, $1 \leq i, j \leq N$,

$$\gamma_{ij} = \{z \in \mathbb{R}| - \Lambda^\alpha|x^i - x^j| \leq z \leq \Lambda^\alpha|x^i - x^j|\} \bigcup \{z \in \mathbb{C}|\ z = \Lambda^\alpha|x^i - x^j|e^{i\theta}, 0 \leq \theta \leq \pi\},$$

we see that

$$\begin{aligned} V_{eff}(x^1, ..., x^N; \Lambda^\alpha) &= -\frac{1}{4\pi^2}\Im\sum_{i<j}^{N}\frac{1}{|x^i - x^j|}\int_{-\Lambda^\alpha|x^i-x^j|}^{\Lambda^\alpha|x^i-x^j|}\frac{u}{u^2 + |x^i - x^j|^2\mu^2}e^{iu}du \\ &= -\frac{2\pi}{4\pi^2}\sum_{i<j}^{N}\frac{1}{|x^i - x^j|}\mathrm{Res}\left(\frac{ue^{iu}}{u^2 + |x^i - x^j|^2\mu^2}; i\mu|x^i - x^j|\right) \\ &\quad +\frac{1}{4\pi^2}\Im\sum_{i<j}^{N}\frac{1}{|x^i - x^j|}\int_0^\pi \frac{\Lambda^{2\alpha}|x^i - x^j|^2 ie^{2i\theta}e^{i\Lambda^\alpha|x^i-x^j|e^{i\theta}}}{\Lambda^{2\alpha}|x^i - x^j|^2e^{2i\theta} + |x^i - x^j|^2\mu^2}d\theta. \end{aligned}$$

It can be seen that

$$\left|\int_0^\pi \frac{\Lambda^{2\alpha}|x^i - x^j|^2 ie^{2i\theta}e^{i\Lambda^\alpha|x^i-x^j|e^{i\theta}}}{\Lambda^{2\alpha}|x^i - x^j|^2e^{2i\theta} + |x^i - x^j|^2\mu^2}d\theta\right| \leq \pi\left|\frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha} - \mu^2}\right|.$$

Hence we have that for each $(x^1, ..., x^N) \in \mathbb{R}^{3N}$,

$$|V_{eff}(x^1, ..., x^N; \Lambda^\alpha)| \leq \frac{1}{4\pi}\sum_{i<j}^{N}\frac{1}{|x^i - x^j|}\left(e^{-\mu|x^i-x^j|} + \left|\frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha} - \mu^2}\right|\right). \qquad (3.\ 12)$$

Since, for $\Lambda > \mu^{\frac{1}{2}}$, the right hand side of (3.12) is infinitesimally small with respect to Δ in $L^2(\mathbb{R}^{3N})$, (3.9) follows. This completes the proof. $\square$

Lemma 3.6 *Assume that α admits as follows*

$$0 < \alpha < \frac{2}{d+1}. \tag{3. 13}$$

Then for any $\Phi \in D(H_0)$,

$$s - \lim_{\Lambda\to\infty} \left(\frac{g}{\Lambda} Q_1(\Lambda^\alpha) + \frac{g^2}{\Lambda^2} Q_2(\Lambda^\alpha) + g^2 V_{eff}(\Lambda^\alpha) \otimes I \right) \Phi = (g^2 V_{eff}(\infty) \otimes I)\Phi,$$

where $V_{eff}(\infty)$ is a multiplication operator in $L^2(\mathbb{R}^{dN})$:

$$V_{eff}(x^1, ..., x^N; \infty) = - \sum_{i=1, j=1, i\neq j}^{N} \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-ik(x_i - x_j)}}{k^2 + \mu^2} dk.$$

In particular, in the case $d = 1$,

$$V_{eff}(x^1, ..., x^N; \infty) = -\frac{1}{2} \sum_{i<j}^{N} \frac{e^{-\mu|x^i - x^j|}}{\mu},$$

in the case $d = 3$,

$$V_{eff}(x^1, ..., x^N; \infty) = -\frac{1}{4\pi} \sum_{i<j}^{N} \frac{e^{-\mu|x^i - x^j|}}{|x^i - x^j|}.$$

Proof: Because of Lemma 3.4, we see that, for $\Phi \in D(H_0)$,

$$g||Q_1(\Lambda^\alpha)\Phi||_{\mathcal{L}} + \frac{g^2}{\Lambda} ||Q_2(\Lambda^\alpha)\Phi||_{\mathcal{L}} \sim \Lambda^{\frac{\alpha(d+1)}{2}}.$$

Hence, due to the assumption (3.13), one can see that

$$\lim_{\Lambda\to\infty} \left(\frac{g}{\Lambda} ||Q_1(\Lambda^\alpha)\Phi||_{\mathcal{L}} + \frac{g^2}{\Lambda^2} ||Q_2(\Lambda^\alpha)\Phi||_{\mathcal{L}} \right) = 0.$$

Next we consider the term $V_{eff}(\Lambda^\alpha)$. In the cases where $d = 1, 2$, by the proof of Lemma 3.5, it can be easily seen that for $\Phi \in D(H_0)$

$$||V_{eff}(\Lambda^\alpha)\Phi - V_{eff}(\infty)\Phi||_{\mathcal{L}} = 0, as\ \Lambda \to \infty. \tag{3. 14}$$

In the case where $d = 3$, from the proof of Lemma 3.5, it follows that there exists a positive constant β so that, for each $(x^1, ..., x^N) \in \mathbb{R}^{3N}$,

$$\begin{aligned} |V_{eff}(x^1, ..., x^N; \Lambda^\alpha) - V_{eff}(x^1, ..., x^N; \infty)| &\leq \frac{\beta}{4\pi} \sum_{i<j}^{N} \frac{1}{|x^i - x^j|} \\ &\equiv \tilde{V}(x^1, ..., x^N). \end{aligned} \quad (3.\ 15)$$

Since $\tilde{V}$ is infinitesimally small with respect to Δ in $L^2(\mathbb{R}^{3N})$, for $\Phi \in D(-\Delta)$,

$$\int_{\mathbb{R}^{3N}} |\tilde{V}(x^1, ..., x^N)|^2 |\Phi(x^1, ..., x^N)|^2 dx < \infty.$$

Let

$$\begin{aligned} \mathbf{H}_{ij} &= \{(x^1, ..., x^N) \in \mathbb{R}^{3N} | x^i = x^j\}, \\ \mathbf{H} &= \bigcup_{i,j=1}^{N} \mathbf{H}_{ij}. \end{aligned}$$

If $x = (x^1, ..., x^N) \notin \mathbf{H}$, from the proof of Lemma 3.5, it follows that

$$\begin{aligned} &|V_{eff}(x^1, ..., x^N; \Lambda^\alpha) - V_{eff}(x^1, ..., x^N; \infty)| \\ &\leq \frac{1}{4\pi^2} \sum_{i<j}^{N} \frac{1}{|x^i - x^j|^2} \frac{\Lambda^\alpha}{|\Lambda^{2\alpha} - \mu^2|} \int_0^\pi \Lambda^\alpha |x^i - x^j| e^{-\Lambda^\alpha |x^i - x^j| \sin\theta} d\theta \\ &\leq \frac{1}{4\pi} \sum_{i<j}^{N} \frac{1}{|x^i - x^j|^2} \frac{\Lambda^\alpha}{|\Lambda^{2\alpha} - \mu^2|} \to 0, \ as\ \Lambda \to \infty. \end{aligned} \quad (3.\ 16)$$

Since the Lebesgue measure of the set $\mathbf{H}$ is zero, combining (3.15) and (3.16), the Lebesgue dominated convergence theorem yields (3.14). This completes the proof. □

Remark 3.7 **(1)** *Assume that* $0 < \alpha < \frac{2}{d+1}$. *From Lemmas 3.4 3.5 and 3.6, it follows that for* $z \in \mathbb{C} \setminus [0, \infty)$, $||C(\Lambda^\alpha)(H_0 - z)^{-1}||_{B(\mathcal{L})}$ *is uniformly bounded in* $\Lambda > \Lambda_0$ *with some* $\Lambda_0 > 0$.

(2) *As is seen in (3.12) in the case where* $d = 3$, *because of the fact that*

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\cos x}{\sqrt{x}},$$

in the case where $d=1$, *we can also estimate* $V_{eff}(x^1,...,x^N;\Lambda^\alpha)$ *as follows*

$$|V_{eff}(x^1,...,x^N;\Lambda^\alpha)| \le \frac{1}{2}\sum_{i<j}^{N}\frac{e^{-\mu|x^i-x^j|}}{\mu} + \frac{1}{2}N(N-1)\left|\frac{\Lambda^\alpha}{\Lambda^{2\alpha}-\mu^2}\right|.$$

Theorem 3.8 *Let* $0<\alpha<\frac{2}{d+1}$. *Then the following (1) and (2) hold.*

(1) *For any* $\epsilon>0$, *there exists* Λ_0 *and* $b(\epsilon)>0$ *such that for all* $\Lambda>\Lambda_0$, $D(C(\Lambda))\supset D(H_0)$ *and for* $\Phi\in D(H_0)$,

$$||C(\Lambda)\Phi||_{\mathcal{L}} \le \epsilon||H_0\Phi||_{\mathcal{L}} + b(\epsilon)||\Phi||_{\mathcal{L}}.$$

(2) *Let* $z\in\mathbb{C}\setminus[0,\infty)$. *Then*

$$s-\lim_{\Lambda\to\infty} C(\Lambda)\left(-\frac{1}{2m}\Delta\otimes I+\Lambda^2 I\otimes H_b - z\right)^{-1} = \left(g^2V_{eff}(\infty)\left(-\frac{1}{2m}\Delta - z\right)^{-1}\right)\otimes P_\Omega.$$

Proof: Because of Lemmas 3.4 and 3.5, (1) follows. By Remark 3.7 and

$$s-\lim_{\Lambda\to\infty}\left(-\frac{1}{2m}\Delta\otimes I + I\otimes H_b - z\right)\left(-\frac{1}{2m}\Delta\otimes I+\Lambda^2 I\otimes H_b - z\right)^{-1} = I\otimes P_\Omega,$$

(2) follows. □

Theorem 3.9 *Let* $0<\alpha<\frac{2}{d+1}$ *and* $0<\Lambda$ *be sufficiently large. Then* $H^D(\Lambda)$ *is self-adjoint on* $D(H_0)$ *and bounded from below uniformly in* Λ *and essentially self-adjoint on any core of* $D(H_0)$. *Moreover the dressing transformation* $U_\Lambda(g)$ *maps* $D(H_0)$ *onto itself and the following unitary equivalence holds on* $D(H_0)$

$$U_\Lambda(g)^{-1}(H(\Lambda)-g^2NE(\Lambda^\alpha))U_\Lambda(g) = H^D(\Lambda). \tag{3. 17}$$

Proof: Note that $H^D(\Lambda)$ is just of the form of the operator K_Λ considered in Proposition 2.1 with the following identifications

$$A=-\frac{1}{2m}\Delta,\quad B=H_b,\quad C_\Lambda=C(\Lambda). \tag{3. 18}$$

By Theorem 3.8 and Proposition 2.1 (1), the first assertion follows. The subset $\mathcal{L}^\infty$ is a core of H_0, same is true for $H^D(\Lambda)$. Since (3.17) holds on $\mathcal{L}^\infty$, it can be easily seen that $U_\Lambda(g)$ maps $D(H^D(\Lambda))$ onto $D(H(\Lambda))$ i.e., $D(H_0)$ onto itself, and (3.17) holds on $D(H_0)$. □

Now we state main theorem in this paper.

Theorem 3.10 *Let* $0 < \alpha < \frac{2}{d+1}$ *and* $z \in \mathbb{C} \setminus [0, \infty)$ *or* $z < 0$ *with* $|z|$ *sufficiently large. Then*

$$s - \lim_{\Lambda\to\infty}(H(\Lambda) - g^2NE(\Lambda^\alpha) - z)^{-1} = \left(-\frac{1}{2m}\Delta + g^2V_{eff}(\infty) - z\right)^{-1} \otimes P_\Omega.$$

Proof: From (3.17) it follows that

$$(H(\Lambda) - g^2NE(\Lambda^\alpha) - z)^{-1} = U_\Lambda(g)(H^D(\Lambda) - z)^{-1}U_\Lambda(g)^{-1}.$$

Since

$$\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\left\|\frac{\Xi_{\Lambda^\alpha}}{\omega^{\frac{3}{2}}}\right\|_{L^2(\mathbb{R}^d)} = 0,$$

it follows that

$$s - \lim_{\Lambda\to\infty} U_\Lambda(g) = I.$$

Note that

$$\text{Ker } H_b = \{\alpha\Omega | \alpha \in \mathbb{C}\}.$$

Thus under the identification as (3.18), by Proposition 2.1 (2), we get the desired result. □

4 GROUND STATE AND SPECTRAL CONCENTRATION

In this section we consider an asymptotic behavior of the ground state for $H(\Lambda)$ and its spectral concentration. Let

$$S(\Lambda) \quad = \quad \inf \sigma(H(\Lambda) - g^2NE(\Lambda)),$$

$$\begin{aligned} S_{eff}(\Lambda) &= \inf \sigma(-\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha)), \\ S_{eff} &= \inf \sigma(-\frac{1}{2m}\Delta + g^2 V_{eff}(\infty)). \end{aligned}$$

Theorem 4.1 *Let* $0 < \alpha < \frac{2}{d+1}$. *Then*

$$\lim_{\Lambda\to\infty} S_{eff}(\Lambda) \le \lim_{\Lambda\to\infty} S(\Lambda) \le S_{eff}. \tag{4. 1}$$

Proof: Noting that $V_{eff}(\Lambda^\alpha)$ commutes $U_\Lambda(g)$, by (3.2), we see that, for $\Phi \in D(H_0)$,

$$\begin{aligned} &(H(\Lambda) - g^2 NE(\Lambda)\Phi, \Phi)_{\mathcal{L}} \\ &= \left(\left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha)\right) \otimes I\Phi, \Phi\right)_{\mathcal{L}} + \left((\Lambda^2 I \otimes H_b)U_\Lambda(g)\Phi, U_\Lambda(g)\Phi\right)_{\mathcal{L}} \\ &\ge \left(\left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha)\right) \otimes I\Phi, \Phi\right)_{\mathcal{L}}, \end{aligned}$$

which implies the first inequality in (4.1). The second inequality follows from (2.1). □

Corollary 4.2 *Assume that* $0 < \alpha < \frac{2}{d+1}$ *and* $d = 1, 2$. *Then*

$$\lim_{\Lambda\to\infty} S(\Lambda) = S_{eff}. \tag{4. 2}$$

Proof: From (3.10) and (3.11) it follows that for $\Phi \in L^2(\mathbb{R}^{dN})$, $z \in \mathbb{C}\setminus[0,\infty)$ and $\epsilon > 0$, there exists $b(\epsilon) > 0$ such that

$$\begin{aligned} &\left\| (V_{eff}(\Lambda^\alpha) - V_{eff}(\infty)) \left(-\frac{1}{2m}\Delta - z\right)^{-1} \Phi \right\|_{L^2(\mathbb{R}^{dN})} \\ &\le \tilde{J}_{\frac{d-2}{2}} \int_{\Lambda^\alpha}^\infty G(r)dr \left(\epsilon + \frac{b(\epsilon)}{|z|}\right) ||\Phi||_{L^2(\mathbb{R}^{dN})}, \end{aligned}$$

which implies that, *in the uniform operator norm*

$$\lim_{\Lambda\to\infty} V_{eff}(\Lambda^\alpha) \left(-\frac{1}{2m}\Delta - z\right)^{-1} = V_{eff}(\infty) \left(-\frac{1}{2m}\Delta - z\right)^{-1}. \tag{4. 3}$$

Let $L_0 = -\frac{1}{2m}\Delta$, $L_\Lambda = L_0 + g^2 V_{eff}(\Lambda^\alpha)$ and $L_\infty = L_0 + g^2 V_{eff}(\infty)$. By using the second resolvent equation for the pair $\{L_\Lambda, L_0\}$ repeatedly, we have

$$\begin{aligned} (L_\Lambda - z)^{-1} &= \sum_{n=0}^N (-1)^n (L_0 - z)^{-1} \left(g^2 V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1}\right)^n \\ &\quad +(-1)^{N+1}(L_\Lambda - z)^{-1} \left(g^2 V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1}\right)^{N+1}. \end{aligned}$$

Since, for $z < 0$ with $|z|$ sufficiently large, uniformly in $\Lambda > \Lambda_0$ with some $\Lambda_0 > 0$,

$$\|(L_\Lambda - z)^{-1}\left(g^2V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1}\right)^N\|_{B(L^2(\mathbb{R}^{dN}))} \to 0, \quad as \;\; N \to \infty,$$

one can easily derive that for such $z < 0$

$$(L_\Lambda - z)^{-1} = \sum_{n=0}^{\infty}(-1)^n(L_0 - z)^{-1}\left(g^2V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1}\right)^n$$

in the uniform operator norm uniformly in $\Lambda > \Lambda_0$ with some $\Lambda_0 > 0$. Thus we can exchange $\sum_{n=0}^{\infty}$ and $\lim_{\Lambda\to\infty}$. Then by (4.3), in the uniform operator norm,

$$\begin{aligned}\lim_{\Lambda\to\infty}(L_\Lambda - z)^{-1} &= \sum_{n=0}^{\infty}(-1)^n(L_0 - z)^{-1}\left(g^2V_{eff}(\infty)(L_0 - z)^{-1}\right)^n \\ &= (L_\infty - z)^{-1}.\end{aligned}$$

Thus

$$||(L_\Lambda - z)^{-1}||_{B(L^2(\mathbb{R}^{dN}))} \to ||(L_\infty - z)^{-1}||_{B(L^2(\mathbb{R}^{dN}))}, \quad as \; \Lambda \to \infty.$$

Hence

$$\lim_{\Lambda\to\infty} S_{eff}(\Lambda) = S_{eff},$$

which, combining Theorem 4.1, implies (4.2). □

We have the following theorem.

Theorem 4.3 ([7,Theorem 2.14]) *Let $R > 0$ and T be the union of a finite number of mutually disjoint, bounded open intervals on $\mathbb{R}$ such that $[-R, R] \cap \sigma(-\frac{1}{2m}\Delta - g^2V_{eff}(\infty)) \subset T$. Let $E_\Lambda(\lambda)$ be the spectral family of $H(\Lambda)$. Then*

$$s - \lim_{\Lambda\to\infty} E_\Lambda([-R, R] \setminus T) = 0.$$

This is called "the part of the spectrum of $H(\Lambda)$ in $[-R, R]$ is asymptotically concentrated on T"([11,P.46]).

5 CONCLUDING REMARKS

(1) In the case where $d = 1$, we do not need to subtract the renormalization term $E(\Lambda^\alpha)$ from the total Hamiltonian $H(\Lambda)$ (see Remark 3.3). In this case Theorem 3.10 is modified under the conditions as in Theorem 3.10 as follows

$$s - \lim_{\Lambda\to\infty}(H(\Lambda) - z)^{-1} = \left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\infty) - z\right)^{-1} \otimes P_\Omega,$$

where

$$V_{eff}(x^1, ..., x^N; \infty) = -\frac{1}{4}\sum_{i,j=1}^{N} \frac{e^{-\mu|x^i - x^j|}}{\mu}.$$

(2) Let V be a $-\Delta$-bounded with a relative bound less than one, e.g., $V \in L^p(\mathbb{R}^d), p \geq \frac{d}{2}, p \geq 2$. Then it is well known that $H(\Lambda) + V \otimes I$ is self-adjoint on $D(H_0)$ and bounded from below. Moreover, as in the case where $V = 0$, one can show that under the conditions as in Theorem 3.10,

$$s - \lim_{\Lambda\to\infty}(H(\Lambda) + V \otimes I - g^2 N E(\Lambda^\alpha) - z)^{-1} = \left(-\frac{1}{2m}\Delta + V + g^2 V_{eff}(\infty) - z\right)^{-1} \otimes P_\Omega.$$

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