



Title	Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field
Author(s)	Hiroshima, F.
Citation	Hokkaido University Preprint Series in Mathematics, 350, 1-21
Issue Date	1996-9-1
DOI	10.14943/83496
Doc URL	<a href="http://hdl.handle.net/2115/69100">http://hdl.handle.net/2115/69100</a>
Type	bulletin (article)
File Information	pre350.pdf



[Instructions for use](#)

**Weak coupling limit with a removal of  
an ultraviolet cut-off for a Hamiltonian  
of particles interacting with  
a massive scalar field**

**Fumio Hiroshima**

Series #350. September 1996

**HOKKAIDO UNIVERSITY**  
**PREPRINT SERIES IN MATHEMATICS**

- #325 K. Sugano, On strongly separable Frobenius extensions, 11 pages. 1995.
- #326 D. Lehmann and T. Suwa, Residues of holomorphic vector fields on singular varieties, 21 pages. 1995.
- #327 K. Tsutaya, Local regularity of non-resonant nonlinear wave equations, 23 pages. 1996.
- #328 T. Ozawa and Y. Tsutsumi, Space-time estimates for null gauge forms and nonlinear Schrödinger equations, 25 pages. 1996.
- #329 O. Ogurusu, Anticommutativity and spin 1/2 Schrödinger operators with magnetic fields, 12 pages. 1996.
- #330 Y. Kurokawa, Singularities for projections of contour lines of surfaces onto planes, 24 pages. 1996.
- #331 M.-H. Giga and Y. Giga, Evolving graphs by singular weighted curvature, 94 pages. 1996.
- #332 M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the  $p$ -laplace diffusion equation, 20 pages. 1996.
- #333 T. Nakazi, The spectra of Toeplitz operators with unimodular symbols, 9 pages. 1996.
- #334 B. Khanedani and T. Suwa, First variation of holomorphic forms and some applications, 11 pages. 1996.
- #335 J. Seade and T. Suwa, Residues and topological invariants of singular holomorphic foliations<sup>1</sup>, 28 pages. 1996.
- #336 Y. Giga, M.E. Gurtin and J. Matias, On the dynamics of crystalline motions, 67 pages. 1996.
- #337 I. Tsuda, A new type of self-organization associated with chaotic dynamics in neural networks, 22 pages. 1996.
- #338 F. Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, 34 pages. 1996.
- #339 N. Tominaga, Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed Dirac operators, 29 pages. 1996.
- #340 A. Inoue, Abel-Tauber theorems for Fourier-Stieltjes coefficients, 17 pages. 1996.
- #341 G. Ishikawa, Topological classification of the tangent developables of space curves, 19 pages. 1996.
- #342 Y. Shimizu, A remark on estimates of bilinear forms of gradients in Hardy space, 8 pages. 1996.
- #343 N. kawazumi and S. Morita, The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes, 11 pages. 1996.
- #344 M.-H. Giga and Y. Giga, A subdifferential interpretation of crystalline motion under nonuniform driving force, 18 pages. 1996.
- #345 A. Douai and H. Terao, The determinant of a hypergeometric period matrix, 20 pages. 1996.
- #346 H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of  $\square u = |u|^p$  for super critical values  $p$  in even space dimensions, 66 pages. 1996.
- #347 T. Nakazi and T. Yamamoto, Weighted Norm Inequalities For Some Singular Integral Operators, 17 pages. 1996.
- #348 Y. Ito and I. Nakamura, Hilbert schemes and simple singularities  $A_n$  and  $D_n$ , 22 pages. 1996
- #349 R. Agemi and K. Yokoyama, The null condition and global existence of solutions to systems of wave equations with different speeds, 42 pages. 1996

# Weak coupling limit with a removal of an ultraviolet cut-off for a Hamiltonian of particles interacting with a massive scalar field

Fumio HIROSHIMA

Department of Mathematics, Faculty of Science, Hokkaido University  
Sapporo 060 Japan

## Abstract

This paper presents a rigorous derivation of an  $N$ -body Schrödinger Hamiltonian from taking a weak coupling limit of a renormalized Hamiltonian describing an interaction of  $N$ -spinless particles and a massive scalar field *at the same time* as an ultraviolet cut-off is removed. In particular, in the case where the space dimension equals three, the Yukawa potential appears in the  $N$ -body Schrödinger Hamiltonian.

# 1 INTRODUCTION

In this paper we are concerned with a model of a quantum  $N$ -particle system interacting with a quantized massive scalar field. The main result is a mathematically rigorous derivation, from the point of view of the operator theory, of an  $N$ -body Schrödinger Hamiltonian by taking a weak coupling (scaling) limit of the Hamiltonian of the model with an ultraviolet cut-off ([1]) *at the same time* as the ultraviolet cut-off is removed ([2]). Such kind of scaling limits of Hamiltonians, but with ultraviolet cut-off fixed, have been discussed in [3,4,5] and references therein. A new feature in our approach is in taking a scaling limit and a removal of the ultraviolet cut-off simultaneously. For mathematical generality, we consider the case where the  $N$ -particles move in  $d$ -dimensional space,  $d=1,2,3$ . The total Hamiltonian  $H$  we analyze in this paper acts in the tensor product of Hilbert spaces  $\underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \cong L^2(\mathbb{R}^{dN})$  and the Boson Fock space  $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^d))$  over  $L^2(\mathbb{R}^d)$ , and has the following form

$$H = -\frac{1}{2m}\Delta \otimes I + I \otimes H_b - g\phi(\Lambda),$$

where  $H_b$  denotes the free Hamiltonian in  $\mathcal{F}$ ,  $\Delta$  the Laplacian in  $L^2(\mathbb{R}^{dN})$ ,  $\phi(\Lambda)$  a scalar field with an ultraviolet cut-off parameterized by  $\Lambda > 0$ , and  $g \in \mathbb{R}$ ,  $m > 0$  a coupling constant, the mass of particles, respectively. Let

$$H(\Lambda) = -\frac{1}{2m}\Delta \otimes I + \Lambda^2 I \otimes H_b - g\Lambda\phi(\Lambda^\alpha), \quad \alpha > 0.$$

We are concerned with the asymptotic behavior of  $H(\Lambda)$  with a renormalization term subtracted as  $\Lambda \rightarrow \infty$  in the strong resolvent sense, and determine a range of the parameter  $\alpha$ , in which  $H(\Lambda)$  converges to an  $N$ -body Schrödinger Hamiltonian. In the case where we take  $\Lambda$ 's in the coefficients of  $I \otimes H_b$  and  $\phi$  infinity with  $\Lambda^\alpha$  in  $\phi(\Lambda^\alpha)$  replaced by a fixed parameter, this limit is called a "weak coupling limit". Conversely, in the case where  $\Lambda^\alpha$  in  $\phi(\Lambda^\alpha)$  tends to infinity with the other  $\Lambda$ 's replaced by fixed parameters, the limit is called a "removal of the ultraviolet cut-off". It is well known that both of these limits exist in some

sense ([1,2]). In this paper we take the weak coupling limit and the removal of the ultraviolet cut-off simultaneously. The basic idea to analyze  $H(\Lambda)$  is to construct a dressing transformation ([2,6]). By using the dressing transformation and an abstract theory discussed in [7], we prove our main result in Theorem 3.10; assume that  $0 < \alpha < \frac{2}{d+1}$ , then

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - g^2 N E(\Lambda^\alpha) - z)^{-1} = \left( -\frac{1}{2m} \Delta + g^2 V_{eff}(\infty) - z \right)^{-1} \otimes P_\Omega, \quad (1.1)$$

where  $E(\Lambda^\alpha)$  is a renormalization term which goes to minus infinity as  $\Lambda \rightarrow \infty$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$  or  $z < 0$  with  $|z|$  sufficiently large,  $P_\Omega$  a projection operator on  $\mathcal{F}$  and  $V_{eff}(\infty)$  a multiplication operator in  $L^2(\mathbb{R}^{dN})$ . In particular, in the case  $d = 3$ ,  $V_{eff}(\infty)$  coincides with the Yukawa potential with the mass of the scalar field  $\mu > 0$ :

$$V_{eff}(x^1, \dots, x^N; \infty) = -\frac{1}{4\pi} \sum_{i < j}^N \frac{e^{-\mu|x^i - x^j|}}{|x^i - x^j|}.$$

Moreover in the case  $d = 1$ ,  $V_{eff}(\infty)$  is as follows:

$$V_{eff}(x^1, \dots, x^N; \infty) = -\frac{1}{2} \sum_{i < j}^N \frac{e^{-\mu|x^i - x^j|}}{\mu}.$$

Consequently the classical Schrödinger Hamiltonian with the Yukawa potential may be interpreted as the approximation in the sense proposed in (1.1).

The article is organized as follows. In section 2, for mathematical maximum generality, we specify our model and consider its asymptotic behavior in an abstract form. In section 3, we construct a dressing transformation which implements a unitary equivalence between the Hamiltonian  $H(\Lambda)$  and a special version of an operator discussed in section 2. Using the dressing transformation, we present an asymptotic behavior of  $H(\Lambda)$  with renormalization term  $E(\Lambda^\alpha)$  subtracted in the strong resolvent sense. In section 4, we consider an asymptotic behavior of the ground state of  $H(\Lambda)$  and spectral concentration. In section 5, we give concluding remarks.

The author would like to thank Professor Asao Arai and Professor Akihiko Inoue for helpful discussions and their continual encouragement throughout this work.

## 2 AN ABSTRACT SCALING LIMIT

In this section, for the succeeding discussion and for the sake of self-consistency of this paper, we review a theory of scaling limit of self-adjoint operators acting in the tensor product of two Hilbert spaces discussed in [7] with a little modification, which is a mathematical generalization that can be applied to our model. To begin with, let us introduce preliminary notations. The scalar product and the associated norm of a Hilbert space  $\mathcal{L}$  are denoted by  $(\cdot, \cdot)_{\mathcal{L}}$  and  $\|\cdot\|_{\mathcal{L}}$ , respectively; the scalar product is linear in  $\cdot$ , antilinear in  $*$ . The domain and the spectrum of an operator  $T$  are denoted by  $D(T)$  and  $\sigma(T)$ , respectively. Moreover the uniform operator norm of bounded operators on  $\mathcal{L}$  is denoted by  $\|\cdot\|_{B(\mathcal{L})}$ .

Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces, respectively. Put  $\mathcal{H} \otimes \mathcal{K} = \mathcal{X}$ . Suppose that an operator  $A$  (resp.  $B$ ) is a non-negative self-adjoint operator in  $\mathcal{H}$  (resp.  $\mathcal{K}$ ), and  $\text{Ker } B \neq \{0\}$ . We denote the projection from  $\mathcal{K}$  onto  $\text{Ker } B$  by  $P_0$ . We suppose that a family of symmetric operators  $\{C_{\Lambda}\}_{\Lambda>0}$  in  $\mathcal{X}$  admits the following conditions:

- (i) For any  $\epsilon > 0$  there exists  $\Lambda'$  such that for all  $\Lambda > \Lambda'$ ,  $D(C_{\Lambda}) \subset D(A \otimes I) \cap D(I \otimes B) = D(A \otimes I + I \otimes B) \equiv D_{AB}$  with

$$\|C_{\Lambda}\Phi\|_{\mathcal{X}} \leq \epsilon\|(A \otimes I + I \otimes B)\Phi\|_{\mathcal{X}} + b(\epsilon)\|\Phi\|_{\mathcal{X}},$$

where  $b(\epsilon) > 0$  is a constant independent of  $\Lambda > \Lambda'$ .

- (ii) There exists a symmetric operator  $C$  in  $\mathcal{X}$  such that for  $z \in \mathbb{C} \setminus [0, \infty)$ ,

$$s - \lim_{\Lambda \rightarrow \infty} C_{\Lambda}(A \otimes I + \Lambda I \otimes B - z)^{-1} = C(A - z)^{-1} \otimes P_0.$$

**Proposition 2.1** ([7, Theorem 2.1]) *Let operators  $A$ ,  $B$ ,  $C_{\Lambda}$  and  $C$  be given as above. Then the following (1)  $\sim$  (4) hold.*

- (1) For  $\Lambda > \Lambda_0$  with some  $\Lambda_0$ ,  $K_{\Lambda} \equiv A \otimes I + \Lambda I \otimes B + C_{\Lambda}$  is self-adjoint on  $D_{AB}$  and

uniformly bounded from below. Moreover it is essentially self-adjoint on any core of  $A \otimes I + I \otimes B$ .

(2)  $K_\infty \equiv A \otimes I + (I \otimes P_0)C(I \otimes P_0)$  is self-adjoint on  $D(A \otimes I)$  and bounded from below.

Moreover it is essentially self-adjoint on any core of  $A \otimes I$ .

(3) For  $z \in \cap_{\Lambda > \Lambda_0} \rho(K_\Lambda) \cap \rho(K_\infty)$  ( $\rho(T)$  denotes the resolvent set of an operator  $T$ ),

$$s - \lim_{\Lambda \rightarrow \infty} (K_\Lambda - z)^{-1} = (K_\infty - z)^{-1}(I \otimes P_0).$$

(4)  $K_\infty$  is reduced by  $\mathcal{H} \otimes \text{Ker} B$ . Moreover the following holds:

$$\overline{\lim}_{\Lambda \rightarrow \infty} \inf \sigma(K_\Lambda) \leq \inf \sigma(K_\infty|_{\mathcal{H} \otimes \text{Ker} B}). \quad (2.1)$$

### 3 DEFINITION OF A HAMILTONIAN AND A DRESSING TRANSFORMATION

In this section we define a Hamiltonian  $H(\Lambda)$  describing an interaction of  $N$ -particles and a scalar field with an ultraviolet cut-off. To analyze an asymptotic behavior of the Hamiltonian, we construct a dressing transformation. The Boson Fock space  $\mathcal{F}$  over  $L^2(\mathbb{R}^d)$  is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} [\otimes_s^n L^2(\mathbb{R}^d)],$$

where  $\otimes_s^0 L^2(\mathbb{R}^d) = \mathbb{C}$  and  $\otimes_s^n L^2(\mathbb{R}^d)$  ( $n \geq 1$ ) is the  $n$ -fold symmetric tensor product of  $L^2(\mathbb{R}^d)$ . The vacuum vector is defined by  $\Omega = \{1, 0, 0, 0, \dots\}$  and the projection from  $\mathcal{F}$  onto the subspace  $\{\alpha\Omega | \alpha \in \mathbb{C}\}$  is denoted by  $P_\Omega$ . Set

$$\mathcal{F}^\infty(\mathcal{W}) = \{ \{\Psi_n\}_{n=0}^\infty \in \mathcal{F} | \Psi_n = 0 \text{ for all but finitely many } n \text{'s} \}.$$

The creation operator and the annihilation operator are denoted by  $a^\dagger(f)$  and  $a(g)$ ,  $f, g \in L^2(\mathbb{R}^d)$ , respectively. Formally we write

$$a^\dagger(f) = \int a^\dagger(k) f(k) dk, \quad f \in L^2(\mathbb{R}^d).$$



The operators  $a^\sharp(f)$  leave  $\mathcal{F}^\infty(\mathcal{W})$  invariant and satisfy

$$\begin{aligned} [a(f), a^\dagger(g)]\Phi &= (\bar{f}, g)_{L^2(\mathbb{R}^d)}\Phi \\ [a^\sharp(f), a^\sharp(g)]\Phi &= 0 \\ (a(f)\Phi, \Psi)_{\mathcal{F}} &= (\Phi, a^\dagger(\bar{f})\Psi)_{\mathcal{F}}, f, g \in L^2(\mathbb{R}^d), \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}). \end{aligned}$$

The scalar field  $\phi(x, f)$ , ( $x \in \mathbb{R}^d$ ) and the momentum conjugate  $\Pi(x, g)$ , ( $x \in \mathbb{R}^d$ ) are defined by

$$\begin{aligned} \phi(x, f) &= \frac{1}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) \frac{f(-k)e^{-ikx}}{\sqrt{\omega(k)}} + a(k) \frac{f(k)e^{ikx}}{\sqrt{\omega(k)}} \right\} dk, f/\sqrt{\omega} \in L^2(\mathbb{R}^d), \\ \Pi(x, g) &= \frac{i}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) f(-k) \sqrt{\omega(k)} e^{-ikx} - a(k) f(k) \sqrt{\omega(k)} e^{ikx} \right\} dk, f\sqrt{\omega} \in L^2(\mathbb{R}^d), \end{aligned}$$

where  $\omega(k) = \sqrt{k^2 + \mu^2}$ ,  $\mu > 0$ . In particular, for an indicator function  $\Xi_\Lambda$ :

$$\Xi_\Lambda(k) = \begin{cases} 1, & |k| < \Lambda, \\ 0, & |k| \geq \Lambda, \end{cases}$$

we set

$$\phi(x, \Xi_\Lambda) = \phi(x, \Lambda).$$

Consider the semi-group generated by the multiplication operator  $\omega$  in  $L^2(\mathbb{R}^d)$ . Its second quantization  $\Gamma(e^{-t\omega})$ ,  $t \geq 0$ , is a strongly continuous contraction semi-group on  $\mathcal{F}$  (cf.[8]).

The free Hamiltonian  $H_b$  is defined to be its generator:

$$\Gamma(e^{-t\omega}) = e^{-tH_b}.$$

Formally we may write

$$H_b = \int \omega(k) a^\dagger(k) a(k) dk.$$

Thus we have the following commutation relations:

$$[\phi(x, f), \phi(y, g)] = 0, \quad f/\sqrt{\omega}, g/\sqrt{\omega} \in L^2(\mathbb{R}^d),$$

$$\begin{aligned} [\Pi(x, f), \Pi(y, g)] &= 0, \quad \sqrt{\omega}f, \sqrt{\omega}g \in L^2(\mathbb{R}^d), \\ [\phi(x, f), \Pi(y, g)] &= \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} f(-k)g(k)e^{-ik(x-y)}dk, \quad f/\sqrt{\omega}, \sqrt{\omega}g \in L^2(\mathbb{R}^d), \end{aligned}$$

on  $\mathcal{F}^\infty(\mathcal{W})$  and on  $D(H_b^{\frac{3}{2}})$

$$\begin{aligned} [H_b, \phi(x, f)] &= -i\Pi(x, f), \quad f/\sqrt{\omega}, \sqrt{\omega}f \in L^2(\mathbb{R}^d), \\ [H_b, \Pi(x, f)] &= i\phi(x, \omega^2 f), \quad \sqrt{\omega}f, \omega\sqrt{\omega}f \in L^2(\mathbb{R}^d). \end{aligned}$$

The total Hamiltonian  $H(\Lambda)$  of our model acts in

$$\mathcal{L} = \underbrace{L^2(\mathbb{R}^d) \otimes \dots \otimes L^2(\mathbb{R}^d)}_N \otimes \mathcal{F} \cong \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F} dx^1 \dots dx^N$$

and is defined as

$$H(\Lambda) = -\frac{1}{2m}\Delta \otimes I + \Lambda^2 I \otimes H_b - g\Lambda\phi(\Lambda^\alpha), \quad \Lambda > 0, \quad \alpha > 0,$$

where

$$\phi(\Lambda^\alpha) = \int_{\mathbb{R}^{dN}}^{\oplus} \sum_{i=1}^N \phi(x^i, \Lambda^\alpha) dx^1 \dots dx^N.$$

For later use, we have introduced the scaling parameters  $\Lambda$  and  $\alpha$ ; a range of  $\alpha$  is determined in the rest of this section. Put

$$H_0 = -\frac{1}{2m}\Delta \otimes I + I \otimes H_b.$$

**Proposition 3.1** ([2]) *The operator  $H(\Lambda)$  is self-adjoint on  $D(H_0)$  and bounded from below. Moreover it is essentially self-adjoint on any core of  $H_0$ .*

It is well known that for each  $(x^1, \dots, x^N) \in \mathbb{R}^{dN}$ ,  $\sum_{j=1}^N \Pi\left(x^j, \frac{\Xi\Lambda^\alpha}{\omega^2}\right)$  is essentially self-adjoint on  $\mathcal{F}^\infty(\mathcal{W})$ . We denote its self-adjoint extension by the same symbol. Let

$$U_\Lambda(g) = \int_{\mathbb{R}^{dN}}^{\oplus} \exp\left(-i\frac{g}{\Lambda} \sum_{j=1}^N \Pi\left(x^j, \frac{\Xi\Lambda^\alpha}{\omega^2}\right)\right) dx^1 \dots dx^N.$$

We define a renormalization term  $E(\Lambda^\alpha)$  by

$$E(\Lambda^\alpha) = -\frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{\Xi_{\Lambda^\alpha}(k)}{k^2 + \mu^2} dk.$$

Set

$$\mathcal{L}^\infty \equiv [C_0^\infty(\mathbb{R}^{dN}) \hat{\otimes} \mathcal{F}^\infty(\mathcal{W})] \cap D(I \otimes H_b),$$

where  $\hat{\otimes}$  denotes the algebraic tensor product. The following lemma is the key lemma in this paper.

**Lemma 3.2** *The unitary operator  $U_\Lambda(g)$  maps  $\mathcal{L}^\infty$  into  $D(H(\Lambda))$  with*

$$\begin{aligned} U_\Lambda(g)^{-1}(H(\Lambda) - g^2 N E(\Lambda^\alpha)) U_\Lambda(g) &= \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left( p_\mu^j \otimes I - \frac{g}{\Lambda} \phi_\mu^j(\Lambda^\alpha) - \frac{g^2}{\Lambda^2} V_\mu^j(\Lambda^\alpha) \otimes I \right)^2 \\ &\quad + \Lambda^2 I \otimes H_b + g^2 V_{eff}(\Lambda^\alpha) \otimes I, \end{aligned} \quad (3.1)$$

where  $p_\mu^j = -iD_{x_\mu^j}$ ,  $\mu = 1, \dots, d, j = 1, \dots, N$ ,  $(x^j = (x_1^j, \dots, x_d^j) \in \mathbb{R}^d)$  denotes the generalized  $L^2$ -derivative.

$$\begin{aligned} \phi_\mu^j(\Lambda^\alpha) &= \int_{\mathbb{R}^{dN}}^\oplus \phi_\mu(x^j, \Lambda^\alpha) dx^1 \dots dx^N, \quad j = 1, \dots, N, \mu = 1, \dots, d, \\ \phi_\mu(x^j, \Lambda^\alpha) &= \frac{1}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) \frac{\Xi_{\Lambda^\alpha}(k) k_\mu e^{-ikx^j}}{\sqrt{\omega(k)^3}} + a(k) \frac{\Xi_{\Lambda^\alpha}(k) k_\mu e^{ikx^j}}{\sqrt{\omega(k)^3}} \right\} dk, \end{aligned}$$

$V_{eff}(\Lambda^\alpha)$  and  $V_\mu^j(\Lambda^\alpha)$  are multiplication operators in  $L^2(\mathbb{R}^{dN})$ :

$$\begin{aligned} V_{eff}(x^1, \dots, x^N; \Lambda^\alpha) &= -\frac{1}{2} \sum_{i=1, j=1, i \neq j}^N \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\Xi_{\Lambda^\alpha}(k) e^{-ik(x^i - x^j)}}{\omega(k)^2} dk, \\ V_\mu^j(x^1, \dots, x^N; \Lambda^\alpha) &= \sum_{i=1, i \neq j}^N \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{k_\mu \Xi_{\Lambda^\alpha}(k) e^{-ik(x^j - x^i)}}{\omega(k)^3} dk, \\ &\quad j = 1, \dots, N, \mu = 1, \dots, d. \end{aligned}$$

*Proof:* Note that  $V_{eff}(\Lambda^\alpha)$  is bounded operator on  $L^2(\mathbb{R}^{dN})$ . By [6, Theorem 3.3],  $U_\Lambda(g)$  maps  $D(I \otimes H_b)$  onto itself with

$$U_\Lambda(g)^{-1}(-g\Lambda\phi(\Lambda^\alpha) + \Lambda^2 I \otimes H_b) U_\Lambda(g) = \Lambda^2 I \otimes H_b + g^2 V_{eff}(\Lambda^\alpha) \otimes I + g^2 N E(\Lambda^\alpha). \quad (3.2)$$

On the other hand, it can be easily seen that on  $\mathcal{L}^\infty$

$$U_\Lambda(g)^{-1}(p_\mu^j \otimes I)U_\Lambda(g) = p_\mu^j \otimes I - \frac{g}{\Lambda}\phi_\mu^j(\Lambda^\alpha) - \frac{g^2}{\Lambda^2}V_\mu^j(\Lambda^\alpha) \otimes I.$$

We have thus proved the lemma.  $\square$

**Remark 3.3** *The renormalization term  $E(\Lambda^\alpha)$  coincides with that of [2]. If  $d = 2, 3$ ,  $\lim_{\Lambda \rightarrow \infty} E(\Lambda^\alpha) = -\infty$ , while, if  $d = 1$ ,  $\lim_{\Lambda \rightarrow \infty} E(\Lambda^\alpha) > -\infty$ . However, to proceed our argument systematically, even in the case where  $d = 1$ , we subtract the renormalization term from the total Hamiltonian, and do not distinguish between them.*

Let the right hand side of (3.1) be  $H^D(\Lambda)$  and define a symmetric operator  $C(\Lambda^\alpha)$  by

$$H^D(\Lambda) = -\frac{1}{2m}\Delta \otimes I + \Lambda^2 I \otimes H_b + C(\Lambda^\alpha).$$

Moreover, for notational simplicity, we introduce the following notations:

$$\begin{aligned} Q_1(\Lambda^\alpha) &= \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left( (p_\mu^j \otimes I) q_\mu^j(\Lambda^\alpha) + q_\mu^j(\Lambda^\alpha) (p_\mu^j \otimes I) \right), \\ Q_2(\Lambda^\alpha) &= \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d q_\mu^j(\Lambda^\alpha)^2, \\ q_\mu^j(\Lambda^\alpha) &= -\phi_\mu^j(\Lambda^\alpha) - \frac{g}{\Lambda} V_\mu^j(\Lambda^\alpha) \otimes I. \end{aligned}$$

Thus in terms of the operators  $Q_1(\Lambda^\alpha)$ ,  $Q_2(\Lambda^\alpha)$  and  $V_{eff}(\Lambda^\alpha)$ , we may write

$$C(\Lambda^\alpha) = g^2 V_{eff}(\Lambda^\alpha) \otimes I + \frac{g}{\Lambda} Q_1(\Lambda^\alpha) + \frac{g^2}{\Lambda^2} Q_2(\Lambda^\alpha).$$

We write  $f(\Lambda) \sim \Lambda^k$  if  $\frac{f(\Lambda)}{\Lambda^k}$  is bounded as  $\Lambda \rightarrow \infty$ .

**Lemma 3.4** *It holds that  $D(Q_1(\Lambda^\alpha)) \supset D(H_0)$  and  $D(Q_2(\Lambda^\alpha)) \supset D(H_0)$ . Moreover the following inequalities hold for  $\Phi \in D(H_0)$*

$$\|Q_1(\Lambda^\alpha)\Phi\|_{\mathcal{L}} \leq (\epsilon_1(\Lambda^\alpha) + \epsilon_2(\Lambda^\alpha))\|(H_0 + I)\Phi\|_{\mathcal{L}}, \quad (3.3)$$

$$\|Q_2(\Lambda^\alpha)\Phi\|_{\mathcal{L}} \leq (\eta_1(\Lambda^\alpha) + \eta_2(\Lambda^\alpha) + \eta_3(\Lambda^\alpha))\|(H_0 + I)\Phi\|_{\mathcal{L}}, \quad (3.4)$$

where  $\epsilon_1(\Lambda^\alpha), \epsilon_2(\Lambda^\alpha), \eta_1(\Lambda^\alpha), \eta_2(\Lambda^\alpha)$  and  $\eta_3(\Lambda^\alpha)$  are positive numbers depending on  $\Lambda$  such that

$$\begin{aligned}\epsilon_1(\Lambda^\alpha) &\sim \Lambda^{\frac{\alpha(d+1)}{2}}, \epsilon_2(\Lambda^\alpha) \sim \Lambda^{\alpha(d-1)-1}, \\ \eta_1(\Lambda^\alpha) &\sim \Lambda^{\frac{\alpha(2d-1)}{2}}, \eta_2(\Lambda^\alpha) \sim \Lambda^{\frac{\alpha(3d-5)}{2}-1}, \eta_3(\Lambda^\alpha) \sim \Lambda^{2\alpha(d-2)-2}.\end{aligned}$$

*Proof:* In the proof we write operators of the form  $A \otimes I$  in  $\mathcal{L}$  as  $A$  and set for  $k = (k_1, \dots, k_d) \in \mathbb{R}^d$

$$\begin{aligned}f_\mu(k) &= \frac{\Xi_{\Lambda^\alpha}(k)k_\mu}{\omega(k)}, \\ f_{\mu\mu}(k) &= \frac{\Xi_{\Lambda^\alpha}(k)k_\mu^2}{\omega(k)}, \mu = 1, \dots, d.\end{aligned}$$

Moreover

$$\begin{aligned}\pi_{\mu\mu}^j(\Lambda^\alpha) &= i \int_{\mathbb{R}^{dN}}^\oplus \pi_{\mu\mu}(x^j, \Lambda^\alpha) dx^1 \dots dx^N, \\ \pi_{\mu\mu}(x^j, \Lambda^\alpha) &= \frac{i}{\sqrt{2(2\pi)^d}} \int \left\{ a^\dagger(k) \frac{\Xi_{\Lambda^\alpha}(k)k_\mu^2 e^{-ikx^j}}{\sqrt{\omega(k)^3}} - a(k) \frac{\Xi_{\Lambda^\alpha}(k)k_\mu^2 e^{ikx^j}}{\sqrt{\omega(k)^3}} \right\} dk, \\ V_{\mu\mu}^j(x^1, \dots, x^N; \Lambda^\alpha) &= \sum_{i=1, i \neq j}^N \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{k_\mu^2 \Xi_{\Lambda^\alpha}(k) e^{-ik(x^j - x^i)}}{\omega(k)^3} dk, \\ & \quad j = 1, \dots, N, \mu = 1, \dots, d.\end{aligned}$$

Set  $\|\sqrt{\omega^n} g\|_{L^2(\mathbb{R}^d)} \equiv \|g\|_n$ . Then note that

$$\begin{aligned}\|f_\mu\|_n &\sim \Lambda^{\frac{\alpha(n+d)}{2}}, (n+d \neq 0), \\ \|f_\mu\|_n &\sim \sqrt{\alpha \log \Lambda}, (n+d = 0), \\ \|f_{\mu\mu}\|_n &\sim \Lambda^{\frac{\alpha(n+d+2)}{2}}, (n+d+2 \neq 0), \\ \|f_{\mu\mu}\|_n &\sim \sqrt{\alpha \log \Lambda}, (n+d+2 = 0).\end{aligned} \tag{3.5}$$

The following inequalities are well known ([9, Lemmas 2.1, 2.4])

$$\|\phi(x, f)\Phi\|_{\mathcal{F}} \leq A \left( 2\|f\|_{-2} \|H_b^{\frac{1}{2}}\Phi\|_{\mathcal{F}} + \|f\|_{-1} \|\Phi\|_{\mathcal{F}} \right),$$

$$\begin{aligned}
\|\pi(x, f)\Phi\|_{\mathcal{F}} &\leq A \left( 2\|f\|_0 \|H_b^{\frac{1}{2}}\Phi\|_{\mathcal{F}} + \|f\|_1 \|\Phi\|_{\mathcal{F}} \right), \quad \Phi \in D(H_b^{\frac{1}{2}}) \\
\|[(H_b + I)^{\frac{1}{2}}, \phi(x, f)]\Psi\|_{\mathcal{F}} &\leq B \left( \|f\|_0 \|H_b^{\frac{1}{2}}\Psi\|_{\mathcal{F}} + \|f\|_1 \|\Psi\|_{\mathcal{F}} \right), \\
\|[(H_b + I)^{\frac{1}{2}}, \pi(x, f)]\Psi\|_{\mathcal{F}} &\leq B \left( \|f\|_2 \|H_b^{\frac{1}{2}}\Psi\|_{\mathcal{F}} + \|f\|_3 \|\Psi\|_{\mathcal{F}} \right), \quad \Psi \in D(H_b), \quad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
A &= \frac{1}{\sqrt{2(2\pi)^d}}, \\
B &= \frac{1}{\sqrt{2(2\pi)^d}} \left( \frac{1}{2\pi} \int_0^\infty \frac{\sqrt{\lambda}}{(\lambda+1)^2} d\lambda \right).
\end{aligned}$$

It can be directly seen that there exists  $\alpha(\Lambda^\alpha)$  and  $\beta(\Lambda^\alpha)$  so that

$$\begin{aligned}
|V_\mu^j(x^1, \dots, x^N; \Lambda^\alpha)| &\leq \alpha(\Lambda^\alpha), \\
|V_{\mu\mu}^j(x^1, \dots, x^N; \Lambda^\alpha)| &\leq \beta(\Lambda^\alpha), \quad j = 1, \dots, N, \mu = 1, \dots, d,
\end{aligned}$$

with  $\alpha(\Lambda^\alpha) \sim \Lambda^{\alpha(d-2)}$  and  $\beta(\Lambda^\alpha) \sim \Lambda^{\alpha(d-1)}$ . We first prove (3.3). Since, on  $D(H_0)$ ,

$$[p_\mu^j, q_\mu^j(\Lambda^\alpha)] = -\pi_{\mu\mu}^j(\Lambda^\alpha) - \frac{g}{\Lambda} V_{\mu\mu}^j(\Lambda^\alpha),$$

in terms of (3.6), we have that for  $\Phi \in D(H_0)$ ,

$$\begin{aligned}
&\|Q_1(\Lambda^\alpha)\Phi\|_{\mathcal{L}} \\
&\leq \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left( 2\|\phi_\mu^j(\Lambda^\alpha)p_\mu^j\Phi\|_{\mathcal{L}} + \frac{2g}{\Lambda} \|V_\mu^j(\Lambda^\alpha)p_\mu^j\Phi\|_{\mathcal{L}} + \|\pi_{\mu\mu}^j(\Lambda^\alpha)\|_{\mathcal{L}} + \frac{g}{\Lambda} \|V_{\mu\mu}^j(\Lambda^\alpha)\Phi\|_{\mathcal{L}} \right) \\
&\leq \sum_{\mu=1}^d \left\{ (\|f_\mu\|_{-2} + \|f_\mu\|_{-1} + \|f_{\mu\mu}\|_{-2} + \|f_{\mu\mu}\|_{-1}) + \left( \frac{\alpha(\Lambda^\alpha)}{\Lambda} + \frac{\beta(\Lambda^\alpha)}{\Lambda} \right) \right\} \\
&\times \text{constant} \times \|(H_0 + I)\Phi\|_{\mathcal{L}}. \quad (3.7)
\end{aligned}$$

Due to (3.5), in the first term on the right hand side of (3.7), the term  $\|f_{\mu\mu}\|_{-1} \sim \Lambda^{\frac{\alpha(d+1)}{2}}$  is the highest order with respect to  $\Lambda$  as  $\Lambda \rightarrow \infty$ , and so is  $\frac{\beta(\Lambda^\alpha)}{\Lambda} \sim \Lambda^{\alpha(d-1)-1}$  in the second term. Then (3.3) follows. Next we prove (3.4). As is seen in the proof of (3.3), one can

easily see that

$$\begin{aligned}
& \|Q_2(\Lambda^\alpha)\Phi\|_{\mathcal{L}} \\
& \leq \frac{1}{2m} \sum_{j=1}^N \sum_{\mu=1}^d \left( \|\phi_\mu^j(\Lambda^\alpha)^2\Phi\|_{\mathcal{L}} + \frac{2g}{\Lambda} \|\phi_\mu^j(\Lambda^\alpha)V_\mu^j(\Lambda^\alpha)\Phi\|_{\mathcal{L}} + \frac{g^2}{\Lambda^2} \|V_\mu^j(\Lambda^\alpha)^2\Phi\|_{\mathcal{L}} \right), \\
& \leq \sum_{\mu=1}^d \left\{ \left( \|f_\mu\|_{-2} \sum_{n=-2}^1 \|f_\mu\|_n + \|f_\mu\|_{-1} \sum_{n=-2}^{-1} \|f_\mu\|_n \right) + \frac{\alpha(\Lambda^\alpha)}{\Lambda} (\|f_\mu\|_{-2} + \|f_\mu\|_{-1}) + \frac{\alpha(\Lambda^\alpha)^2}{\Lambda^2} \right\} \\
& \times \text{constant} \times \|(H_0 + I)\Phi\|_{\mathcal{L}}. \tag{3.8}
\end{aligned}$$

Due to (3.5), in the first term on the right hand side of (3.8), the term  $\|f_\mu\|_{-2}\|f_\mu\|_1 \sim \Lambda^{\frac{\alpha(2d-1)}{2}}$  is the highest order, and so is  $\frac{\alpha(\Lambda^\alpha)}{\Lambda}\|f_\mu\|_{-1} \sim \Lambda^{\frac{\alpha(3d-5)}{2}-1}$  in the second term. Moreover in the third term,  $\frac{\alpha(\Lambda^\alpha)^2}{\Lambda^2} \sim \Lambda^{2\alpha(d-2)-2}$ . Thus (3.4) follows. This completes the proof.  $\square$

**Lemma 3.5** *It holds that  $D(V_{eff}(\Lambda^\alpha)) \supset D(-\Delta)$  and, for any  $\epsilon > 0$ , there exists  $b(\epsilon) > 0$  such that for  $\Phi \in D(-\Delta)$  and for any  $\Lambda > \Lambda_0$  with some  $\Lambda_0 > 0$ ,*

$$\|V_{eff}(\Lambda^\alpha)\Phi\|_{L^2(\mathbb{R}^{dN})} \leq \epsilon \|-\Delta\Phi\|_{L^2(\mathbb{R}^{dN})} + b(\epsilon)\|\Phi\|_{L^2(\mathbb{R}^{dN})}. \tag{3.9}$$

*Proof:* For each  $(x^1, \dots, x^N) \in \mathbb{R}^{dN}$ , due to the spherically symmetry of  $\frac{\Xi\Lambda^\alpha}{\omega^2}$ , one can see that

$$V_{eff}(x^1, \dots, x^N; \Lambda^\alpha) = -\frac{1}{\sqrt{(2\pi)^d}} \sum_{i < j}^N \frac{1}{|x^i - x^j|^{\frac{d-1}{2}}} \int_0^{\Lambda^\alpha} \frac{r^{\frac{d-1}{2}}}{r^2 + \mu^2} \sqrt{r|x^i - x^j|} J_{\frac{d-2}{2}}(r|x^i - x^j|) dr, \tag{3.10}$$

where  $J_\nu$  is the Bessel function:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}.$$

Let

$$G(r) = \frac{r^{\frac{d-1}{2}}}{r^2 + \mu^2}.$$

At first, we prove (3.9) in the cases where  $d = 1, 2$ . Then, since  $G \in L^1(\mathbb{R}^d)$ , by a fundamental estimate of the Bessel functions (see [10, § 7.21])

$$\sup_{u \in [0, \infty)} \left| \sqrt{u} J_\nu(u) \right| \equiv \tilde{J}_\nu < \infty,$$

we see that

$$|V_{eff}(x^1, \dots, x^N; \Lambda^\alpha)| \leq \frac{1}{\sqrt{(2\pi)^d}} \tilde{J}_{\frac{d-2}{2}} \sum_{i < j}^N \frac{1}{|x^i - x^j|^{\frac{d-1}{2}}} \int_0^\infty G(r) dr. \quad (3.11)$$

Since, with the help of a direct extension of [8, Theorem X.16], one can easily see that the right hand side of (3.11) is infinitesimally small with respect to the Laplacian  $\Delta$  in  $L^2(\mathbb{R}^{dN})$ ,  $d = 1, 2$ . Thus (3.9) follows. Next we consider the case where  $d = 3$ . We note that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sin x}{\sqrt{x}}.$$

Hence, changing the variables in the integrand (3.10) and using the contour integrals along with closed curves  $\gamma_{ij}$ ,  $1 \leq i, j \leq N$ ,

$$\gamma_{ij} = \{z \in \mathbb{R} \mid -\Lambda^\alpha |x^i - x^j| \leq z \leq \Lambda^\alpha |x^i - x^j|\} \cup \{z \in \mathbb{C} \mid z = \Lambda^\alpha |x^i - x^j| e^{i\theta}, 0 \leq \theta \leq \pi\},$$

we see that

$$\begin{aligned} V_{eff}(x^1, \dots, x^N; \Lambda^\alpha) &= -\frac{1}{4\pi^2} \Im \sum_{i < j}^N \frac{1}{|x^i - x^j|} \int_{-\Lambda^\alpha |x^i - x^j|}^{\Lambda^\alpha |x^i - x^j|} \frac{u}{u^2 + |x^i - x^j|^2 \mu^2} e^{iu} du \\ &= -\frac{2\pi}{4\pi^2} \sum_{i < j}^N \frac{1}{|x^i - x^j|} \operatorname{Res} \left( \frac{u e^{iu}}{u^2 + |x^i - x^j|^2 \mu^2}; i\mu |x^i - x^j| \right) \\ &\quad + \frac{1}{4\pi^2} \Im \sum_{i < j}^N \frac{1}{|x^i - x^j|} \int_0^\pi \frac{\Lambda^{2\alpha} |x^i - x^j|^2 i e^{2i\theta} e^{i\Lambda^\alpha |x^i - x^j| e^{i\theta}}}{\Lambda^{2\alpha} |x^i - x^j|^2 e^{2i\theta} + |x^i - x^j|^2 \mu^2} d\theta. \end{aligned}$$

It can be seen that

$$\left| \int_0^\pi \frac{\Lambda^{2\alpha} |x^i - x^j|^2 i e^{2i\theta} e^{i\Lambda^\alpha |x^i - x^j| e^{i\theta}}}{\Lambda^{2\alpha} |x^i - x^j|^2 e^{2i\theta} + |x^i - x^j|^2 \mu^2} d\theta \right| \leq \pi \left| \frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha} - \mu^2} \right|.$$

Hence we have that for each  $(x^1, \dots, x^N) \in \mathbb{R}^{3N}$ ,

$$|V_{eff}(x^1, \dots, x^N; \Lambda^\alpha)| \leq \frac{1}{4\pi} \sum_{i < j}^N \frac{1}{|x^i - x^j|} \left( e^{-\mu |x^i - x^j|} + \left| \frac{\Lambda^{2\alpha}}{\Lambda^{2\alpha} - \mu^2} \right| \right). \quad (3.12)$$



Since, for  $\Lambda > \mu^{\frac{1}{2}}$ , the right hand side of (3.12) is infinitesimally small with respect to  $\Delta$  in  $L^2(\mathbb{R}^{3N})$ , (3.9) follows. This completes the proof.  $\square$

**Lemma 3.6** *Assume that  $\alpha$  admits as follows*

$$0 < \alpha < \frac{2}{d+1}. \quad (3.13)$$

Then for any  $\Phi \in D(H_0)$ ,

$$s - \lim_{\Lambda \rightarrow \infty} \left( \frac{g}{\Lambda} Q_1(\Lambda^\alpha) + \frac{g^2}{\Lambda^2} Q_2(\Lambda^\alpha) + g^2 V_{eff}(\Lambda^\alpha) \otimes I \right) \Phi = (g^2 V_{eff}(\infty) \otimes I) \Phi,$$

where  $V_{eff}(\infty)$  is a multiplication operator in  $L^2(\mathbb{R}^{dN})$ :

$$V_{eff}(x^1, \dots, x^N; \infty) = - \sum_{i=1, j=1, i \neq j}^N \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-ik(x_i - x_j)}}{k^2 + \mu^2} dk.$$

In particular, in the case  $d = 1$ ,

$$V_{eff}(x^1, \dots, x^N; \infty) = -\frac{1}{2} \sum_{i < j}^N \frac{e^{-\mu|x^i - x^j|}}{\mu},$$

in the case  $d = 3$ ,

$$V_{eff}(x^1, \dots, x^N; \infty) = -\frac{1}{4\pi} \sum_{i < j}^N \frac{e^{-\mu|x^i - x^j|}}{|x^i - x^j|}.$$

*Proof:* Because of Lemma 3.4, we see that, for  $\Phi \in D(H_0)$ ,

$$g \|Q_1(\Lambda^\alpha) \Phi\|_{\mathcal{L}} + \frac{g^2}{\Lambda} \|Q_2(\Lambda^\alpha) \Phi\|_{\mathcal{L}} \sim \Lambda^{\frac{\alpha(d+1)}{2}}.$$

Hence, due to the assumption (3.13), one can see that

$$\lim_{\Lambda \rightarrow \infty} \left( \frac{g}{\Lambda} \|Q_1(\Lambda^\alpha) \Phi\|_{\mathcal{L}} + \frac{g^2}{\Lambda^2} \|Q_2(\Lambda^\alpha) \Phi\|_{\mathcal{L}} \right) = 0.$$

Next we consider the term  $V_{eff}(\Lambda^\alpha)$ . In the cases where  $d = 1, 2$ , by the proof of Lemma 3.5, it can be easily seen that for  $\Phi \in D(H_0)$

$$\|V_{eff}(\Lambda^\alpha) \Phi - V_{eff}(\infty) \Phi\|_{\mathcal{L}} = 0, \text{ as } \Lambda \rightarrow \infty. \quad (3.14)$$

In the case where  $d = 3$ , from the proof of Lemma 3.5, it follows that there exists a positive constant  $\beta$  so that, for each  $(x^1, \dots, x^N) \in \mathbb{R}^{3N}$ ,

$$\begin{aligned} |V_{eff}(x^1, \dots, x^N; \Lambda^\alpha) - V_{eff}(x^1, \dots, x^N; \infty)| &\leq \frac{\beta}{4\pi} \sum_{i < j}^N \frac{1}{|x^i - x^j|} \\ &\equiv \tilde{V}(x^1, \dots, x^N). \end{aligned} \quad (3.15)$$

Since  $\tilde{V}$  is infinitesimally small with respect to  $\Delta$  in  $L^2(\mathbb{R}^{3N})$ , for  $\Phi \in D(-\Delta)$ ,

$$\int_{\mathbb{R}^{3N}} |\tilde{V}(x^1, \dots, x^N)|^2 |\Phi(x^1, \dots, x^N)|^2 dx < \infty.$$

Let

$$\begin{aligned} \mathbf{H}_{ij} &= \{(x^1, \dots, x^N) \in \mathbb{R}^{3N} | x^i = x^j\}, \\ \mathbf{H} &= \bigcup_{i,j=1}^N \mathbf{H}_{ij}. \end{aligned}$$

If  $x = (x^1, \dots, x^N) \notin \mathbf{H}$ , from the proof of Lemma 3.5, it follows that

$$\begin{aligned} &|V_{eff}(x^1, \dots, x^N; \Lambda^\alpha) - V_{eff}(x^1, \dots, x^N; \infty)| \\ &\leq \frac{1}{4\pi^2} \sum_{i < j}^N \frac{1}{|x^i - x^j|^2} \frac{\Lambda^\alpha}{|\Lambda^{2\alpha} - \mu^2|} \int_0^\pi \Lambda^\alpha |x^i - x^j| e^{-\Lambda^\alpha |x^i - x^j| \sin \theta} d\theta \\ &\leq \frac{1}{4\pi} \sum_{i < j}^N \frac{1}{|x^i - x^j|^2} \frac{\Lambda^\alpha}{|\Lambda^{2\alpha} - \mu^2|} \rightarrow 0, \text{ as } \Lambda \rightarrow \infty. \end{aligned} \quad (3.16)$$

Since the Lebesgue measure of the set  $\mathbf{H}$  is zero, combining (3.15) and (3.16), the Lebesgue dominated convergence theorem yields (3.14). This completes the proof.  $\square$

**Remark 3.7 (1)** Assume that  $0 < \alpha < \frac{2}{d+1}$ . From Lemmas 3.4 3.5 and 3.6, it follows that for  $z \in \mathbb{C} \setminus [0, \infty)$ ,  $\|C(\Lambda^\alpha)(H_0 - z)^{-1}\|_{B(\mathcal{L})}$  is uniformly bounded in  $\Lambda > \Lambda_0$  with some  $\Lambda_0 > 0$ .

(2) As is seen in (3.12) in the case where  $d = 3$ , because of the fact that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2 \cos x}{\pi \sqrt{x}}},$$

in the case where  $d = 1$ , we can also estimate  $V_{eff}(x^1, \dots, x^N; \Lambda^\alpha)$  as follows

$$|V_{eff}(x^1, \dots, x^N; \Lambda^\alpha)| \leq \frac{1}{2} \sum_{i < j}^N \frac{e^{-\mu|x^i - x^j|}}{\mu} + \frac{1}{2} N(N-1) \left| \frac{\Lambda^\alpha}{\Lambda^{2\alpha} - \mu^2} \right|.$$

**Theorem 3.8** Let  $0 < \alpha < \frac{2}{d+1}$ . Then the following (1) and (2) hold.

- (1) For any  $\epsilon > 0$ , there exists  $\Lambda_0$  and  $b(\epsilon) > 0$  such that for all  $\Lambda > \Lambda_0$ ,  $D(C(\Lambda)) \supset D(H_0)$  and for  $\Phi \in D(H_0)$ ,

$$\|C(\Lambda)\Phi\|_{\mathcal{L}} \leq \epsilon \|H_0\Phi\|_{\mathcal{L}} + b(\epsilon) \|\Phi\|_{\mathcal{L}}.$$

- (2) Let  $z \in \mathbb{C} \setminus [0, \infty)$ . Then

$$s - \lim_{\Lambda \rightarrow \infty} C(\Lambda) \left( -\frac{1}{2m} \Delta \otimes I + \Lambda^2 I \otimes H_b - z \right)^{-1} = \left( g^2 V_{eff}(\infty) \left( -\frac{1}{2m} \Delta - z \right)^{-1} \right) \otimes P_\Omega.$$

*Proof:* Because of Lemmas 3.4 and 3.5, (1) follows. By Remark 3.7 and

$$s - \lim_{\Lambda \rightarrow \infty} \left( -\frac{1}{2m} \Delta \otimes I + I \otimes H_b - z \right) \left( -\frac{1}{2m} \Delta \otimes I + \Lambda^2 I \otimes H_b - z \right)^{-1} = I \otimes P_\Omega,$$

(2) follows. □

**Theorem 3.9** Let  $0 < \alpha < \frac{2}{d+1}$  and  $0 < \Lambda$  be sufficiently large. Then  $H^D(\Lambda)$  is self-adjoint on  $D(H_0)$  and bounded from below uniformly in  $\Lambda$  and essentially self-adjoint on any core of  $D(H_0)$ . Moreover the dressing transformation  $U_\Lambda(g)$  maps  $D(H_0)$  onto itself and the following unitary equivalence holds on  $D(H_0)$

$$U_\Lambda(g)^{-1} (H(\Lambda) - g^2 N E(\Lambda^\alpha)) U_\Lambda(g) = H^D(\Lambda). \quad (3.17)$$

*Proof:* Note that  $H^D(\Lambda)$  is just of the form of the operator  $K_\Lambda$  considered in Proposition 2.1 with the following identifications

$$A = -\frac{1}{2m} \Delta, \quad B = H_b, \quad C_\Lambda = C(\Lambda). \quad (3.18)$$

By Theorem 3.8 and Proposition 2.1 (1), the first assertion follows. The subset  $\mathcal{L}^\infty$  is a core of  $H_0$ , same is true for  $H^D(\Lambda)$ . Since (3.17) holds on  $\mathcal{L}^\infty$ , it can be easily seen that  $U_\Lambda(g)$  maps  $D(H^D(\Lambda))$  onto  $D(H(\Lambda))$  i.e.,  $D(H_0)$  onto itself, and (3.17) holds on  $D(H_0)$ .  $\square$

Now we state main theorem in this paper.

**Theorem 3.10** *Let  $0 < \alpha < \frac{2}{d+1}$  and  $z \in \mathbb{C} \setminus [0, \infty)$  or  $z < 0$  with  $|z|$  sufficiently large.*

*Then*

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - g^2 NE(\Lambda^\alpha) - z)^{-1} = \left( -\frac{1}{2m} \Delta + g^2 V_{eff}(\infty) - z \right)^{-1} \otimes P_\Omega.$$

*Proof:* From (3.17) it follows that

$$(H(\Lambda) - g^2 NE(\Lambda^\alpha) - z)^{-1} = U_\Lambda(g)(H^D(\Lambda) - z)^{-1}U_\Lambda(g)^{-1}.$$

Since

$$\lim_{\Lambda \rightarrow \infty} \frac{1}{\Lambda} \left\| \frac{\Xi_{\Lambda^\alpha}}{\omega^{\frac{3}{2}}} \right\|_{L^2(\mathbb{R}^d)} = 0,$$

it follows that

$$s - \lim_{\Lambda \rightarrow \infty} U_\Lambda(g) = I.$$

Note that

$$\text{Ker } H_b = \{\alpha \Omega \mid \alpha \in \mathbb{C}\}.$$

Thus under the identification as (3.18), by Proposition 2.1 (2), we get the desired result.  $\square$

## 4 GROUND STATE AND SPECTRAL CONCENTRATION

In this section we consider an asymptotic behavior of the ground state for  $H(\Lambda)$  and its spectral concentration. Let

$$S(\Lambda) = \inf \sigma(H(\Lambda) - g^2 NE(\Lambda)),$$

$$\begin{aligned} S_{eff}(\Lambda) &= \inf \sigma\left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha)\right), \\ S_{eff} &= \inf \sigma\left(-\frac{1}{2m}\Delta + g^2 V_{eff}(\infty)\right). \end{aligned}$$

**Theorem 4.1** *Let  $0 < \alpha < \frac{2}{d+1}$ . Then*

$$\lim_{\Lambda \rightarrow \infty} S_{eff}(\Lambda) \leq \lim_{\Lambda \rightarrow \infty} S(\Lambda) \leq S_{eff}. \quad (4.1)$$

*Proof:* Noting that  $V_{eff}(\Lambda^\alpha)$  commutes  $U_\Lambda(g)$ , by (3.2), we see that, for  $\Phi \in D(H_0)$ ,

$$\begin{aligned} & (H(\Lambda) - g^2 N E(\Lambda) \Phi, \Phi)_{\mathcal{L}} \\ &= \left( \left( -\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha) \right) \otimes I \Phi, \Phi \right)_{\mathcal{L}} + \left( (\Lambda^2 I \otimes H_b) U_\Lambda(g) \Phi, U_\Lambda(g) \Phi \right)_{\mathcal{L}} \\ &\geq \left( \left( -\frac{1}{2m}\Delta + g^2 V_{eff}(\Lambda^\alpha) \right) \otimes I \Phi, \Phi \right)_{\mathcal{L}}, \end{aligned}$$

which implies the first inequality in (4.1). The second inequality follows from (2.1).  $\square$

**Corollary 4.2** *Assume that  $0 < \alpha < \frac{2}{d+1}$  and  $d = 1, 2$ . Then*

$$\lim_{\Lambda \rightarrow \infty} S(\Lambda) = S_{eff}. \quad (4.2)$$

*Proof:* From (3.10) and (3.11) it follows that for  $\Phi \in L^2(\mathbb{R}^{dN})$ ,  $z \in \mathbb{C} \setminus [0, \infty)$  and  $\epsilon > 0$ , there exists  $b(\epsilon) > 0$  such that

$$\begin{aligned} & \left\| (V_{eff}(\Lambda^\alpha) - V_{eff}(\infty)) \left( -\frac{1}{2m}\Delta - z \right)^{-1} \Phi \right\|_{L^2(\mathbb{R}^{dN})} \\ & \leq \tilde{J}_{\frac{d-2}{2}} \int_{\Lambda^\alpha}^\infty G(r) dr \left( \epsilon + \frac{b(\epsilon)}{|z|} \right) \|\Phi\|_{L^2(\mathbb{R}^{dN})}, \end{aligned}$$

which implies that, *in the uniform operator norm*

$$\lim_{\Lambda \rightarrow \infty} V_{eff}(\Lambda^\alpha) \left( -\frac{1}{2m}\Delta - z \right)^{-1} = V_{eff}(\infty) \left( -\frac{1}{2m}\Delta - z \right)^{-1}. \quad (4.3)$$

Let  $L_0 = -\frac{1}{2m}\Delta$ ,  $L_\Lambda = L_0 + g^2 V_{eff}(\Lambda^\alpha)$  and  $L_\infty = L_0 + g^2 V_{eff}(\infty)$ . By using the second resolvent equation for the pair  $\{L_\Lambda, L_0\}$  repeatedly, we have

$$\begin{aligned} (L_\Lambda - z)^{-1} &= \sum_{n=0}^N (-1)^n (L_0 - z)^{-1} \left( g^2 V_{eff}(\Lambda^\alpha) (L_0 - z)^{-1} \right)^n \\ &\quad + (-1)^{N+1} (L_\Lambda - z)^{-1} \left( g^2 V_{eff}(\Lambda^\alpha) (L_0 - z)^{-1} \right)^{N+1}. \end{aligned}$$

Since, for  $z < 0$  with  $|z|$  sufficiently large, uniformly in  $\Lambda > \Lambda_0$  with some  $\Lambda_0 > 0$ ,

$$\|(L_\Lambda - z)^{-1} (g^2 V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1})^N\|_{B(L^2(\mathbb{R}^{dN}))} \rightarrow 0, \quad \text{as } N \rightarrow \infty,$$

one can easily derive that for such  $z < 0$

$$(L_\Lambda - z)^{-1} = \sum_{n=0}^{\infty} (-1)^n (L_0 - z)^{-1} (g^2 V_{eff}(\Lambda^\alpha)(L_0 - z)^{-1})^n$$

in the uniform operator norm uniformly in  $\Lambda > \Lambda_0$  with some  $\Lambda_0 > 0$ . Thus we can exchange  $\sum_{n=0}^{\infty}$  and  $\lim_{\Lambda \rightarrow \infty}$ . Then by (4.3), in the uniform operator norm,

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} (L_\Lambda - z)^{-1} &= \sum_{n=0}^{\infty} (-1)^n (L_0 - z)^{-1} (g^2 V_{eff}(\infty)(L_0 - z)^{-1})^n \\ &= (L_\infty - z)^{-1}. \end{aligned}$$

Thus

$$\|(L_\Lambda - z)^{-1}\|_{B(L^2(\mathbb{R}^{dN}))} \rightarrow \|(L_\infty - z)^{-1}\|_{B(L^2(\mathbb{R}^{dN}))}, \quad \text{as } \Lambda \rightarrow \infty.$$

Hence

$$\lim_{\Lambda \rightarrow \infty} S_{eff}(\Lambda) = S_{eff},$$

which, combining Theorem 4.1, implies (4.2).  $\square$

We have the following theorem.

**Theorem 4.3** ([7, Theorem 2.14]) *Let  $R > 0$  and  $T$  be the union of a finite number of mutually disjoint, bounded open intervals on  $\mathbb{R}$  such that  $[-R, R] \cap \sigma(-\frac{1}{2m}\Delta - g^2 V_{eff}(\infty)) \subset T$ . Let  $E_\Lambda(\lambda)$  be the spectral family of  $H(\Lambda)$ . Then*

$$s - \lim_{\Lambda \rightarrow \infty} E_\Lambda([-R, R] \setminus T) = 0.$$

*This is called "the part of the spectrum of  $H(\Lambda)$  in  $[-R, R]$  is asymptotically concentrated on  $T$ " ([11, P.46]).*

## 5 CONCLUDING REMARKS

- (1) In the case where  $d = 1$ , we do not need to subtract the renormalization term  $E(\Lambda^\alpha)$  from the total Hamiltonian  $H(\Lambda)$  (see Remark 3.3). In this case Theorem 3.10 is modified under the conditions as in Theorem 3.10 as follows

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) - z)^{-1} = \left( -\frac{1}{2m} \Delta + g^2 V_{eff}(\infty) - z \right)^{-1} \otimes P_\Omega,$$

where

$$V_{eff}(x^1, \dots, x^N; \infty) = -\frac{1}{4} \sum_{i,j=1}^N \frac{e^{-\mu|x^i-x^j|}}{\mu}.$$

- (2) Let  $V$  be a  $-\Delta$ -bounded with a relative bound less than one, e.g.,  $V \in L^p(\mathbb{R}^d)$ ,  $p \geq \frac{d}{2}$ ,  $p \geq 2$ . Then it is well known that  $H(\Lambda) + V \otimes I$  is self-adjoint on  $D(H_0)$  and bounded from below. Moreover, as in the case where  $V = 0$ , one can show that under the conditions as in Theorem 3.10,

$$s - \lim_{\Lambda \rightarrow \infty} (H(\Lambda) + V \otimes I - g^2 N E(\Lambda^\alpha) - z)^{-1} = \left( -\frac{1}{2m} \Delta + V + g^2 V_{eff}(\infty) - z \right)^{-1} \otimes P_\Omega.$$

## 6 REFERENCES

- [1] E.B.Davies, Particle-boson interactions and the weak coupling limit, J. Math. Phys. 20 (1979)345-351.
- [2] E.Nelson, Interaction of nonrelativistic particles with a quantized scalar field, J. Math. Phys. 5 (1964)1190-1197.
- [3] F.Hiroshima, Scaling limit of a model in quantum electrodynamics, J. Math. Phys. 34 (1993)4478-4518.
- [4] F.Hiroshima, Functional integral representation of a model in quantum electrodynamics, submitted to J.Funct.Anal.

- [5] F.Hiroshima, A scaling limit of a Hamiltonian of many nonrelativistic particles interacting with a quantized radiation field, to appear in *Rev.Math.Phys.*.
- [6] F.Hiroshima, Diamagnetic inequalities for systems of nonrelativistic particles with a quantized field, *Rev.Math.Phys.No.8 Vol.2 (1996)185-203*.
- [7] A.Arai, An asymptotic analysis and its applications to the nonrelativistic limit of the Pauli-Fierz and spin-boson model, *J.Math.Phys.31(1990)2653-2663*.
- [8] M.Reed, B.Simon, *Method of Modern Mathematical Physics II; Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1978.
- [9] A.Arai, Perturbation of embedded eigenvalues: A general class of exactly soluble models in Fock spaces, *Hokkaido Mathematical Journal Vol.XIX No.1 (1990)1-34*.
- [10] G.N.Watson, *Theory of Bessel functions*, 2nd ed., Cambridge Univ. Press, 1944.
- [11] M.Reed, B.Simon, *Method of Modern Mathematical Physics IV; Analysis of Operators*, Academic Press, New York, 1978.